Semilinear damped wave equation in locally uniform spaces

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Abstract

We study a damped wave equation with nonlinear damping in the locally uniform spaces and prove well-posedness of the equation and existence of a locally compact attractor. An upper bound on the Kolmogorov's ε-entropy is also established using the method of trajectories.

1 Introduction

We study the semilinear damped wave equation

$$u_{tt} + g(u_t) - \Delta u + \alpha u + f(u) = h, \qquad t > 0, x \in \mathbb{R}^d, \tag{1.1}$$

where f and g are nonlinear continuous functions described in more detail in Section 3, with initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \qquad x \in \mathbb{R}^d. \tag{1.2}$$

We focus on proving the well-posedness of the problem in the context of locally uniform spaces, the existence of a locally compact attractor and mainly on establishing an upper bound on the Kolmogorov's ε-entropy. We use the method of trajectories introduced in [15], which has been previously used in a similar context for showing the finite dimensionality of the global attractor of (1.1) in bounded domains in [16]. However, the approach applied to the bounded domain problem cannot be used directly due to a different nature of embeddings in weighted spaces and requires a slightly different technique. To this end, we introduce a hyperbolic variant of locally uniform spaces which seems suitable for equations with a finite speed of propagation. Also as usual in locally uniform spaces, the problem has an inherent non-compactness and non-separability. In order to obtain the dissipation of energy, we formulate additional assumptions that allow for the nonlinearities in the equation to be superlinear. One could expect that a suitable control of dispersion could yield dissipative estimates under weak growth restrictions on the nonlinearities.

This equation has been intensely studied in the setting of bounded domains. The existence of a global attractor with supercritical nonlinearities has been shown in [8] and for critical nonlinearities [18] with less restrictive conditions on the damping. The finite dimensionality has been discussed in [16] and has been achieved even for critical nonlinearities in [4] and [11].

In the context of locally uniform spaces, a linearly damped wave equation has been studied in [7] and [20]. In [20] the author has also established an upper bound on the Kolmogorov's ε-entropy of the locally compact attractor. Some results, including well-posedness and the existence of a locally compact attractor, have also been shown for a strongly damped wave equation in [19] and recently for a wave equation with fractional damping in [17]. To the best of our knowledge, nonlinear damping in this setting has not yet been studied.

The paper is organized as follows: In Section 2 we review the locally uniform spaces. In Section 3, the well-posedness of the equation (1.1) is established. In Section 4, we discuss

MSC 2010: Primary: 37L30; Secondary: 35B41, 35L70.

Keywords: Damped wave equations, nonlinear damping, unbounded domains, locally compact attractor, Kolmogorov's entropy.

additional assumptions that lead to dissipative estimates. In Section 5, we introduce the trajectory setting and prove a local variant of squeezing property, which is used in Section 6 to establish the existence of the locally compact attractor and an upper bound on its Kolmogorov's ε-entropy .

2 Function spaces

In this section we review the basic facts about weighted Sobolev spaces and locally uniform spaces. These spaces and their relation have been studied in [21] (TODO: check) and [2].

By an *admissible weight function of growth rate* $\nu \le 0$ we understand $\phi : \mathbb{R}^d \to (0, +\infty)$ measurable and bounded satisfying

$$C_\phi^{-1} e^{-\nu|x-y|} \le \phi(x)/\phi(y) \le C_\phi e^{\nu|x-y|} \tag{2.1}$$

for some $C_\phi \ge 1$ and every $x, y \in \mathbb{R}^d$.

For $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$ we define the *weight function* $\phi_{\bar{x},\varepsilon}$ *with center in* $\bar{x}$ *and decay rate* ε by

$$\phi_{\bar{x},\varepsilon}(x) = e^{-\varepsilon|x-\bar{x}|}. \tag{2.2}$$

Then clearly for every multiindex α there exists $C_\alpha > 0$ such that

$$|D^\alpha \phi_{\bar{x},\varepsilon}| \le C_\alpha \varepsilon^{|\alpha|} \phi_{\bar{x},\varepsilon}. \tag{2.3}$$

We emphasize that, thanks to [2, Proposition 4.1], the particular choice of the weight function (2.2) does not play any role in the definition of the locally uniform spaces below as long as a certain decay properties are met. Also note that by the above definition, $\phi_{\bar{x},\varepsilon}$ is an admissible weight function with growth ε.

