ON SEMICONCAVITY VIA THE SECOND DIFFERENCE

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Let f be a continuous real function on a convex subset of a Banach space. We study what can be said about the semiconcavity (with a general modulus) of f, if we know that the estimate $\Delta_h^2(f,x) \leq \omega(\|h\|)$ holds, where $\Delta_h^2(f,x) = f(x+2h) - 2f(x-h) + f(x)$ and $\omega : [0,\infty) \to [0,\infty)$ is a nondecreasing function right continuous at 0 with $\omega(0) = 0$. A partial answer to this question was given by P. Cannarsa and C. Sinestrari (2004); we prove versions of their result, which are in a sense best possible. We essentially use methods of A. Marchaud, S.B. Stechkin and others, whose results clarify when the inequality $|\Delta_h^2(f,x)| \leq \omega(\|h\|)$ implies that f is a C^1 function (and f' is uniformly continuous with a corresponding modulus of continuity).

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1. Introduction

For a real function f on $[a,b]$, we consider the second difference $\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$. The second modulus of continuity of f is defined as $\omega_2(f,t) := \sup\{|\Delta_h^2(f,x)| : 0 \leq h \leq t, x, x+2h \in [a,b]\}$, $t \geq 0$.

The following result is a special case of the classical Marchaud theorem (see [10] or, e.g., [4]) which deals with $\omega_n(f,t), n \geq 2$. (For a generalization to Banach spaces see [9].)

Theorem M. *Let f be a continuous function on $[a,b]$ and*

$$\int_0^1 \frac{\omega_2(f,t)}{t^2}\, dt \;<\; \infty.$$

Then $f \in C^1[a,b]$ and f' is uniformly continuous with modulus of continuity

$$\eta(\delta) := C \int_0^\delta \frac{\omega_2(f,t)}{t^2}\, dt, \quad \delta \geq 0,$$

where $C > 0$ is an absolute constant.

Theorem M$^+$ below is a "semi-analogon" of Theorem M. Indeed, it works with the second "semi-modulus" of continuity $\omega_2^+(f,t) := \sup\{\Delta_h^2(f,x) : 0 \le h \le t, x, x+2h \in [a,b]\}$, $t \ge 0$, instead of $\omega_2(f,t)$, and with the class of functions which are semiconcave with modulus ω instead of the class of C^1 functions having f' uniformly continuous with modulus ω.

Because the notion of a semiconcave function is basic for the present article, we give the definition just now. (For the theory of semiconcave functions in $\mathbb{R}^n$ see [3].)

Definition 1.1. (i) We denote by $\mathcal{M}$ the set of all functions $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ which are nondecreasing on $[0,\infty)$ and right continuous at 0.

(ii) Let X be a real Banach space. We say that a real continuous function u on a convex set $D \subset X$ is *semiconcave with modulus* $\omega \in \mathcal{M}$ (or briefly ω*-semiconcave*) if

$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \le \lambda(1-\lambda)\omega(\|x-y\|)\|x-y\|, \tag{1.1}$$

whenever $\lambda \in [0,1]$ and $x, y \in \Omega$.

A function is called *semiconcave on* Ω if it is semiconcave on Ω with some modulus $\omega \in \mathcal{M}$.

Theorem M$^+$. *Let f be a continuous function on $[a,b]$ and*

$$\int_0^1 \frac{\omega_2^+(f,t)}{t^2}\, dt \;<\; \infty.$$

Then f is semiconcave on $[a,b]$ with modulus

$$\gamma(\delta) := C \int_0^\delta \frac{\omega_2^+(f,t)}{t^2}\, dt, \quad \delta \ge 0,$$

where $C > 0$ is an absolute constant.

It seems that Theorem M$^+$ is new. However, it is closely related to the following slight reformulation of a result of P. Cannarsa and C. Sinestrari (see [3, Theorem 2.1.10]).

Theorem CS. *Let $\tilde{\omega} : [0,\infty) \to [0,\infty)$ be a nondecreasing upper semicontinuous function such that $\lim_{\rho\to 0+} \tilde{\omega}(\rho) = 0$. Let $S \subset \mathbb{R}^n$ be a convex set and $u : S \to \mathbb{R}$ a continuous function such that*

$$u(x) + u(y) - 2u\left(\frac{x+y}{2}\right) \le \frac{\|x-y\|}{2}\tilde{\omega}(\|x-y\|), \quad x, y \in S. \tag{1.2}$$

Then u is semiconcave on S with modulus

$$\beta(\rho) := \sum_{k=0}^{\infty} \tilde{\omega}\left(\frac{\rho}{2^k}\right), \quad \rho \ge 0, \tag{1.3}$$

provided the right-hand side is finite.

Observe that condition (1.2) coincides with condition (1.1) with $\lambda = 1/2$ and $\omega = \tilde{\omega}$. Therefore the following terminology from [14] is natural.

Definition 1.2. Let $\tilde{\omega} \in \mathcal{M}$. We say that a real function f on a convex subset D of a Banach space is *Jensen $\tilde{\omega}$-semiconcave*, if condition (1.2) holds.

Under this terminology, the above observation means that

(1.4) ω-semiconcavity implies Jensen ω-semiconcavity.

Further, condition (1.2) can be easily formulated using the "semi-modulus" $\omega_2^+(u,\cdot)$. Namely, it is equivalent to the inequality $\omega_2^+(u,t) \le t\tilde{\omega}(2t),\ t \ge 0$ (see Lemma 4.1).

In Section 4, we will infer Theorem M$^+$ from Theorem CS. In fact, we prove Theorem 4.8, which generalizes Theorem M$^+$ to Banach spaces. In the proof, we use the following basic observations.

(O1) In the proof of Theorem CS, the assumption that $\tilde{\omega}$ is nondecreasing and upper semicontinuous is not used. Moreover, the proof works without any change also for functions on Banach spaces, cf. Theorem CS2 in Section 4.

(O2) The "semi-modulus" $\omega_2^+(f,t)$ has the property that the function $t^{-2}\omega_2^+(f,t)$ is 4-almost decreasing on $(0,\infty)$ (see (3.1) for the definition).

To show that $\omega_2^+(f,t)$ has the above property, it is sufficient (see Lemma 3.1) to modify in an obvious way the well-known proof (from [10] or [11]) which shows that $\omega_2(f,t)$ has this property.

It is natural to ask (cf. [3, Remark 2.1.11]) the following questions:

(Q1) Is the result of Theorem CS sharp ?

(Q2) For which $\eta \in \mathcal{M}$ Jensen η-semiconcavity and continuity of a function implies its $C\eta$-semiconcavity for some $C = C(\eta)$?

For the most interesting moduli $\omega_p(t) = t^p$ $(0 < p \le 1)$, Theorem CS gives that each continuous Jensen ω_p-semiconcave function is $(2/(2-2^{1-p}))\omega_p$-semiconcave. This fact with (1.4) shows that, for these moduli, Theorem CS is "sharp up to a multiplicative constant" and the implication from (Q2) holds.

However, [14, Theorem 4] gives that we can write above $1/(2-2^{1-p})$ instead of $2/(2-2^{1-p})$ (and this constant is optimal). Thus Theorem CS is not "absolutely sharp".

We will show that, for some $\tilde{\omega}$, Theorem CS is not sharp in any sense.

The reason is that observation (O2) rather easily implies (see Proposition 4.2) that each Jensen $\tilde{\omega}$-semiconcave function is Jensen $4\tilde{\omega}_*$-semiconcave, where $\tilde{\omega}_*$ is the "Stechkin regularization of $\tilde{\omega}$" defined by

$$\tilde{\omega}_*(h) := h \inf\{\tilde{\omega}(u)/u :\ 0 < u \le h\} \text{ for } h > 0 \ \text{ and } \ \tilde{\omega}_*(0) := 0\ . \tag{1.5}$$

So, applying Theorem CS to $4\tilde{\omega}_*$, we obtain that u is semiconcave with modulus

$$\beta^*(\rho) = 4\sum_{k=0}^{\infty} \tilde{\omega}_*\left(\frac{\rho}{2^k}\right), \quad \rho \ge 0, \tag{1.6}$$

provided the right-hand side is finite. Thus, since $\beta^*(\rho)$ can be essentially smaller than $\beta(\rho)$ (it can even happen that $\beta(\rho)$ is infinite and $\beta^*(\rho)$ is identically 0, see Example 5.1), for some $\tilde{\omega}$ the result of Theorem CS is not sharp in any sense.

Further, we show (see Theorem 5.13 for a precise formulation) that Theorem 4.4 (i.e., the result with modulus $\beta^*(\rho)$) is "sharp up to a multiplicative constant" for many D. Consequently, since $\tilde{\omega}_* = \tilde{\omega}$ for concave $\tilde{\omega}$, the result of Theorem CS is "sharp up to a multiplicative constant" for many D if $\tilde{\omega}$ is concave.

The proofs of sharpness results are based on well-known properties of so called "extremal function", which is important in the classical "smooth case".

For bounded D and all $D \subset \mathbb{R}^n$, we give a complete answer to question (Q2) in Theorems 6.1 and 6.2. In particular, we prove (see Remark 6.4(i)) that if D is bounded and a nonzero η is concave, then the implication of (Q2) holds if and only if there exists $K > 1$ such that $\liminf_{t\to 0+} \eta(Kt)/\eta(t) > 1$.

2. Preliminaries

In the following, X will always be a real Banach space. By $B(x, r)$ we denote the open ball with center x and radius r. By $\langle x^*, x\rangle$ we will denote the usual dual pairing between $x^* \in X^*$ and $x \in X$. For $t \in \mathbb{R}$, we set $t^+ := \max(t, 0)$. The integer part of $x \in \mathbb{R}$ is denoted by $[x]$. The integral is always the Lebesgue integral.

We write $f(t) = O(g(t)),\ t \to 0+$, provided there exist $C > 0$ and $\delta > 0$ such that $|f(t)| \leq C|g(t)|$ for $t \in (0, \delta)$. We write $f(t) \asymp g(t),\ t \to 0+$, provided $f(t) = O(g(t)),\ t \to 0+$, and $g(t) = O(f(t)),\ t \to 0+$.

If f is a function on on open subset G of a Banach space, then $f'(x)$ denotes the Fréchet derivative of f at $x \in G$.

If X is a Banach space, $A \neq \emptyset$ is a convex subset of X, $h \in X$ and f is a function on A, then we denote by $\Delta_h f$ the function defined on $A \cap (A - h)$ by $\Delta_h f(x) := f(x+h) - f(x)$. We set $\Delta_h^2 f := \Delta_h(\Delta_h f)$. So $\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$ for $x \in A \cap (A - 2h)$. We define the second modulus of continuity of f by

$$\omega_2(f, t) := \sup\{|\Delta_h^2(f, x)| : \|h\| \leq t, x \in A \cap (A - 2h)\}.$$

Further we define the second "semi-modulus of continuity" by

$$\begin{aligned} \omega_2^+(f, t) :&= \sup\{(\Delta_h^2(f, x))^+ : \|h\| \leq t, x \in A \cap (A - 2h) \\ &= \sup\{\Delta_h^2(f, x) : \|h\| \leq t, x \in A \cap (A - 2h)\}. \end{aligned} \tag{2.1}$$

Obviously,

$$\omega_2^+(cf, t) = c\omega_2^+(f, t),\ t \geq 0, \quad \text{for each } c > 0. \tag{2.2}$$

Further, if $v \in X$ and $f_v(x) := f(x - v),\ x \in A + v$, then clearly

$$\omega_2^+(f_v, t) = \omega_2^+(f, t),\ t \geq 0. \tag{2.3}$$

The set $\mathcal{M}$ and the notion of a semiconcave function (with modulus $\omega \in \mathcal{M}$) were defined in Definition 1.1 above. A function is called *semiconvex* (with modulus $\omega \in \mathcal{M}$) if $-f$ is a semiconcave function (with modulus $\omega \in \mathcal{M}$). We will need the following well-known properties of semiconcave functions.