For $p \in [1, \infty)$ we define the *weighted Lebesgue space* $L^p_{\bar{x},\varepsilon}(\mathbb{R}^d)$ by

$$L^p_{\bar{x},\varepsilon}(\mathbb{R}^d) = \{u \in L^p_{loc}(\mathbb{R}^d); \|u\|_{L^p_{\bar{x},\varepsilon}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |u|^p \phi_{\bar{x},\varepsilon}\, dx\right)^{1/p} < \infty\}.$$

In the special case $p = 2$ we use the notation

$$\|u\|_{\bar{x},\varepsilon} \equiv \|u\|_{L^2_{\bar{x},\varepsilon}(\mathbb{R}^d)}.$$

We denote the scalar product on $L^2_{\bar{x},\varepsilon}(\mathbb{R}^d)$ by $(\cdot,\cdot)_{\bar{x},\varepsilon}$. The scalar product on $L^2(\mathbb{R}^d)$ will be denoted by $(\cdot,\cdot)$.

Clearly the embedding

$$L^p_{\bar{x},\varepsilon_1}(\mathbb{R}^d) \hookrightarrow L^p_{\bar{x},\varepsilon_2}(\mathbb{R}^d) \tag{2.4}$$

holds for $\varepsilon_1 \le \varepsilon_2$.

The weighted Sobolev spaces are defined in an obvious manner and allow the continuous embedding

$$W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow W^{l,q}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$$

with $k \ge l$ and $q \ge p$ such that $W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{l,q}(\mathbb{R}^d)$ and $\tilde{\varepsilon} = \varepsilon q/p$. We stress out that the embedding

$$W^{1,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow L^q_{\bar{x},\varepsilon}(\mathbb{R}^d)$$

does not hold for any $q > p$.

The weighted spaces also allow certain compact embeddings. More precisely, let $k \ge l$ and $q \ge p$ be such that $W^{k,p}(B(0,1)) \hookrightarrow\hookrightarrow W^{l,q}(B(0,1))$. Then we have the compact embedding

$$W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow\hookrightarrow W^{l,q}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$$

with $\tilde{\varepsilon} > \varepsilon p/q$, which gives that for example the embedding

$$\{u \in L^\infty(0,T_0; W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d)), u_t \in L^\infty(0,T_0; L^2_{\bar{x},\varepsilon}(\mathbb{R}^d))\} \hookrightarrow\hookrightarrow L^m(0,T_0; L^s_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)), \tag{2.5}$$

where $1 < m < \infty$ and $1 \le s < 2d/(d-2)$, is compact, and continuous for $s = 2d/(d-2)$.

Let C_k denote a closed unit cube in $\mathbb{R}^d$ centered at $x_k \in (\mathbb{Z}/2)^d$, i.e.

$$C_k = \prod_{i=1}^{d} [x_{k,i} - 1/2, x_{k,i} + 1/2], \qquad k \in \mathbb{N}.$$

The weighted *locally uniform Lebesgue space* $L^p_{b,\phi}$ for $p \in [1, \infty)$ and an admissible weight function ϕ is defined by

$$L^p_{b,\phi}(\mathbb{R}^d) = \{u \in L^p_{loc}(\mathbb{R}^d); \|u\|_{L^p_{b,\phi}(\mathbb{R}^d)} = \sup_{k \in \mathbb{N}} \phi^{1/p}(x_k)\|u\|_{L^p(C_k)} < \infty\}.$$

If $\phi \equiv 1$, we omit the subscript and write for example L^2_b instead of $L^2_{b,\phi}$. For $p = 2$ we use a simplified notation

$$\|u\|_{b,\phi} \equiv \|u\|_{L^2_{b,\phi}(\mathbb{R}^d)}.$$

The spaces $L^p_{b,\phi}(\mathbb{R}^d)$ are neither separable nor reflexive.

Locally uniform Sobolev spaces are again constructed in a straightforward manner. The standard embeddings holding on bounded domains also hold for locally uniform spaces, namely

$$W^{1,2}_b(\mathbb{R}^d) \hookrightarrow L^{2d/(d-2)}_b(\mathbb{R}^d). \tag{2.6}$$

However, none of these embeddings is compact.

The weighted Lebesgue spaces and the locally uniform spaces are connected through the equivalence of the locally uniform norm. The following lemma is standard and the proof can be found e.g. in [2] or [9].

Lemma 2.1. *Let $\varepsilon > 0$, $1 \leq p < \infty$, $k \in \mathbb{N}_0$ and let ϕ be an admissible weight function with growth $\nu < \varepsilon$. Then $u \in W^{k,p}_{b,\phi}(\mathbb{R}^d)$ if and only if*

$$\sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x})\|u\|^p_{W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d)} < \infty.$$

Moreover, the norm

$$\|u\|^p_{\tilde{W}^{k,p}_{b,\phi}(\mathbb{R}^d)} := \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x})\|u\|^p_{W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d)}$$

is equivalent with the original $W^{k,p}_{b,\phi}(\mathbb{R}^d)$norm.

Finally we define so-called *parabolic locally uniform space* $L^p_b(0, T; L^p(\mathbb{R}^d))$ by

$$L^p(0, T; L^p(\mathbb{R}^d)) = \{u : (0, T) \times \mathbb{R}^d \to \mathbb{R}; \|u\|_{L^p_b(0,T;L^p)} := \sup_{k \in \mathbb{N}} \|u\|_{L^p(0,T;L^p(C_k))} < \infty\}.$$

These spaces and their weighted variants have been studied in [9].

3 Well-posedness for locally uniform data

In this section we prove the existence and uniqueness of weak solutions of (1.1) for infinite energy data. We will make use of the following energy spaces which arise in the case of (1.1) in unbounded domains

$$\Phi_{\bar{x},\varepsilon} = W^{1,2}_{\bar{x},\varepsilon}(\Omega) \times L^2_{\bar{x},\varepsilon}(\mathbb{R}^d), \quad \Phi_b = W^{1,2}_b(\mathbb{R}^d) \times L^2_b(\mathbb{R}^d), \quad \Phi_{loc} = W^{1,2}_{loc}(\mathbb{R}^d) \times L^2_{loc}(\mathbb{R}^d).$$

We consider Φ_b as the phase space for the asymptotic analysis. However, it is well known that the locally uniform spaces are not separable, hence there are problems with attaining the initial conditions and approximating less regular data. There are at least two ways how to overcome this inconvenience. The first one is to consider Sobolev spaces with the weight functions like $\phi_{\bar{x},\varepsilon}$ with better properties. The second way is to use a phase space which is defined as closure of smooth functions in $\|\cdot\|_{\Phi_b}$ (such approach was considered e.g. in [14]). Both settings combined with the finite speed of propagation property of wave equations lead to the uniqueness and existence result. We have chosen the second approach.

Let us denote $L^p_{loc}(I; W^{k,p}_{loc}(\Omega))$ the set of measurable functions u on $I \times \Omega$ such that for any compact $J \subseteq I$ and $K \subseteq \Omega$ we have $u \in L^p(J; W^{k,p}(K))$. Particularly, $u \in L^\infty_{loc}(0, \infty; X)$ means that u is strongly (Bochner) measurable in (0,∞) and $u \in L^\infty_{loc}(0, T; X)$ for any $T > 0$.

We impose the following requirements on the nonlinearities of studied equation:

$$f \in C^1(\mathbb{R}), \tag{F1}$$
$$\forall r \in \mathbb{R} : |f'(r)| \le \gamma_1(|r|^{p-1}+1), \tag{F2}$$
$$f' \ge -\beta, \tag{F3}$$
$$\liminf_{|r|\to\infty} f(r)/r \ge 0, \tag{F4}$$

$$g \in C^1(\mathbb{R}), g(0)=0, g'>0, \tag{G1}$$
$$\liminf_{|z|\to\infty} g'(z) > 0, \tag{G2}$$
$$\forall r \in \mathbb{R} : \quad \gamma_2|r|^{\mu+1} - \gamma_3 \le g(r)r \le \gamma_4(|r|^{\mu+1}+1), \tag{G3}$$

where $\gamma_j, \beta > 0$. In what follows, we consider the following set of parameters:

$$p \in \left(0, \frac{2^*}{2}\right] \quad \text{for } d>2, \qquad p \in (0,\infty) \quad \text{for } d=2, \quad \mu \in [1,\infty). \tag{3.1}$$