Lemma 2.1. *Let f be a continuous function on $[a, b] \subset \mathbb{R}$ and $\omega \in \mathcal{M}$. Then the following assertions hold.*

- (i) *f is ω-semiconcave on $[a, b]$ if and only if f is ω-semiconcave on (a, b).*
- (ii) *If f' is uniformly continuous with modulus ω on (a, b), then f is semiconcave with modulus ω.*
- (iii) *If f is semiconcave on (a, b) with modulus ω, then f has a finite right derivative $f'_+(x)$ at each $x \in (a, b)$, and*

$$f'_+(x + \delta) - f'_+(x) \leq 2\omega(\delta) \quad \textit{whenever} \quad x, x + \delta \in (a, b) \quad \textit{and} \quad \delta > 0. \tag{2.4}$$

Proof. Assertion (i) easily follows from (1.1), continuity of f and monotonicity of ω. To prove (ii), we can use [3, Proposition 2.1.2] and (i). For assertion (iii) see [7, Proposition 2.8] (it also easily follows from [3, Theorem 3.2.1 and Proposition 3.3.10]). $\square$

Definition 2.2. Let f be a continuous real valued function on a convex subset D of a Banach space X. Put $\eta_f(0) := 0$ and, for $t > 0$, set

$$\begin{aligned}\eta_f(t) := \sup\Big\{ \Big(\frac{\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y)}{\lambda(1-\lambda)\|x-y\|} \Big)^+ : \\ x, y \in D, 0 < \|x-y\| \le t, \lambda \in (0,1) \Big\}.\end{aligned} \tag{2.5}$$

It is easy to see that $\eta_f : [0,\infty) \to [0,\infty]$ is nondecreasing and f is semiconcave if and only if $\eta_f \in \mathcal{M}$. Then clearly η_f is the minimal modulus of semiconcavity of f. Obviously,

$$\eta_{cf}(t) = c\eta_f(t),\ t \ge 0, \quad \text{for each } c > 0. \tag{2.6}$$

Further, if $v \in X$ and $f_v(x) := f(x-v),\ x \in D + v$, then clearly

$$\eta_{f_v}(t) = \eta_f(t), \quad t \ge 0. \tag{2.7}$$

We will need also the the following obvious corollary of [6, Corollary 3.6].

Lemma 2.3. *Let f be a semiconcave function with modulus $\eta \in \mathcal{M}$ on an open convex subset Ω of a Banach space X. Then f is semiconcave with a concave modulus $\tilde{\eta} \in \mathcal{M}$ for which $\tilde{\eta} \le 4\eta$.*

3. Moduli

We will say (following S.N. Bernstein, see [1, p. 493]) that a function $g : (0,\infty) \to [0,\infty]$ is *almost decreasing*, if there exists $C > 0$ such that

$$g(t) \le C g(u) \quad \text{whenever} \quad 0 < u \le t. \tag{3.1}$$

If (3.1) holds, we say that g is *C-almost decreasing.*

The proof of statement (ii) of the following lemma is an obvious modification of the corresponding proof ([10], [11]) concerning $\omega_2(f,t)$.

Lemma 3.1. *Let f be a function defined on a nonempty convex subset D of a Banach space X. Then:*

(i) *$\omega_2^+(f,0) = 0$ and $\omega_2^+(f,\cdot) : [0,\infty) \to [0,\infty]$ is nondecreasing.*
(ii) *The function $t \mapsto t^{-2}\omega_2^+(f,t)$ is 4-almost decreasing on $(0,\infty)$.*
(iii) *If $\omega_2^+(f,t_0) \in \mathbb{R}$ for some $t_0 \in (0,\infty)$, then $\omega_2^+(f,\cdot)$ is finite.*
(iv) *If $\int_0^1 \frac{\omega_2^+(f,t)}{t^2}\, dt < \infty$, then $\omega_2^+(f,\cdot) \in \mathcal{M}$.*

Proof. Property (i) is obvious.

(ii) First consider arbitrary $p \in \mathbb{N}$, $h \in X$ and observe that

$$\Delta_{ph} f(x) = \sum_{i=0}^{p-1} \Delta_h f(x + ih) \quad \text{for} \quad x \in D \cap (D - ph)$$

and so, for $x \in D \cap (D - 2ph)$,

$$\Delta_{ph}^2 f(x) = \sum_{i=0}^{p-1}\sum_{j=0}^{p-1} \Delta_h^2 f(x + ih + jh) \le p^2 \omega_2^+(f, \|h\|).$$

Therefore,

$$\omega_2^+(f, pt) \le p^2 \omega_2^+(f,t) \quad \text{for each} \quad p \in \mathbb{N} \quad \text{and} \quad t \ge 0. \tag{3.2}$$

To prove our assertion, we must show that

$$t^{-2}\omega_2^+(f,t) \le 4u^{-2}\omega_2^+(f,u) \quad \text{whenever} \quad 0<u<t. \tag{3.3}$$

To this end fix $0<u<t$. For $p := [tu^{-1}+1]$ we have $tu^{-1} < p \le tu^{-1}+1 < 2tu^{-1}$. Therefore $t<pu$, and so monotonicity of $\omega_2^+(f,\cdot)$ and (3.2) imply

$$\omega_2^+(f,t) \le \omega_2^+(f,pu) \le p^2\omega_2^+(f,u) \le 4t^2u^{-2}\omega_2^+(f,u)$$

and (3.3) follows.

Property (iii) easily follows from (i) and (ii).

(iv) If $\int_0^1 \frac{\omega_2^+(f,t)}{t^2}\, dt < \infty$, then $\omega_2^+(f,\cdot)$ is finite by (iii). By (i), $\omega_2^+(f,\cdot)$ is nondecreasing and also $L := \lim_{t\to 0+}\omega_2^+(f,t) = 0$, since otherwise $\int_0^1 \frac{\omega_2^+(f,t)}{t^2}\, dt \ge L\int_0^1 t^{-2}\, dt = \infty$. ☐

Definition 3.2. (i) For $k=1,2$, we will denote by $\mathcal{M}_k$ the set of $\omega \in \mathcal{M}$, for which $t^{-k}\omega(t)$ is nonincreasing on $(0,\infty)$.

(ii) For $\omega \in \mathcal{M}$, we define (Stechkin's regularizations) $\omega_* = \omega_{1*}$, $\omega_{**} = \omega_{2*}$ by $\omega_*(0) = \omega_{**}(0) = 0$,

$$\omega_*(t) = \omega_{1*}(t) := t \cdot \inf\{\omega(u)/u : \ 0<u\le t\}, \quad t\in(0,\infty), \quad \text{and}$$

$$\omega_{**}(t) = \omega_{2*}(t) := t^2 \cdot \inf\{\omega(u)/u^2 : \ 0<u\le t\}, \quad t\in(0,\infty).$$

Remark 3.3. (i) Functions from $\mathcal{M}_k$ are called k-majorants (see e.g. [4, p. 203, Definition 4.4'], where also $k>2$ are considered).

(ii) Stechkin's regularizations were introduced (and denoted by ω_k^{**}) in [12, p. 232] for any $k\in\mathbb{N}$ and for (not necessarily nondecreasing) $\omega : (0,\pi] \to (0,\infty)$. Roughly speaking, for $\omega\in\mathcal{M}$, our ω_{k*} coincides with ω_k^{**} from [12] and our $\omega_{1*} = \omega_*$ is denoted by ω_1 in [1, p. 519].

We will use many times some properties of $\mathcal{M}_k$ and Stechkin's regularizations. The following properties follow quite easily from the definitions and will be repeatedly used without a reference:

(i) If $\omega\in\mathcal{M}_k$ $(k=1,2)$ and $\lambda>0$, then the functions $\lambda\omega$ and $\omega(\lambda t)$ belong to $\mathcal{M}_k$.
(ii) $0\le\omega_*\le\omega$ and $0\le\omega_{**}\le\omega$.
(iii) If $k\in\{1,2\}$, $\omega_1\in\mathcal{M}$, $\omega_2\in\mathcal{M}$ and $\omega_1\le\omega_2$, then $(\omega_1)_{k*}\le(\omega_2)_{k*}$. Further, $(c\omega)_{k*} = c(\omega)_{k*}$ whenever $\omega\in\mathcal{M}$ and $c\ge 0$.

In the following lemma we recall some known (deeper) properties.

Lemma 3.4.

(i) *If $\omega\in\mathcal{M}$ is concave, then $\omega\in\mathcal{M}_1$.*
(ii) *If $\omega\in\mathcal{M}_1$, then ω is subadditive.*
(iii) *If $\omega\in\mathcal{M}$ and $k=1,2$, then $\omega_{k*}\in\mathcal{M}_k$.*
(iv) *If $\omega\in\mathcal{M}$ and $k=1,2$, then $\omega\in\mathcal{M}_k$ if and only if $\omega=\omega_{k*}$.*
(v) *If $\omega\in\mathcal{M}$, $C>0$ and $k=1,2$, then $t^{-k}\omega(t)$ is C-almost decreasing on $(0,\infty)$ if and only if $C\omega_{k*}\ge\omega$.*

Proof. Assertion (i) is easy to see. For (ii) see [4, p. 173, Remark 1.4].

Concerning (iii), observe that the proofs of [12, 2) and 3), p. 233] (on $(0,\pi]$) works also for $\omega\in\mathcal{M}$ and give that $\omega_{k*}(t)/t^k$ is nonincreasing on $(0,\infty)$ and ω_{k*} is nondecreasing. Since $\omega_{k*}\le\omega$, we obtain (iii).

To prove (iv), observe that if $\omega \in \mathcal{M}_k$ and $0 < u \leq t$, then $t^k \cdot (\omega(u)/u^k) \geq \omega(t)$, and so $\omega_{k*}(t) = \omega(t)$ by the definition of ω_{k*}. The converse implication follows from (iii).

Important property (v) is implicitly contained in the proof of [12, 4), p. 233]. For completeness, we present a proof:

First suppose that $C\omega_{k*} \geq \omega$. Since $\omega_{k*} \in \mathcal{M}_k$ and $\omega_{k*} \leq \omega$, we obtain that, for $0 < u \leq t$,

$$\frac{\omega(t)}{t^k} \leq \frac{C\omega_{k*}(t)}{t^k} \leq C\frac{\omega_{k*}(u)}{u^k} \leq C\frac{\omega(u)}{u^k}.$$

So $t^{-k}\omega(t)$ is C-almost decreasing on $(0,\infty)$. To prove the converse implication, suppose that $t^{-k}\omega(t) \leq Cu^{-k}\omega(u)$ whenever $0 < u \leq t$. Then, if $0 < u \leq t$, we have $\omega(t) \leq Ct^k u^{-k}\omega(u)$. Consequently $\omega(t) \leq C\omega_{k*}(t)$. □

Remark 3.5. (i) It is easy to show that ω_{k*} is the maximal k-majorant (i.e., an element of $\mathcal{M}_k$) majorized by ω.