The assumptions (F1), (F3) and (F4) allow us to find a decomposition $f(s) = f_1(s)+f_2(s)$ such that $f_1, f_2 \in C^1(\mathbb{R})$ satisfy

$$f_1(r)r \ge 0,\ f_1(0)=0, \qquad |f_1'| \ge -\beta,$$
$$f_1'(r) \le \gamma_1\left(|r|^{p-1}+1\right), \qquad |f_2(r)| + |f_2'(r)| \le \gamma_5,$$

where $\gamma_5 > 0$ depends on the function f.

Also from (G1) and (G2) we observe that for every $\delta > 0$ the estimate

$$|u-v|^2 \le \delta + c(\delta)\left(g(u)-g(v)\right)(u-v) \tag{3.2}$$

holds for every $u, v \in \mathbb{R}$ (cf. [10, Lemma 1] or [8]) .

We use the notation

$$E[u](t) = \frac{1}{2}\left(|\partial_t u(t)|^2 + |\nabla u(t)|^2 + \alpha|u(t)|^2\right),$$
$$F[u](t) = E[u](t,x) + F(u(t)),$$

where

$$F(r) = F_1(r) + F_2(r) = \int_0^r f_1(s)\,ds + \int_0^r f_2(s)\,ds.$$

Also note that

$$F_1(r) = \int_0^r f_1(s) + \beta s\,ds - \frac{\beta}{2}r^2 \le f_1(r)(r) + \frac{\beta}{2}r^2. \tag{3.3}$$

Definition. *Let $u_0,\ u_1 \in \Phi_b$ and $h \in L^2_b(0,\infty; L^2)$. We call $u : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ a* weak solution *of* (1.1) *if for every $\varepsilon > 0$ and $\bar{x} \in \mathbb{R}^d$ we have*

$$(u,u_t) \in C([0,\infty), \Phi_{\bar{x},\varepsilon}), \qquad u_t \in L^{\mu+1}_{loc}(0,\infty; L^{\mu+1}_{\bar{x},\varepsilon}(\mathbb{R}^d)),$$
$$u(0)=u_0, \quad u_t(0)=u_1, \qquad \|u\|^2_{W^{1,2}_b(\mathbb{R}^d)} + \|u_t\|^2_{L^2_b(\mathbb{R}^d)} \in L^\infty_{loc}((0,\infty)) \tag{3.4}$$

and the equality

$$-\int_0^\infty (u_t(t),\psi_t(t,\cdot))\,dt + \int_0^\infty (\nabla u(t),\nabla\psi(t,\cdot))\,dt + \int_0^\infty (\alpha u(t),\psi(t,\cdot))\,dt$$
$$+ \int_0^\infty (g(u_t(t)),\psi(t,\cdot))\,dt + \int_0^\infty (f(u(t)),\psi(t,\cdot))\,dt = \int_0^\infty (h(t),\psi(t,\cdot))\,dt \tag{3.5}$$

holds for every test function $\psi \in \mathcal{D}((0,\infty)\times\mathbb{R}^d)$ (or equivalently for every Lipschitz compactly supported function ψ).

The equality (3.5) has an equivalent version closely connected to the energy space $\Phi_{\varepsilon,\bar{x}}$. By using $\psi\phi_{\varepsilon,\bar{x}}$ with $\psi \in \mathcal{D}((0,\infty)\times\mathbb{R}^d)$ as a test function in (3.5), we obtain

$$\begin{aligned}
&-\int_0^T \Big(u'(t),\partial_t\psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \int_0^T \Big(\nabla u(t),\nabla\psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \int_0^T \Big(\alpha u(t),\psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt \\
&\quad + \int_0^T \Big(g(u_t(t)),\psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \int_0^T \Big(f(u(t)),\psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt \\
&\quad = \int_0^T \Big(h(t),\psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt - \int_0^T \Big(\nabla u(t),\psi(t,\cdot)\nabla\phi_{\bar{x},\varepsilon}\Big)\, dt.
\end{aligned} \tag{3.6}$$