(ii) Each k-majorant is continuous (see [4, the note before Definition 4.4']).

However, we will not use these facts.

On the other hand, we will need also the following facts. (In their formulation we use the classical notation, denoting, e.g., the function $t \mapsto \omega(ct)$ by $\omega(ct)$.)

Lemma 3.6.

(i) *If $\omega \in \mathcal{M}$, then $(t\omega(t))_{**} = t\omega_*(t)$.*

(ii) *If $\omega \in \mathcal{M}$, $k = 1, 2$ and $c > 0$, then $(\omega(ct))_{k*} = \omega_{k*}(ct)$.*

(iii) *If $\omega \in \mathcal{M}_1$, then the function $t\omega(t)$ belongs to $\mathcal{M}_2$.*

Proof. To prove (i), denote $\mu(t) := t\omega(t)$. Then, for each $t > 0$,

$$\mu_{**}(t) = t^2 \inf\{u\omega(u)/u^2 : \ 0 < u \leq t\} = t \cdot t \inf\{\omega(u)/u : \ 0 < u \leq t\} = t\omega_*(t).$$

To prove (ii), denote $\alpha(t) = \omega(ct)$. Then, for each $t > 0$,

$$\begin{aligned}\alpha_{k*}(t) = t^k \inf\{\omega(cu)/u^k : \ 0 < u \leq t\} &= c^k t^k \inf\{\omega(cu)/(cu)^k : \ 0 < u \leq t\} \\ &= (ct)^k \inf\{\omega(z)/z^k : \ 0 < z \leq ct\} = \omega_{k*}(ct).\end{aligned}$$

Property (iii) immediately follows from definitions. □

Now we can formulate the following important easy consequence of Lemma 3.1.

Lemma 3.7. *Let f be a function defined on a nonempty convex subset D of a Banach space X.*

(i) *For every $\omega \in \mathcal{M}$,*

$$\omega_2^+(f,t) \leq \omega(t), \ t \geq 0 \qquad \Rightarrow \qquad \omega_2^+(f,t) \leq 4\omega_{**}(t), \ t \geq 0.$$

(ii) *If $\omega_2^+(f,\cdot) \in \mathcal{M}$, then there exists $\tilde{\omega} \in \mathcal{M}_2$ such that*

$$\omega_2^+(f,t) \leq \tilde{\omega}(t) \leq 4\omega_2^+(f,t), \ t \geq 0.$$

Proof. To prove (i), suppose $\omega_2^+(f,t) \leq \omega(t), \ t \geq 0$. Since the function $t^{-2}\omega_2^+(f,t)$ is 4-almost decreasing by Lemma 3.1, we obtain $\omega_2^+(f,\cdot) \leq 4(\omega_2^+(f,\cdot))_{**} \leq 4\omega_{**}$ by Lemma 3.4(v).

To prove (ii), apply (i) with $\omega := \omega_2^+(f,\cdot)$, set $\tilde{\omega} := 4(\omega_2^+(f,\cdot))_{**}$ and apply Lemma 3.4(iii). □

Proposition 3.8. *Let f be a function defined on a nonempty convex subset D of a Banach space X. If f is semiconcave with modulus $\eta \in \mathcal{M}$, then f is semiconcave with modulus $4\eta_*$.*

Proof. By Lemma 2.3 f is semiconcave with a concave modulus $\tilde{\eta}$ for which $\tilde{\eta} \leq 4\eta$. Since $\tilde{\eta} \in \mathcal{M}_1$ by Lemma 3.4(i), we obtain $\tilde{\eta} = \tilde{\eta}_* \leq 4\eta_*$ by Lemma 3.4(iv), and our assertion follows. ☐

Lemma 3.9. *For each $\omega \in \mathcal{M}_1$ there exists a concave $\tilde{\omega} \in \mathcal{M}$ such that $\omega \leq \tilde{\omega} \leq 2\omega$.*

Proof. By Lemma 3.4(ii), ω is subadditive, and therefore $\omega \leq \tilde{\omega} \leq 2\omega$, where $\tilde{\omega}$ is the least concave majorant of ω. This inequality was at first observed by S.B. Stechkin (cf. [6, p. 241]). Stechkin's lemma is usually formulated for a continuous subadditive $\omega \in \mathcal{M}$. However, a subadditive $\omega \in \mathcal{M}$ is clearly continuous. Moreover, the standard proof (see e.g. [5, p. 43]) does not use the continuity of ω. ☐

Now we can prove the following version of Lemma 2.3 which can be of some independent interest.

Proposition 3.10. *Let $\eta \in \mathcal{M}$. Then there exists a concave $\eta_c \in \mathcal{M}$ with $\eta_c \leq \eta$ and the following property:*

If f is a function defined on a nonempty convex subset D of a Banach space X which is semiconcave with modulus η, then f is semiconcave with modulus $8\eta_c$.

Proof. By Lemma 3.4(iii) and Lemma 3.9 there exists a concave $\tilde{\eta}$ such that $\eta_* \leq \tilde{\eta} \leq 2\eta_*$. Set $\eta_c := \tilde{\eta}/2$; then $\eta_c \leq \eta_* \leq \eta$. If f is as above (semiconcave with modulus η), then Proposition 3.8 implies that f is semiconcave with modulus $4\eta_*$. Since $4\eta_* \leq 4\tilde{\eta} = 8\eta_c$, the assertion follows. ☐

We will use also the following result (see [1, Lemma 2, p. 494, and the footnote, p. 493]).

Lemma 3.11. ([1]) Let $\eta \in \mathcal{M}$ and $\eta(t) > 0$ for $t > 0$. Then the following conditions are equivalent.

(i) $\int_0^t \frac{\eta(u)}{u}\, du = O(\eta(t)),\ t \to 0+$.

(ii) $\exists K > 1 :\ \liminf_{t \to 0+} \frac{\eta(Kt)}{\eta(t)} > 1$.

4. Sufficient conditions for semiconcavity

In this section we prove a more general and (for some $\tilde{\omega}$) sharper version (Theorem 4.4) of Theorem CS and a generalization (Theorem 4.8) of Theorem M$^+$ from the introduction.

First we present the generalization of Theorem CS which was mentioned in observation (O1) in the introduction.

Theorem CS2. *Let u be a continuous function defined on a nonempty convex subset D of a Banach space X. Let, for a $\tilde{\omega} : [0,\infty) \to [0,\infty)$,*

$$u(x) + u(y) - 2u\left(\frac{x+y}{2}\right) \leq \frac{\|x-y\|}{2}\tilde{\omega}(\|x-y\|) \quad \textit{whenever} \quad x, y \in D. \tag{4.1}$$

Let $\eta \in \mathcal{M}$ and

$$\sum_{k=0}^{\infty} \tilde{\omega}\left(\frac{\rho}{2^k}\right) \leq \eta(\rho), \quad \rho \geq 0. \tag{4.2}$$

Then u is semiconcave on D with modulus η.

Proof. It is sufficient to repeat word by word the proof of [3, Theorem 2.1.10], writing only D instead of S and changing in an obvious way the final argument (the only formal problem is that the function $\sum_{k=0}^{\infty} \tilde{\omega}\left(\frac{\rho}{2^k}\right)$ need not belong to $\mathcal{M}$). $\square$

The following easy lemma shows how the condition (4.1) of Theorem CS2 is related to an estimate of the "semi-modulus" ω_2^+.

Lemma 4.1. *Let u be a function defined on a nonempty convex subset D of a Banach space.*

(i) *Let $\omega \in \mathcal{M}$ and $\omega_2^+(u,t) \le \omega(t)$, $t \ge 0$. Set $\tilde{\omega}(t) := (2/t)\omega(t/2)$, $t > 0$, and $\tilde{\omega}(0) := 0$. Then*

$$u(x) + u(y) - 2u\left(\frac{x+y}{2}\right) \le \frac{\|x-y\|}{2}\tilde{\omega}(\|x-y\|) \quad \text{whenever} \quad x, y \in D. \tag{4.3}$$

(ii) *Let $\tilde{\omega} \in \mathcal{M}$. Set $\omega(t) := t\tilde{\omega}(2t)$, $t \ge 0$. Then the condition (4.3) holds (i.e., u is Jensen $\tilde{\omega}$-semiconcave) if and only if*

$$\omega_2^+(u,t) \le \omega(t),\ t \ge 0. \tag{4.4}$$

Proof. (i) Observe that

$$u(x) + u(y) - 2u((x+y)/2) = \Delta^2_{(y-x)/2}(u,x), \quad \text{whenever} \quad x, y \in D.$$

Consequently, for each $x, y \in D$,

$$u(x)+u(y)-2u((x+y)/2) \le \omega_2^+\left(u, \frac{\|y-x\|}{2}\right) \le \omega\left(\frac{\|y-x\|}{2}\right) = \frac{\|y-x\|}{2}\tilde{\omega}(\|y-x\|),$$

so (4.3) holds.

(ii) First suppose that (4.4) holds. Since clearly $\omega \in \mathcal{M}$ and $\tilde{\omega}(t) = (2/t)\omega(t/2)$, $t > 0$, (4.3) follows by (i). Now suppose that (4.3) holds. Consider arbitrary $t \ge 0$, $x \in D$ and $h \in X$ with $\|h\| \le t$ such that $\Delta_h^2(u,x) = u(x)+u(x+2h)-2u(x+h)$ is defined. Setting $y := x + 2h$, we have $h := (y-x)/2$ and

$$\Delta_h^2(u,x) = u(x) + u(y) - 2u((x+y)/2) \le \|h\|\tilde{\omega}(2\|h\|).$$

Since $\tilde{\omega}$ is nondecreasing, we have $\Delta_h^2(u,x) \le t\tilde{\omega}(2t) = \omega(t)$, and consequently $\omega_2^+(u,t) \le \omega(t)$. $\square$

Let us note that the function $\tilde{\omega}$ from (i) need not be nondecreasing. From this reason, observation (O1) from the introduction is important.

Now we can prove a precise analogon of Proposition 3.8.

Proposition 4.2. *Let $\tilde{\omega} \in \mathcal{M}$ and let D be a nonempty convex subset of a Banach space. Then each Jensen $\tilde{\omega}$-semiconcave function on D is Jensen $4\tilde{\omega}_*$-semiconcave.*

Proof. Set $\omega(t) := t\tilde{\omega}(2t)$, $t \ge 0$. By Lemma 4.1(ii) we obtain $\omega_2^+(u,t) \le \omega(t)$, $t \ge 0$. So, using Lemma 3.7(i) and Lemma 3.6(i),(ii), we obtain

$$\omega_2^+(u,t) \le 4\omega_{**}(t) = 4t\tilde{\omega}_*(2t).$$

Consequently, using Lemma 4.1(ii) (with $4\tilde{\omega}_*$ instead of $\tilde{\omega}$) we obtain that f is Jensen $4\tilde{\omega}_*$-semiconcave. $\square$

We will use also the following consequence of Lemma 4.1 and Proposition 4.2.

Corollary 4.3. *Let $\eta \in \mathcal{M}$ and let u be a function defined on a nonempty convex subset D of a Banach space.*

(i) *If $\omega_2^+(u,t) \le t\eta(t),\ t \ge 0$, then u is Jensen η-semiconcave.*
(ii) *If u is Jensen η-semiconcave, then $\omega_2^+(u,t) \le 8t\eta_*(t),\ t \ge 0$.*

Proof. (i) By Lemma 4.1(i) (with $\omega(t) = t\eta(t)$), we obtain that (4.3) holds with

$$\tilde{\omega}(t) = (2/t) \cdot (t/2) \cdot \eta(t/2) \le \eta(t), \quad t \ge 0,$$

and so u is Jensen η-semiconcave.

(ii) By Proposition 4.2, u is Jensen $4\eta_*$-semiconcave. Using Lemma 4.1(ii) (with $\tilde{\omega} = 4\eta_*$) and subadditivity of η_* (see Lemma 3.4(ii),(iii)), we obtain

$$\omega_2^+(u,t) \le t4\eta_*(2t) \le 8t\eta_*(t),\ t \ge 0.$$

□

Now we are ready to prove the following theorem, which is more general and (for some $\tilde{\omega}$) sharper that Theorem CS.

Theorem 4.4. *Let $\tilde{\omega} \in \mathcal{M}$ and let D be a convex subset of a Banach spaces. Let u be a continuous Jensen $\tilde{\omega}$-semiconcave function on D. Then u is semiconcave with the modulus*

$$\beta^*(\rho) = 4\sum_{k=0}^{\infty} \tilde{\omega}_*\left(\frac{\rho}{2^k}\right), \quad \rho \ge 0,$$

provided the right-hand side is finite.

Proof. Set $\hat{\omega} := 4\tilde{\omega}_*$. Then, by Proposition 4.2, the condition (4.1) of Theorem CS2 holds with $\hat{\omega}$ instead of $\tilde{\omega}$. It is easy to see that if β^* is finite, then $\beta^* \in \mathcal{M}$. Thus the assertion of the theorem follows from Theorem CS2. □

We will need the following easy well-known estimates.

Lemma 4.5. *Let $h \ge 0$ be a nonincreasing function on $(0,\infty)$ and $\delta > 0$. Then*

(4.5)
$$\frac{1}{2}\sum_{k=0}^{\infty} h\left(\frac{\delta}{2^{k+1}}\right)\frac{\delta}{2^{k+1}} \le \sum_{k=0}^{\infty} h\left(\frac{\delta}{2^{k}}\right)\frac{\delta}{2^{k+1}} \le \int_0^{\delta} h(u)\ du \le \sum_{k=0}^{\infty} h\left(\frac{\delta}{2^{k+1}}\right)\frac{\delta}{2^{k+1}}.$$

Proof. Clearly

$$h\left(\frac{\delta}{2^{k}}\right)\frac{\delta}{2^{k+1}} \le \int_{\delta 2^{-(k+1)}}^{\delta 2^{-k}} h(u)\ du \le h\left(\frac{\delta}{2^{k+1}}\right)\frac{\delta}{2^{k+1}}.$$

Summing these inequalities, we obtain

$$\sum_{k=0}^{\infty} h\left(\frac{\delta}{2^{k}}\right)\frac{\delta}{2^{k+1}} \le \int_0^{\delta} h(u)\ du \le \sum_{k=0}^{\infty} h\left(\frac{\delta}{2^{k+1}}\right)\frac{\delta}{2^{k+1}}.$$

So (4.5) follows, since

$$\frac{1}{2}\sum_{k=0}^{\infty} h\left(\frac{\delta}{2^{k+1}}\right)\frac{\delta}{2^{k+1}} = \sum_{k=0}^{\infty} h\left(\frac{\delta}{2^{k}}\right)\frac{\delta}{2^{k+1}} - h(\delta)\frac{\delta}{2}.$$

□

As a consequence, we easily obtain the following facts.

Lemma 4.6. *Let $\omega \in \mathcal{M}_2$. Set $\tilde{\omega}(t) := (2/t)\omega(t/2),\ t > 0$ and $\tilde{\omega}(0) := 0$. Then, for each $\delta > 0$,*

$$\int_0^\delta \frac{\omega(t)}{t^2}\, dt \le \sum_{k=0}^{\infty} \tilde{\omega}\left(\frac{\delta}{2^k}\right) \le 2\int_0^\delta \frac{\omega(t)}{t^2}\, dt. \tag{4.6}$$

Proof. Set $h(t) := t^{-2}\omega(t),\ t > 0$, and fix an arbitrary $\delta > 0$. Then

$$\sum_{k=0}^{\infty} \tilde{\omega}\left(\frac{\delta}{2^k}\right) = \sum_{k=0}^{\infty} \frac{2^{k+1}}{\delta}\omega\left(\frac{\delta}{2^{k+1}}\right) = \sum_{k=0}^{\infty} h\left(\frac{\delta}{2^{k+1}}\right)\frac{\delta}{2^{k+1}}.$$

Since h is nonincreasing, (4.5) implies (4.6). ☐

Lemma 4.7. *Let $\tilde{\omega} \in \mathcal{M}_1$ and $\delta > 0$. Then*

$$\frac{1}{2}\sum_{k=0}^{\infty} \tilde{\omega}\left(\frac{\delta}{2^k}\right) \le \int_0^\delta \frac{\tilde{\omega}(t)}{t}\, dt \le \sum_{k=0}^{\infty} \tilde{\omega}\left(\frac{\delta}{2^k}\right) \tag{4.7}$$

Proof. For the nonnegative nonincreasing function $h(t) := \tilde{\omega}(t)/t$, we have

$$\frac{1}{2}\sum_{k=0}^{\infty} \tilde{\omega}\left(\frac{\delta}{2^k}\right) = \sum_{k=0}^{\infty} h\left(\frac{\delta}{2^k}\right)\frac{\delta}{2^{k+1}} \quad \text{and}$$

$$\sum_{k=0}^{\infty} h\left(\frac{\delta}{2^{k+1}}\right)\frac{\delta}{2^{k+1}} = \sum_{k=0}^{\infty} \tilde{\omega}\left(\frac{\delta}{2^{k+1}}\right) \le \sum_{k=0}^{\infty} \tilde{\omega}\left(\frac{\delta}{2^k}\right),$$

and so (4.5) implies (4.7). ☐

Now we infer from Theorem CS2 the following generalization of Theorem M$^+$, which was mentioned in the introduction.

Theorem 4.8. *Let u be a continuous function defined on a nonempty convex subset D of a Banach space X and*

$$\int_0^1 \frac{\omega_2^+(u,t)}{t^2}\, dt \ < \ \infty.$$

Then u is semiconcave on D with modulus

$$\gamma(\delta) = 8\int_0^\delta \frac{\omega_2^+(u,t)}{t^2}\, dt.$$

Proof. By Lemma 3.1(iv) and Lemma 3.7(ii) we can choose $\omega \in \mathcal{M}_2$ such that $\omega_2^+(u,t) \le \omega(t) \le 4\omega_2^+(u,t),\ t \ge 0$. Set $\tilde{\omega}(t) := (2/t)\omega(t/2),\ t > 0$ and $\tilde{\omega}(0) := 0$. By Lemma 4.1 we obtain that, for every $x, y \in D$,

$$u(x) + u(y) - 2u\left(\frac{x+y}{2}\right) \le \frac{\|x-y\|}{2}\tilde{\omega}(\|x-y\|).$$

By Lemma 4.6, we obtain, for $\delta > 0$,

$$\sum_{n=0}^{\infty} \tilde{\omega}\left(\frac{\delta}{2^n}\right) \le 2\int_0^\delta \frac{\omega(t)}{t^2}\, dt \le 8\int_0^\delta \frac{\omega_2^+(u,t)}{t^2}\, dt = \gamma(\delta).$$

So, using Theorem CS2, we obtain our assertion. ☐

Theorem 4.8 easily implies the following analogon of Theorem 4.4.

Theorem 4.9. *Let $\omega \in \mathcal{M}$ with $\int_0^1 \frac{\omega_{**}(t)}{t^2}\, dt < \infty$. Let u be a continuous function defined on a nonempty convex subset D of a Banach space X such that*

$$\omega_2^+(u,t) \le \omega(t), \ t \ge 0.$$

Then u is semiconcave on D with modulus

$$\eta(\delta) = 32 \int_0^\delta \frac{\omega_{**}(t)}{t^2}\, dt.$$

Proof. Since Lemma 3.7(i) implies $\omega_2^+(u,t) \le 4\omega_{**}(t), t \ge 0$, we have

$$8 \int_0^\delta \frac{\omega_2^+(u,t)}{t^2}\, dt \le 32 \int_0^\delta \frac{\omega_{**}(t)}{t^2}\, dt = \eta(\delta).$$

Using Theorem 4.8 we obtain our assertion. □

5. Sharpness

First we present an example of $\tilde{\omega} \in \mathcal{M}$ for which Theorem CS is not sharp in any sense.

Example 5.1. There exists a continuous $\tilde{\omega} \in \mathcal{M}$ such that any continuous Jensen $\tilde{\omega}$-semiconcave function on a convex subset of a Banach space is concave and

$$\beta(\rho) := \sum_{k=0}^{\infty} \tilde{\omega}(\rho 2^{-k}) = \infty, \quad \rho > 0. \tag{5.1}$$

Proof. First observe that we can inductively define an increasing sequence $(k_n)_{n=1}^\infty$ of natural numbers and a decreasing sequence $(y_n)_{n=1}^\infty$ of positive real numbers such that $k_1 = 1$, $y_1 = 1$ and, for each $n \in \mathbb{N}$,

$$2^{k_n} y_{n+1} < \frac{1}{n} \quad \text{and} \tag{5.2}$$

$$(k_{n+1} - k_n) y_{n+1} > 1. \tag{5.3}$$

Further we can clearly choose a continuous $\tilde{\omega} \in \mathcal{M}$ such that

$$\tilde{\omega}(2^{-i}) = y_{n+1}, \quad \text{whenever} \quad i \in \mathbb{N} \quad \text{and} \quad k_n \le i < k_{n+1}.$$

Using (5.3), we obtain

$$\sum_{i=1}^{\infty} \tilde{\omega}(2^{-i}) = \sum_{n=1}^{\infty} \sum_{i=k_n}^{k_{n+1}-1} \tilde{\omega}(2^{-i}) = \sum_{n=1}^{\infty} (k_{n+1} - k_n) y_{n+1} = \infty,$$

and (5.1) easily follows.

Fix $t > 0$. For each n such that $2^{-k_n} < t$, the definition of $\tilde{\omega}_*$ and (5.2) give

$$\tilde{\omega}_*(t) \le \frac{t}{2^{-k_n}} \tilde{\omega}(2^{-k_n}) = t 2^{k_n} y_{n+1} < t/n. \tag{5.4}$$

Therefore $\tilde{\omega}_*(t) = 0$ and so Theorem 4.4 implies the concavity of u. □

Remark 5.2. Let us note that, for the above example, it is not necessary to use Theorem 4.4, since it would be possible to use a result of [13] (see Theorem TT in Section 9 below). In fact, that result easily implies that u is concave whenever u is continuous Jensen $\tilde{\omega}$-semiconcave and $\liminf_{t \to 0+} \ \tilde{\omega}(t)/t = 0$, and the latter condition for our $\tilde{\omega}$ holds by (5.4).