By a standard density argument, the equality (3.6) holds also for any function $\psi\theta$ where $\theta = \theta(t)$ is a smooth function compactly supported in $(0,\infty)$ and

$$\psi \in L^{\infty}_{loc}(0,\infty; W^{1,2}_{\bar{x},\varepsilon}(\Omega)) \cap W^{1,2}_{loc}(0,\infty; L^2_{\bar{x},\varepsilon}(\mathbb{R}^d)) \cap L^{\mu+1}_{loc}(0,\infty; L^{\mu+1}_{\bar{x},\varepsilon}(\mathbb{R}^d)).$$

Moreover, if

$$\psi \in C([0,\infty); W^{1,2}_{\bar{x},\varepsilon}(\Omega)) \cap L^{\mu+1}_{loc}(0,\infty; L^{\mu+1}_{\bar{x},\varepsilon}(\mathbb{R}^d)) \tag{3.7}$$

then

$$\begin{aligned} -\int_0^T \Big(u'(t), \partial_t \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \Big(u'(T), \psi(T,\cdot)\Big)_{\bar{x},\varepsilon} - \Big(u'(0), \psi(0,\cdot)\Big)_{\bar{x},\varepsilon} \\ + \int_0^T \Big(\nabla u(t), \nabla \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \int_0^T \Big(g(u_t(t)), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt \\ + \int_0^T \Big(\alpha u(t), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \int_0^T \Big(f(u(t)), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt \\ = \int_0^T \Big(h(t), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt - \int_0^T \Big(\nabla u(t), \psi(t,\cdot)\nabla\phi_{\bar{x},\varepsilon}\Big)\, dt. \end{aligned} \tag{3.8}$$

is satisfied for every $T \in (0,\infty)$. To this end, we test (3.6) by $\psi\theta_n$ with

$$\theta_n'(t) = n\eta\left(n\left(t - \frac{1}{n}\right)\right)\chi_{\left(0,\frac{2}{n}\right)} - n\eta\left(n\left(t - T + \frac{1}{n}\right)\right)\chi_{\left(T-\frac{2}{n},T\right)}, \quad \theta_n(0) = 0$$

where η is the standard non-negative mollifier compactly supported in $(-1,1)$ and χ_I denotes the characteristic function of $I \subset \mathbb{R}$. Observe that $\theta' \rightharpoonup^* \delta_0 - \delta_T$ in the space of Radon measures on $[0,T]$ and $\theta \to 1$ in $L^s([0,T])$ for all $s \in [1,\infty)$. Hence, we conclude (3.8) by letting $n \to \infty$ and using the continuity of ψ with respect to time. The weak formulation is therefore equivalent to (3.8) with test functions (3.7).

The lack of regularity of u_t with respect to the space variables prevents us from using it as a test function in (3.8). On the other hand, one can test the weak formulation (3.8) by the time difference

$$D_\tau u(t,x) = \frac{u(t+\tau,x) - u(t-\tau,x)}{2\tau}$$

where we take $u(s,x) = u(0,x)$ for $s < 0$. Indeed, as $u \in AC([0,\infty),L^2_{\bar{x},\varepsilon})$ we have for a fixed $t \in (0,\infty)$

$$\left\|\frac{u(t+\tau,x) - u(t,x)}{\tau}\right\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}} = \left\|\frac{1}{\tau}\left(\int_t^{t+\tau} u'(s)\, ds\right)\right\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}} \le \frac{1}{\tau}\int_t^{t+\tau} \|u'(s)\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}}\, ds,$$

thus $D_\tau u \in L^{\mu+1}_{loc}\left(0,\infty; L^{\mu+1}_{\bar{x},\varepsilon}\right)$. In the rest of the paper, with an obvious abuse of terminology, we will use the phrase "testing by u_t" instead of taking the time differences as test functions and sending $\tau \to 0^+$. For more details see e.g. [12].