Now we will consider sharpness of theorems from the preceding section. Our results are based on the use of so called "extremal functions" (see [4, p. 203]), which were applied in several articles which (essentially) concern the sharpness of Marchaud-type theorems (cf. Section 8).

We need and define the extremal function φ for $k = 2$ only:

Definition 5.3. Let $\omega \in \mathcal{M}_2$. Then we denote by $\varphi = \varphi_\omega$ the odd function on $\mathbb{R}$ for which

$$\varphi(x) := x \int_1^x \frac{\omega(t)}{t^2}\, dt, \quad x > 0.$$

It is easy to show (see [4, Lemma 4.2(i), p. 204]) that φ is continuous on $\mathbb{R}$. More difficult are the following inequalities:

$$\omega_2(\varphi \restriction_{[0,\infty)}, t) \le 2\omega(t), \quad \omega_2(\varphi, t) \le 6\omega(t), \quad t \ge 0. \tag{5.5}$$

The first inequality of (5.5) follows from [4, Theorem 4.2, p. 206] and the second one then from [4, Theorem 1.1, p. 300] (cf. also [8], where slightly weaker inequalities were proved and applied). We will use also the following "semiconcavity properties" of φ.

Lemma 5.4. *Let $\omega \in \mathcal{M}_2$ and φ be as in Definition 5.3. Then the following hold.*

(i) *If $\int_0^1 \omega(t)/t^2\, dt = \infty$ and $0 < a < \infty$, then the function $\psi := \varphi \restriction_{(-a,a)}$ is not semiconcave.*

(ii) *If $\int_0^1 \omega(t)/t^2\, dt < \infty$ and $0 < a < \infty$, then*

$$\eta_{\varphi\restriction_{(0,a)}}(\delta) \ge \frac{1}{2} \int_0^\delta \frac{\omega(t)}{t^2}\, dt, \quad 0 \le \delta \le a. \tag{5.6}$$

Proof. To prove (i), observe that

$$\psi'_+(0) = \varphi'_+(0) = \lim_{x \to 0+} \frac{\varphi(x)}{x} = \lim_{x\to 0+} \int_1^x \frac{\omega(t)}{t^2}\, dt = -\infty.$$

Therefore Lemma 2.1(iii) implies that ψ is not semiconcave.

To prove (ii), first observe that

$$\lim_{\delta \to 0+} \omega(\delta)/\delta = 0. \tag{5.7}$$

Indeed, since $\omega \in \mathcal{M}_2$, we easily obtain $\int_0^\delta \omega(t)/t^2\, dt \ge \delta \cdot (\omega(\delta)/\delta^2)$, $\delta > 0$, which implies (5.7).

Now fix an arbitrary $\delta \in (0, a]$ and consider $0 < h < \delta/2$. For $x > 0$, we clearly have $\varphi'(x) = \int_1^x \omega(t)/t^2\, dt + \omega(x)/x$, and consequently

$$\varphi'(\delta - h) - \varphi'(h) = \int_1^{\delta-h} \omega(t)/t^2\, dt + \frac{\omega(\delta-h)}{\delta - h} - \int_1^h \omega(t)/t^2\, dt - \frac{\omega(h)}{h}.$$

Therefore (2.4) gives

$$\eta_{\varphi\restriction_{(0,a)}}(\delta) \ge (1/2)(\varphi'(\delta-h) - \varphi'(h)) \ge (1/2)\left(\int_h^{\delta-h} \omega(t)/t^2\, dt - \frac{\omega(h)}{h}\right),$$

and so (5.7) easily implies (5.6). $\square$

To prove sharpness results in multidimensional spaces, we will use a standard construction via the following lemma.

Lemma 5.5. *Let D be a convex subset of a Banach space X, $a \in D$ and $v \in X$ with $\|v\| = 1$. Let $x^* \in X^*$ be a functional such that $\|x^*\| = x^*(v) = 1$. Let, for a $0 < d \leq \infty$, we have*

$$S := \{a + tv : \ t \in [0, d] \cap \mathbb{R}\} \subset D.$$

Let f be a continuous function on $\mathbb{R}$ and

$$\tilde{f}(x) := f(x^*(x)), \quad x \in D.$$

Then the following assertions hold.

(i) $\omega_2^+(\tilde{f}, t) \leq \omega_2^+(f, t), \quad t \geq 0.$

(ii) *Denoting $I := x^*(S)$, then $\eta_{\tilde{f}}(t) \geq \eta_{f\restriction_I}(t), \ t \geq 0.$*

Proof. To prove (i), fix an arbitrary $t \geq 0$ and consider $x \in X$ and $h \in X$ with $\|h\| \leq t$ and $x \in D \cap (D - 2h)$. Then

$$\tilde{f}(x + 2h) - 2\tilde{f}(x + h) + \tilde{f}(x) = f(x^*(x) + 2x^*(h)) - 2f(x^*(x) + x^*(h)) + f(x^*(x)),$$

i.e. $\Delta_h^2 \tilde{f}(x) = \Delta^2_{x^*(h)} f(x^*(x))$. Since $|x^*(h)| \leq \|h\| \leq t$, (i) follows.

To prove (ii), observe that $I = x^*(S) = [x^*(a), x^*(a) + d]$ if $d \in \mathbb{R}$ and $I = x^*(S) = [x^*(a), \infty)$ if $d = \infty$. Now fix an arbitrary $t \geq 0$ and consider $x, y \in I$ with $0 < |x - y| \leq t$. Let $X, Y \in S$ be the points with $x^*(X) = x$, $x^*(Y) = y$. Since $x^*(v) = 1$, clearly $\|X - Y\| = |x - y| \leq t$ and, for each $\lambda \in (0, 1)$,

$$\frac{\lambda\tilde{f}(X) + (1-\lambda)\tilde{f}(Y) - \tilde{f}(\lambda X + (1-\lambda)Y)}{\lambda(1-\lambda)\|X - Y\|} = \frac{\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y)}{\lambda(1-\lambda)|x - y|}.$$

Hence (ii) easily follows. □

Theorem 5.6. *Let $\omega \in \mathcal{M}$ and let D be an infinite convex subset of a Banach space. Then the following assertions are equivalent.*

(i) *If f is a continuous function on D with $\omega_2^+(f, t) \leq \omega(t), \ t \geq 0$, then f is semiconcave on I.*

(ii) $\int_0^1 \frac{\omega_{**}(t)}{t^2} \, dt \ < \ \infty.$

Proof. The implication (ii) $\Rightarrow$ (i) holds by Theorem 4.9.

Now suppose that (ii) does not hold. Choose $a \in D$, $v \in X$ with $\|v\| = 1$ and $0 < d < \infty$ such that $S := \{a + tv : \ t \in [0, d]\} \subset D$. By the Hahn-Banach theorem we can choose $x^* \in X^*$ with $\|x^*\| = x^*(v) = 1$. Let $I := x^*(S)$ and $\tilde{a} := x^*(a)$; clearly $I = [\tilde{a}, \tilde{a} + d]$. Set $f(t) := \varphi(t - (\tilde{a} + d/2))$, $t \in \mathbb{R}$, where $\varphi = \varphi_{\omega_{**}}$ is the (extremal) function from Definition 5.3.

Further set $\tilde{f}(x) := f(x^*(x))$, $x \in D$. By Lemma 5.5(i), (2.3) and (5.5) we obtain

$$\omega_2^+(\tilde{f}, t) \leq \omega_2^+(f, t) = \omega_2^+(\varphi, t) \leq 6\omega_{**}(t), \quad t \geq 0. \tag{5.8}$$

Lemma 5.5(ii), (2.7) imply

$$\eta_{\tilde{f}}(t) \geq \eta_{f\restriction_I}(t) = \eta_{\varphi\restriction_{[-d/2,d/2]}}(t) \geq \eta_{\varphi\restriction_{(-d/2,d/2)}}(t), \quad t \geq 0. \tag{5.9}$$

Therefore Lemma 5.4(i) (recall that $\omega_{**} \in \mathcal{M}_2$) implies that $\tilde{f}$ is not semiconcave and so (2.6) implies that $g := (1/6)\tilde{f}$ is not semiconcave as well. Further (2.2) and (5.8) imply $\omega_2^+(g, t) \leq \omega_{**}(t) \leq \omega(t)$, $t \geq 0$. Consequently (i) does not hold and we are done. □

Theorem 5.7. *Let $\omega \in \mathcal{M}$, $\eta \in \mathcal{M}$ and let D be an infinite convex subset of a Banach space. Suppose that, for every continuous f on D, the condition $\omega_2^+(f,t) \le \omega(t)$, $t \ge 0$, implies that f is semiconcave with modulus η. Denote $\gamma^*(\delta) := \int_0^\delta \frac{\omega_{**}(t)}{t^2}\, dt$. Then the following assertions hold.*

(i) $\gamma^*(\delta) = O(\eta(\delta))$, $\delta \to 0+$.
(ii) *If D contains a ray, then* $\eta(\delta) \ge \frac{1}{12}\gamma^*(\delta)$, $0 \le \delta < \infty$.
(iii) *If D is compact, then* $\eta(\delta) \ge \frac{1}{12}\gamma^*(\delta)$, $0 \le \delta \le \operatorname{diam} D$.

Proof. We can clearly choose $a \in D$, $v \in X$ with $\|v\| = 1$ and $0 < d \le \infty$ such that $S := \{a + tv : t \in [0,d] \cap \mathbb{R}\} \subset D$. Moreover, we can suppose that $d = \infty$ if D contains a ray and $d = \operatorname{diam} D$ if D is compact. By the Hahn-Banach theorem we can choose $x^* \in X^*$ with $\|x^*\| = x^*(v) = 1$. Let $I := x^*(S)$ and $\tilde{a} := x^*(a)$. Set $f(t) := \varphi(t - \tilde{a})$, $t \in \mathbb{R}$, where $\varphi = \varphi_{\omega_{**}}$ is the (extremal) function from Definition 5.3.

Further set $\tilde{f}(x) := f(x^*(x))$, $x \in D$. By Lemma 5.5(i), (2.3) and (5.5) we obtain

$$\omega_2^+(\tilde{f},t) \le \omega_2^+(f,t) = \omega_2^+(\varphi,t) \le 6\omega_{**}(t), \quad t \ge 0. \tag{5.10}$$

Lemma 5.5(ii), (2.7) and Lemma 5.4(ii) (recall that $\omega_{**} \in \mathcal{M}_2$) easily imply

$$\eta_{\tilde{f}}(\delta) \ge \eta_{f\restriction_I}(\delta) = \eta_{\varphi\restriction_{[0,d]\cap\mathbb{R}}}(\delta) \ge \frac{1}{2}\gamma^*(\delta), \quad \delta \in [0,d] \cap \mathbb{R}. \tag{5.11}$$

Set $g := (1/6)\tilde{f}$. Using (5.11) and (2.6), we obtain $\eta_g(\delta) \ge \frac{1}{12}\gamma^*(\delta)$, $\delta \in [0,d] \cap \mathbb{R}$. Since (5.10) and (2.2) imply $\omega_2^+(g,t) \le \omega_{**}(t) \le \omega(t)$, $t \ge 0$, the assumptions of the theorem imply that g is η-semiconcave, and so

$$\eta(\delta) \ge \eta_g(\delta) \ge \frac{1}{12}\gamma^*(\delta), \quad \delta \in [0,d] \cap \mathbb{R}.$$

Consequently the assertions (i), (ii), (iii) hold. ☐

We will need below the following easy lemma.

Lemma 5.8. *Let $\alpha, \beta \in \mathcal{M}$, $\beta(t) > 0$ for $t > 0$, and $\alpha(t) = O(\beta(t)), t \to 0+$. Then*

$$\text{for each } d > 0 \text{ there exists } C_d > 0 \text{ such that } \alpha(t) \le C_d\beta(t),\ 0 \le t \le d. \tag{5.12}$$

Further, if D is a bounded convex subset of a Banach space, then there exists $C > 0$ such that, for each function f on D, the following implications hold:

(i) *f is α-semiconcave $\Rightarrow$ f is $C\beta$-semiconcave.*
(ii) $\omega_2^+(f,t) \le \alpha(t)$, $t \ge 0$ $\Rightarrow$ $\omega_2^+(f,t) \le C\beta(t)$, $t \ge 0$.

Proof. We will omit a (quite easy) proof of (5.12). If we set $d := \operatorname{diam} D$ and $C := C_d$, then (i) clearly hold. Since $\omega_2^+(f,t) = \omega_2^+(f,d)$ for $t \ge d$, also (ii) holds. ☐

Theorem 5.9. *Let $\omega \in \mathcal{M}$, $\eta \in \mathcal{M}$ and let D be an infinite bounded convex subset of a Banach space. Denote $\gamma^*(\delta) := \int_0^\delta \frac{\omega_{**}(t)}{t^2}\, dt$. Then the following assertions are equivalent.*

(i) *For every continuous f on D, the condition $\omega_2^+(f,t) \le \omega(t)$, $t \ge 0$, implies that f is $(C_1\eta)$-semiconcave for some $C_1 = C_1(\eta, f) > 0$.*
(ii) *There exists $C_2 > 0$ such that, for every continuous f on D, the condition $\omega_2^+(f,t) \le \omega(t)$, $t \ge 0$, implies that f is $(C_2\eta)$-semiconcave.*
(iii) $\gamma^*(\delta) = O(\eta(\delta))$, $\delta \to 0+$.

Proof. The implication $(ii) \Rightarrow (i)$ is trivial.

To prove $(iii) \Rightarrow (ii)$, suppose that (iii) holds. Consider an arbitrary continuous f on D with $\omega_2^+(f,t) \le \omega(t),\ t \ge 0$. Since γ^* is finite by (iii), Theorem 4.9 implies that f is $(32\,\gamma^*)$-semiconcave. Since the case $\omega_{**} \equiv 0$ is trivial and $\omega_{**} \in \mathcal{M}_2$, we can suppose that $\omega_{**}(t) > 0$ for $t > 0$, and so (iii) implies that $\eta(t) > 0$ for $t > 0$. Consequently, applying the second part of Lemma 5.8 with $\alpha := 32\,\gamma^*$ and $\beta := \eta$, we easily obtain (ii).

To prove $(i) \Rightarrow (iii)$, let a, v, d, S, x^*, I, $\tilde{a}$, f, φ, $\tilde{f}$, g be as in the proof of Theorem 5.7. Then

$$C_1(\eta,g)\eta(\delta) \ge \eta_g(\delta) \ge (1/12)\gamma^*(\delta), \quad \delta \in [0,d] \cap \mathbb{R}, \tag{5.13}$$

and (iii) follows. □

Quite similarly we obtain the following version.

Theorem 5.10. *Let $\omega \in \mathcal{M}$, $\eta \in \mathcal{M}$ and let D be a convex subset of a Banach space which contains a ray. Denote $\gamma^*(\delta) := \int_0^\delta \frac{\omega_{**}(t)}{t^2}\,dt$. Then the following assertions are equivalent.*

- (i) *For every continuous f on D, the condition $\omega_2^+(f,t) \le \omega(t),\ t \ge 0$, implies that f is $(C_1\eta)$-semiconcave for some $C_1 = C_1(\eta, f) > 0$.*
- (ii) *There exists $C_2 > 0$ such that, for every continuous f on D, the condition $\omega_2^+(f,t) \le \omega(t),\ t \ge 0$, implies that f is $(C_2\eta)$-semiconcave.*
- (iii) *There exists $C_3 > 0$ such that $\eta(\delta) \ge C_3\gamma^*(\delta), \quad \delta \ge 0$.*

Proof. The implication $(ii) \Rightarrow (i)$ is trivial and $(iii) \Rightarrow (ii)$ follows clearly by Theorem 4.9. The implication $(i) \Rightarrow (iii)$ follows as in the proof of Theorem 5.9. Note that now $d = \infty$, and so (5.13) implies (iii). □

Remark 5.11. It is well-known (see, e.g., [2]) that each convex unbounded $D \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ contains a ray.

From the above results on the sharpness of Theorem 4.9, we easily infer analogous results on the sharpness of Theorem 4.4. In the proofs we will use the following facts.

Observation. Let $\tilde{\omega} \in \mathcal{M}$ and let D be an infinite convex subset of a Banach space. Set $\omega(t) := t\tilde{\omega}_*(t),\ t \ge 0$. Then $\omega \in \mathcal{M}_2$ by Lemma 3.4(iii) and Lemma 3.6(iii), so $\omega_{**} = \omega$ by Lemma 3.4(iv) and therefore

$$\frac{\omega_{**}(t)}{t^2} = \frac{\tilde{\omega}_*(t)}{t}, \quad t \ge 0. \tag{5.14}$$

Further, let f be a continuous function on D. Then Corollary 4.3 implies that

$$\text{if } \omega_2^+(f,t) \le \omega(t),\ t \ge 0, \text{ then } f \text{ is Jensen } \tilde{\omega}\text{-semiconcave and} \tag{5.15}$$

$$\text{if } f \text{ is Jensen } \tilde{\omega}\text{-semiconcave, then } \omega_2^+(f,t) \le 8\omega(t),\ t \ge 0. \tag{5.16}$$

Finally, denote $\beta^*(\delta) := \sum_{k=0}^\infty \tilde{\omega}_*\left(\frac{\delta}{2^k}\right)$ and $\gamma^*(\delta) = \int_0^\delta \frac{\omega_{**}(t)}{t^2}\,dt = \int_0^\delta \tilde{\omega}_*(t)/t\,dt$, $t \ge 0$. Then, since $\tilde{\omega}_* \in \mathcal{M}_1$ by Lemma 3.4(iii), Lemma 4.7 implies

$$(1/2)\beta^*(\delta) \le \gamma^*(\delta) \le \beta^*(\delta). \tag{5.17}$$

Theorem 5.12. *Let $\omega \in \mathcal{M}$ and let D be an infinite convex subset of a Banach space. Then the following assertions are equivalent.*

(i) *If u is a continuous Jensen $\tilde{\omega}$-semiconcave function on D, then u is semiconcave on D.*
(ii) $\sum_{k=0}^{\infty} \tilde{\omega}_* \left(\frac{1}{2^k}\right) < \infty.$
(iii) $\int_0^1 \frac{\tilde{\omega}_*(t)}{t}\, dt < \infty.$

Proof. Since $\tilde{\omega}_* \in \mathcal{M}_1$ by Lemma 3.4(iii), we have $(ii) \Leftrightarrow (iii)$ by Lemma 4.7. The condition (ii) clearly implies (i) by Theorem 4.4.

Now suppose that (i) holds. Set $\omega(t) := t\tilde{\omega}_*(t),\ t \geq 0$. If u is an arbitrary continuous function on D with $\omega_2^+(u,t) \leq \omega(t),\ t \geq 0$, then u is Jensen $\tilde{\omega}_*$-semiconcave by (5.15), and consequently semiconcave by (i). Hence the assumption (i) of Theorem 5.6 holds, so condition (ii) of Theorem 5.6 (which is equivalent to (iii) by (5.14)) holds. $\square$

Theorem 5.13. *Let $\tilde{\omega} \in \mathcal{M}$, $\eta \in \mathcal{M}$ and let D be an infinite convex subset of a Banach space. Suppose that every continuous Jensen $\tilde{\omega}$-semiconcave function on D is η-semiconcave. Denote $\beta^*(\delta) := \sum_{k=0}^{\infty} \tilde{\omega}_* \left(\frac{\delta}{2^k}\right)$. Then the following assertions hold.*

(i) $\beta^*(\delta) = O(\eta(\delta)),\quad \delta \to 0+.$
(ii) *If D contains a ray, then* $\eta(\delta) \geq (1/24)\beta^*(\delta), \quad 0 \leq \delta < \infty.$
(iii) *If D is compact, then* $\eta(\delta) \geq (1/24)\beta^*(\delta), \quad 0 \leq \delta \leq \operatorname{diam} D.$

Proof. Set $\omega(t) := t\tilde{\omega}_*(t),\ t \geq 0$. If u is an arbitrary continuous function on D with $\omega_2^+(f,t) \leq \omega(t),\ t \geq 0$, then u is Jensen $\tilde{\omega}$-semiconcave by (5.15), and consequently u is semiconcave with modulus η by the assumption of the theorem. Thus the assumption of Theorem 5.7 holds. Consequently, the assertion of Theorem 5.7 holds. Using (5.17), we obtain (i), (ii) and (iii). $\square$

Theorem 5.14. *Let $\tilde{\omega} \in \mathcal{M}$, $\eta \in \mathcal{M}$ and let D be an infinite bounded convex subset of a Banach space. Denote $\beta^*(\delta) := \sum_{k=0}^{\infty} \tilde{\omega}_* \left(\frac{\delta}{2^k}\right)$. Then the following assertions are equivalent.*

(i) *Every continuous Jensen $\tilde{\omega}$-semiconcave function on D is $(C_1\eta)$-semiconcave for some $C_1 = C_1(\eta, f) > 0$.*
(ii) *There exists $C_2 > 0$ such that every continuous Jensen $\tilde{\omega}$-semiconcave function on D is $(C_2\eta)$-semiconcave.*
(iii) $\beta^*(\delta) = O(\eta(\delta)),\quad \delta \to 0+.$

Proof. Set $\omega(t) := t\tilde{\omega}_*(t),\ t \geq 0$. Using (5.15), (5.16), (5.17), (2.2) and (2.6), it is easy to see that (i) (resp. (ii), resp. (iii)) is equivalent to (i) (resp. (ii), resp. (iii)) of Theorem 5.9. So Theorem 5.9 implies our theorem. $\square$

By the same way Theorem 5.10 implies the following theorem.

Theorem 5.15. *Let $\tilde{\omega} \in \mathcal{M}$, $\eta \in \mathcal{M}$ and let D be a convex subset of a Banach space which contains a ray. Denote $\beta^*(\delta) := \sum_{k=0}^{\infty} \tilde{\omega}_* \left(\frac{\delta}{2^k}\right)$. Then the following assertions are equivalent.*

(i) *Every continuous Jensen $\tilde{\omega}$-semiconcave function on D is $(C_1\eta)$-semiconcave for some $C_1 = C_1(\eta, f)$.*
(ii) *There exists $C_2 > 0$ such that every continuous Jensen $\tilde{\omega}$-semiconcave function on D is $(C_2\eta)$-semiconcave.*
(iii) *There exists $C_3 > 0$ such that $\eta(\delta) \geq C_3\beta^*(\delta), \quad \delta \geq 0$.*

Remark 5.16. Lemma 4.7 implies that Theorems 5.13, 5.14 and 5.15 remain hold, if we everywhere replace $\beta^*(\delta)$ by $\gamma^*(\delta) = \int_0^\delta \tilde{\omega}_*(t)/t\ dt$.

6. For which η Jensen η-semiconcavity implies $C\eta$-semiconcavity ?

We present two theorems; the first for an unbounded D and the second for a bounded D.

Theorem 6.1. *Let $\eta \in \mathcal{M}$ and let D be a convex subset of a Banach space which contains a ray. Denote $\mu^*(\delta) = \int_0^\delta \eta_*(t)/t\ dt$. Then the following assertions are equivalent.*

(i) *Every continuous Jensen η-semiconcave function on D is $(C_1\eta)$-semiconcave for some $C_1 = C_1(\eta, f) > 0$.*
(ii) *There exists $C_2 > 0$ such that every continuous Jensen η-semiconcave function on D is $(C_2\eta)$-semiconcave.*
(iii) *There exists $C_3 > 0$ such that $\eta(\delta) \geq C_3\mu^*(\delta), \quad \delta \geq 0$.*
(iv) *There exists $C_4 > 0$ such that $\eta_*(\delta) \geq C_4\mu^*(\delta), \quad \delta \geq 0$.*

Proof. Applying Theorem 5.15 and Remark 5.16 with $\tilde{\omega} = \eta$, we obtain $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Observe that if μ^* is a finite function, then $\mu^* \in \mathcal{M}_1$. Indeed, the function

$$\psi(\delta) := \frac{\mu^*(\delta)}{\delta} = \frac{1}{\delta}\int_0^\delta \eta_*(t)/t\ dt$$

is clearly nonincreasing on $(0, \infty)$, since the function $t \mapsto \eta_*(t)/t,\ t > 0$ is nonincreasing (as $\eta_* \in \mathcal{M}_1$ by Lemma 3.4(iii)). Consequently $(\mu^*)_* = \mu^*$, which easily implies $(iii) \Leftrightarrow (iv)$. □

Theorem 6.2. *Let $\eta \in \mathcal{M}$ and let D be an infinite bounded convex subset of a Banach space. Denote $\mu^*(\delta) = \int_0^\delta \eta_*(t)/t\ dt$. Then the following conditions are equivalent.*

(i) *Every continuous Jensen η-semiconcave function on D is $(C_1\eta)$-semiconcave for some $C_1 = C_1(\eta, f) > 0$.*
(ii) *There exists $C_2 > 0$ such that every continuous Jensen η-semiconcave function on D is $(C_2\eta)$-semiconcave.*
(iii) $\mu^*(\delta) = O(\eta(\delta)), \quad \delta \to 0+$.
(iv) $\mu^*(\delta) = O(\eta_*(\delta)), \quad \delta \to 0+$.

If moreover $\eta(t) > 0$ for $t > 0$, then these conditons are equivalent to

(v) *There exists $K > 1$ such that*

$$\liminf_{t\to 0+} \frac{\eta_*(Kt)}{\eta_*(t)} > 1.$$

Proof. Applying Theorem 5.14 and Remark 5.16 with $\tilde{\omega} = \eta$, we obtain $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

The argument from the proof of Theorem 6.1 gives $(\mu^*)_* = \mu^*$, which implies $(iii) \Leftrightarrow (iv)$. Indeed, $(iv) \Rightarrow (iii)$ is trivial. Further, if (iii) holds, then there exists $\delta_0 > 0$ and $K > 0$ such that $\mu^*(\delta) \leq K\eta(\delta)$, $0 \leq \delta < \delta_0$. So, by Definition 3.2(ii), $\mu^*(\delta) = (\mu^*)_*(\delta) \leq (K\eta)_*(\delta) = K\eta_*(\delta)$, $0 \leq \delta < \delta_0$, and consequently (iv) holds.

Finally, $(iv) \Leftrightarrow (v)$ follows from Lemma 3.11. □

Remark 6.3. Both Theorem 6.1 and Theorem 6.2 remain hold, if we replace at all places $\mu^*(\delta)$ by $\nu^*(\delta) := \sum_{k=0}^{\infty} \eta_* \left(\frac{\delta}{2^k}\right)$. This follows easily from (4.7) (since $\eta_* \in \mathcal{M}_1$).

Remark 6.4. Theorems 6.1 and 6.2, Remark 6.3, Lemma 3.4 and Remark 5.11 imply that, if $\eta \in \mathcal{M}$ is nonzero concave, then the implication from question (Q2) in the introduction holds if and only

(i) there exists $K > 1$ such that $\liminf_{t\to 0+} \eta(Kt)/\eta(t) > 1$ in the case when the domain D is an infinite bounded convex subset of a Banach space, and
(ii) there exists $C > 0$ such that $\eta(\delta) \geq C \int_0^\delta \eta(t)/t \, dt$ (respectively $\eta(\delta) \geq C \sum_{k=0}^{\infty} \eta(2^{-k}\delta))$, $\delta \geq 0$, if $D \subset \mathbb{R}^n$ is an unbounded convex set.

7. Which properties of $\omega_2^+(f,\cdot)$ follow from η-semiconcavity of f ?

In preceding sections we studied the question when a property of $\omega_2^+(f,\cdot)$ or the Jensen $\tilde{\omega}$-semiconcavity of f implies semiconcavity of f.

Now we will address the converse question. In particular, we ask whether (when) η-semiconcavity of f implies $\int_0^1 \frac{\omega_2^+(f,t)}{t^2} \, dt < \infty$.

If f is η-semiconcave, then f is trivially Jensen η-semiconcave (see (1.4)), which holds (by Lemma 4.1(ii)) if and only if $\omega_2^+(f,t) \leq t\eta(2t)$, $t \geq 0$. However, these consequences are not the best possible; using Proposition 3.8 (or Proposition 4.2), we obtain consequences which are the best possible (up a multiplicative constant).

Proposition 7.1. (i) *Let D be a convex subset of a Banach space, $\eta \in \mathcal{M}$, and f an η-semiconcave function on D. Then*

$$\omega_2^+(f,t) \leq 8t\eta_*(t), \quad t \geq 0.$$

(ii) *If $\eta \in \mathcal{M}$ and $a > 0$, then there exists on $I = [-a,a]$ an η_*-semiconcave (and so also η-semiconcave) function f such that*

$$\omega_2^+(f,t) \geq (1/4)t\eta_*(t), \quad 0 \leq t \leq a. \tag{7.1}$$

Proof. (i) It follows immediately from Corollary 4.3(ii).

(ii) By Lemma 3.4(iii) and Lemma 3.9 we can choose a concave $\omega \in \mathcal{M}$ such that $\eta_* \leq \omega \leq 2\eta_*$. Let f be the even function on $[-a,a]$ with $f(x) = (1/4)\int_0^x \omega(u)\, du$, $x \in [0,a]$. Clearly $f'(x) = (1/4)\operatorname{sgn}(x)\omega(|x|)$, $x \in [-a,a]$. Since ω is concave, it is easy to check that f' is uniformly continuous with the modulus $(1/4)\omega$ both on $[0,a]$ and on $[-a,0]$, and consequently f' is uniformly continuous on $[-a,a]$ with modulus $(1/2)\omega$. Consequently f is semiconcave with modulus $(1/2)\omega$ by Lemma 2.1(i),(ii). Since $(1/2)\omega \leq \eta_*$, we obtain that f is η_*-semiconcave. For $0 < t \leq a$ we clearly have $\Delta_t^2(f,-t) = 2f(t)$ and so, using concavity of ω, we obtain

$$\omega_2^+(f,t) \geq 2f(t) = (1/2)\int_0^t \omega(u)\, du \geq (1/4)t\omega(t) \geq (1/4)t\eta_*(t).$$

□

The following corollary is obvious.

Corollary 7.2. *If $\eta \in \mathcal{M}$ and $a > 0$, then there exists a function f semiconcave on $[-a,a]$ with modulus η such that*

$$\omega_2^+(f,t) \asymp t\eta_*(t), \ t \to 0+ .$$

Corollary 7.3. *Let $\eta \in \mathcal{M}$ and $a > 0$. Then the following are equivalent.*

(i) *If f is a function on $I = [-a, a]$ which is semiconcave with modulus η, then $\int_0^1 \frac{\omega_2^+(f,t)}{t^2}\, dt < \infty$.*

(ii) $\int_0^1 \frac{\eta_*(t)}{t}\, dt < \infty$.

Proof. If (ii) holds, then (i) clearly follows by Proposition 7.1(i).

Further suppose that (ii) does not hold. Let f be a function from Proposition 7.1(ii). Then (7.1) clearly implies that (i) does not hold. □

Corollary 7.4. *The condition $\int_0^1 \frac{\omega_2^+(f,t)}{t^2}\, dt < \infty$ is not necessary for the semiconcavity of f on $[-a, a]$ ($a > 0$).*

Proof. It is easy to check that the function $\varphi(t) = -(\ln t)^{-1}$ is increasing and concave on $(0, e^{-2})$. We can clearly find a concave $\eta \in \mathcal{M}$ such that $\eta = \varphi$ on $(0, e^{-2})$. Since $\eta = \eta_*$ by Lemma 3.4(i),(iv) and $\int_0^1 \frac{\eta(t)}{t}\, dt = \infty$, Corollary 7.3 gives our assertion. □

Proposition 7.5. (i) *Let D be a convex subset of a Banach space, $\eta \in \mathcal{M}$, and f an η-semiconcave function on D. Then f is Jensen $4\eta_*$-semiconcave.*

(ii) *Let $\eta \in \mathcal{M}$, $\tilde{\omega} \in \mathcal{M}$ and $a > 0$. Let, for each function f on $I = [-a, a]$, η-semiconcavity of f implies Jensen $\tilde{\omega}$-semiconcavity. Then*

$$(1/8)\eta_*(t) \le \tilde{\omega}(t), \quad 0 \le t \le 2a.$$

Proof. Property (i) follows from (1.4) and Proposition 3.8 (or Proposition 4.2).

To prove (ii), we find by Proposition 7.1(ii) an η_*-semiconcave function f on I such that $\omega_2^+(f,t) \ge (1/4)t\eta_*(t)$, $0 \le t \le a$. Since f is η-semiconcave, it is Jensen $\tilde{\omega}$-semiconcave, and so $\omega_2^+(f,t) \le t\tilde{\omega}(2t)$, $t \ge 0$, by Lemma 4.1(ii). Consequently $(1/4)t\eta_*(t) \le t\tilde{\omega}(2t)$ for $0 \le t \le a$. Since η_* is subadditive by Lemma 3.4(ii),(iii), $\eta_*(t) \ge (1/2)\eta_*(2t)$. So $(1/8)t\eta_*(2t) \le t\tilde{\omega}(2t)$ for $0 \le t \le a$ and our assertion follows. □

Note that assertion (ii) shows that (on $[-a, a]$) both Proposition 3.8 and Proposition 4.2 are the best possible (up a multiplicative constant).

8. Semi-Zygmund classes

There exist several works which (essentially) concern the sharpness of Marchaud-type theorems (in particular Theorem M from the introduction). The corresponding theorems were formulated as relations between generalized Hölder (Lipschitz) classes $W^r H_k^{\varphi}(J)$ (for different r,k,φ); see [4, p. 247]. Closely related results, which concern (generalized) Zygmund spaces $\Lambda_{\varphi}([-1,1]^n)$ were proved in [8, pp. 30,31]. Here $\Lambda_{\varphi}([-1,1]^n)$ ($\varphi \in \mathcal{M}$) is the set of all bounded functions on $[-1,1]^n$ for which

$$\omega_2(f,h) \le C\varphi(h), \quad h \ge 0, \qquad \text{for some } \ C > 0.$$

For $J := [-1, 1]$, we have $\Lambda_{\varphi}(J) = W^0 H_2^{\varphi}(J) = H_2^{\varphi}$ and so, for $n = 1$, results of [8, pp. 30,31] follow from [4, Theorem 7, p. 247 and Corollary 7.2, p. 251]. These results concern inclusions between spaces

$$\Lambda_{\omega}(J), \quad C^{1,\eta}(J), \quad C^1(J),$$

where $\omega \in \mathcal{M}_2$, $\eta \in \mathcal{M}_1$, and $f \in C^{1,\eta}(J)$ iff $f \in C^1(J)$ and f' is uniformly continuous with modulus Cη for some $C > 0$, .

The results of preceding sections easily imply analogous results on inclusions between classes

$$\Lambda^+_\omega(D), \quad S_\eta(D), \quad S(D),$$

where D is an infinite bounded convex subset of a Banach spaces and, if f is a function on D, then by definition

(i) $f \in \Lambda^+_\omega(D)$ if $\omega^+_2(f,h) \le C\,\varphi(h),\ \ h \ge 0$, for some $C > 0$,
(ii) $f \in S_\eta(D)$ if f is $C\eta$-semiconcave for some $C > 0$ and
(iii) $f \in S(D)$ if f is semiconcave.

It is natural to label classes $\Lambda^+_\omega(D)$ as semi-Zygmund classes.

As in [4], we will consider only the case of functions on a compact interval in $\mathbb{R}$. So we fix $a > 0$ and, in the rest of the section, we set $\Lambda^+_\omega := \Lambda^+_\omega([-a,a])$, $S_\eta := S_\eta([-a,a])$ and $S := S([-a,a])$.

Observe that Lemma 3.8 and Lemma 3.7(i) imply that, for every $\omega, \eta \in \mathcal{M}$,

$$S_\eta = S_{\eta_*} \quad \text{and} \quad \Lambda^+_\omega = \Lambda^+_{\omega_{**}}.$$

Therefore (using Lemma 3.4(iv)) we can (and will, as is usual in the "smooth results") in the study of systems S_η and Λ^+_ω suppose that $\eta \in \mathcal{M}_1$ and $\omega \in \mathcal{M}_2$, respectively.

Proposition 8.1. *Let $\omega \in \mathcal{M}_2$. Then*

$$\Lambda^+_\omega \subset S \qquad \Leftrightarrow \qquad \int_0^1 \frac{\omega(u)}{u^2}\,du < \infty.$$

Proof. Since $\omega_{**} = \omega$ by Lemma 3.4(iv), both implications easily follow from Theorem 5.6. □

Proposition 8.2. *Let $\eta \in \mathcal{M}_1$ and $\omega \in \mathcal{M}_2$. Then*

$$\Lambda^+_\omega \subset S_\eta \qquad \Leftrightarrow \qquad \int_0^t \frac{\omega(u)}{u^2}\,du = O(\eta(t)),\ \ t \to 0+\,.$$

Proof. Since $\omega_{**} = \omega$ by Lemma 3.4(iv), both implications easily follow from Theorem 5.9. □

Proposition 8.3. *Let $\eta \in \mathcal{M}_1$ and $\omega \in \mathcal{M}_2$. Then*

$$S_\eta \subset \Lambda^+_\omega \qquad \Leftrightarrow \qquad t\,\eta(t) = O(\omega(t)),\ \ t \to 0+\,.$$

Proof. First note that $\eta = \eta_*$ by Lemma 3.4(iv).

To prove "$\Rightarrow$", suppose $S_\eta \subset \Lambda^+_\omega$. By Proposition 7.1(ii) we can choose $f \in S_\eta$ such that $t\,\eta(t) = O(\omega^+_2(f,t)),\ t \to 0+$. Since $f \in \Lambda^+_\omega$, we have $\omega^+_2(f,t) = O(\omega(t)),\ t \to 0+$, and so $t\,\eta(t) = O(\omega(t)),\ t \to 0+$, follows.

To prove "$\Leftarrow$", suppose that $t\,\eta(t) = O(\omega(t)),\ t \to 0+$, and consider an arbitrary $f \in S_\eta$. By Proposition 7.1(i) there exists $M > 0$ such that $\omega^+_2(f,t) \le Mt\eta(t),\ t \ge 0$. Applying Lemma 5.8 to $\alpha(t) = Mt\eta(t),\ t \ge 0$, and $\beta = \omega$, we obtain $f \in \Lambda^+_\omega$. □

Theorem 8.4. *Let $\eta \in \mathcal{M}_1$ and $\omega \in \mathcal{M}_2$. Then the following conditions are equivalent.*

(i) $S_\eta = \Lambda^+_\omega$.
(ii) $t\eta(t) = O(\omega(t)), t \to 0+$, *and* $\int_0^t \frac{\omega(u)}{u^2}\,du = O(\eta(t)), t \to 0+\,$.
(iii) $t\eta(t) \asymp \omega(t), t \to 0+$, *and* $\int_0^t \frac{\eta(u)}{u}\,du \asymp \eta(t), t \to 0+$.
(iv) $t\eta(t) \asymp \omega(t), t \to 0+$, *and* $\int_0^t \frac{\omega(u)}{u^2}\,du \asymp \frac{\omega(t)}{t}, t \to 0+\,$.

If moreover $\eta(t) > 0$ *for* $t > 0$*, then these conditions are quivalent to*

(v) $\exists K > 1 :\ \liminf_{t\to 0+} \frac{\eta(Kt)}{\eta(t)} > 1$ *and* $t\eta(t) \asymp \omega(t), t \to 0+$.

Proof. Propositions 8.2 and 8.3 immediately imply $(i) \Leftrightarrow (ii)$.

To prove $(ii) \Rightarrow (iii)$, suppose that (ii) holds. Since the function $\omega(u)/u^2$ is nonincreasing, we have $\int_0^t \frac{\omega(u)}{u^2}\, du \geq (\omega(t)/t^2) \cdot t$, and consequently $\omega(t)/t = O(\eta(t)), t \to 0+$, which with the first relation of (ii) clearly implies $t\eta(t) \asymp \omega(t), t \to 0+$. Consequently $\int_0^t \frac{\eta(u)}{u}\, du \asymp \int_0^t \frac{\omega(u)}{u^2}\, du = O(\eta(t)), t \to 0+$. Since the function $\eta(u)/u$ is nonincreasing, we have

$$\int_0^t \frac{\eta(u)}{u}\, du \geq (\eta(t)/t) \cdot t = \eta(t), \quad t > 0. \tag{8.1}$$

The above estimates give the second relation of (iii), and so (iii) holds.

The proofs of $(iii) \Leftrightarrow (iv)$ and $(iii) \Rightarrow (ii)$ are obvious.

Finally, $(iii) \Leftrightarrow (v)$ follows by Lemma 3.11 and (8.1). ☐

9. Connection to $\alpha(\cdot)$-midconvex functions

In this last short section we present some results concerning $\alpha(\cdot)$-midconvex functions which follow from our results. The following definition is a non-essential modification of [13, Definition 1.1].

Definition 9.1. Let $\alpha : [0, \infty) \to [0, \infty)$ be a nondecreasing function and D be a convex subset of a Banach space. A function $f : D \to \mathbb{R}$ is called $\alpha(\cdot)$-midconvex if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} + \alpha(\|x - y\|), \qquad x, y \in D. \tag{9.1}$$

The authors of [13] investigate properties of $\alpha(\cdot)$-midconvex functions. Let us note that in [13]

(i) the condition $\lim_{t\to 0+} \alpha(t) = 0$ *is not assumed* and

(ii) also *discontinuous* $\alpha(\cdot)$-midconvex functions are studied.

It is easy to see that (9.1) is equivalent to $\Delta^2_{(y-x)/2}(-f, x) \leq 2\alpha(\|x-y\|),\ x, y \in D$, and consequently to

$$\omega_2^+(-f, t) \leq 2\alpha(2t), \quad t \geq 0. \tag{9.2}$$

Therefore, if $\alpha \in \mathcal{M}$, Lemma 3.7 has the following consequence.

Proposition 9.2. *Let* $\alpha \in \mathcal{M}$ *and* D *be a convex subset of a Banach space. Then every* $\alpha(\cdot)$*-midconvex function on* D *is a* $4\alpha_{**}(\cdot)$*-midconvex function.*

Proof. Set $\omega(t) := 2\alpha(2t)$, $t \geq 0$. Let f be an $\alpha(\cdot)$-midconvex function on D. Then (see (9.2)) $\omega_2^+(-f, t) \leq \omega(t)$, $t \geq 0$, and Lemma 3.7 gives $\omega_2^+(-f, t) \leq 4\omega_{**}(t)$, $t \geq 0$. Since $\omega_{**}(t) = 2\alpha_{**}(2t)$ by Lemma 3.6(ii), we have

$$\omega_2^+(-f, t) \leq 2(4\alpha)_{**}(2t), \quad t \geq 0,$$

and so f is a $4\alpha_{**}(\cdot)$-midconvex function. ☐

The above result shows that, if $\alpha \in \mathcal{M}$, some known results on $\alpha(\cdot)$-midconvex functions (e.g. [13, Theorem 3.1]) have immediate consequences which can (if $\alpha \notin \mathcal{M}_2$) improve original results, since $4\alpha_{**}$ can be essentially smaller than α.

As a concrete application of Proposition 9.2, we will present an alternative proof of the following result which is essentially contained in [13, Theorem 5.1] (the case $A = 0$).

Theorem TT. *Let $\alpha : [0, \infty) \to [0, \infty)$ be a nondecreasing function for which*

$$\liminf_{r \to 0+} \frac{\alpha(r)}{r^2} = 0. \tag{9.3}$$

Let D be a convex subset of a Banach space X and f an $\alpha(\cdot)$-midconvex function on D such that, for each line L in X with $L \cap D \neq \emptyset$, the restriction $f \restriction_{L \cap D}$ is locally bounded at each point of $L \cap D$. Then f is convex on D.

Proof. Condition (9.3) clearly implies that $\alpha \in \mathcal{M}$ and

$$\alpha_{**}(t) = t^2 \cdot \inf \left\{ \alpha(u)/u^2 : \ 0 < u \leq t \right\} = 0, \quad t \in (0, \infty).$$

Hence Proposition 9.2 implies that f is midconvex (= Jensen convex) on D. Therefore each function

$$f_{x,v} : t \mapsto f(x + tv), \quad x + tv \in D, \qquad \text{where } x \in D, v \in X, \|v\| = 1,$$

is midconvex and locally bounded on its domain $D_{x,v}$ (which is an interval or a singleton). Consequently the Bernstein-Doetsch theorem (see [13, p. 656] for the formulation and references) implies that each $f_{x,v}$ is convex on the interior of $D_{x,v}$. Since $f_{x,v}$ is midconvex, a straightforward elementary argument gives that $f_{x,v}$ is convex. Consequently, f is convex. $\square$

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