Theorem 3.1. *For every $(u_0, u_1) \in \Phi_b$ and $h \in L^2_b(0,\infty; L^2)$ there exists a unique weak solution of* (1.1) *which satisfies the energy equality*

$$\begin{aligned} \int_{\mathbb{R}^d} F[u](t_2)\phi_{\bar{x},\varepsilon}\, dx - \int_{\mathbb{R}^d} F[u](t_1)\phi_{\bar{x},\varepsilon}\, dx \\ + \int_{t_1}^{t_2} \Big(g(u_t), u_t\Big)_{\bar{x},\varepsilon} dt + \int_{t_1}^{t_2} \Big(\nabla u, u_t \nabla\phi_{\bar{x},\varepsilon}\Big) = \int_{t_1}^{t_2} \Big(h(t), u_t\Big)_{\bar{x},\varepsilon} dt \end{aligned} \tag{3.9}$$

for every $0 \le t_1 < t_2 < \infty$, $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$.

We remark that both existence and uniqueness of solutions can be shown even in the so-called super-critical case, particularly when

$$\mu \in [1,\infty),\ p \in \left(\frac{2^*}{2}, 2^* - 1\right),\ p \le \frac{2^*\mu}{\mu+1}, \quad f \in C^2(\mathbb{R}), \quad |f''(r)| \le \gamma_1(|r|^{p-2} + 1). \tag{3.10}$$

The existence part remains unchanged. Uniqueness follows by combination of the approach presented in [3] for bounded domains with the localisation technique developed in [1, Section 7].

As usual in the context of locally uniform spaces, one cannot expect the strong time continuity of solutions in the phase space Φ_b. Taking $d = 1$, one can check that

$$u(t,x) = e^{-t/2}\theta(x - t)$$

with

$$\theta(x) = \int_0^x \sum_{n=1}^{\infty} (-1)^{n+1} n \chi_{n,n+\frac{1}{n^2}}(y)\, dy$$

is a weak solution of

$$u_{tt} - u_{xx} + u_t + \frac{1}{4}u = 0,$$
$$(u(0,x),u_t(0,x)) = \Big(\theta(x), \frac{\theta}{2}(x) - \theta'(x)\Big) \in \Phi_b.$$

However,

$$\|u(t_1) - u(t_2)\|_{W_b^{1,2}} \geq \frac{1}{2e},$$

holds for all t_1, $t_2 \in (0,\delta)$, $t_1 \neq t_2$, provided $\delta > 0$ is small enough. Hence, $u\colon [0,\delta) \to W_b^{1,2}$ is not continuous as the range of u is not separable. Moreover, the function u is not strongly (Bochner) measurable.

Proof of Theorem 3.1. Assume that $\varepsilon > 0$ and $\bar{x}$ are given. It is sufficient to show existence of solutions on $(0,T)$ for fixed $T \in (0,\infty)$ independent on the initial data together with time continuity, particularly $(u,u_t) \in C([0,T]; \Phi_{\bar{x},\varepsilon})$. The existence of global solutions then follows from a continuation argument.

Step 1 - approximations and solutions on bounded domains. We approximate the nonlinear term f by Lipschitz functions and the initial data by compactly supported data. Let $\{f^k\}_k$ be a sequence of functions such that for every $k \in \mathbb{N}$ the function $f^k \in C^1(\mathbb{R})$ is globally Lipschitz, satisfies (F1)-(F4), $f^k \to f$ pointwise, $f^k(t) = f(t)$ for $t \in (-k,k)$ and $|f^k| \leq |f|$.

For $k \in \mathbb{N}$, we define function

$$\phi^k(x) = \begin{cases} 1 & \text{for } x \in B(0,k), \\ k + 1 - |x| & \text{for } x \in B(0,k+1) \setminus B(0,k), \\ 0 & \text{for } x \in B(0,k+1)^c. \end{cases} \tag{3.11}$$

Let $u_0^k = \eta_k * (u_0\phi^k)$, $u_1^k = \eta_k * (u_1\phi^k)$, $h^k = \eta_k * (h\phi^k)$ where $\eta_k = k^d\eta(k|x|)$ and η is the standard mollifier. We get

$$(u_0^k,u_1^k) \to (u_0,u_1) \quad \text{in } \Phi_{\bar{x},\varepsilon}, \qquad \|(u_0^k,u_1^k)\|_{\Phi_{\bar{x},\varepsilon}} \leq \|(u_0,u_1)\|_{\Phi_{\bar{x},\varepsilon}} \tag{3.12}$$
$$h^k \to h \quad \text{in } L^2((0,T); L^2_{\bar{x},\varepsilon}), \qquad \|h^k\|_{L^2_{\bar{x},\varepsilon}} \leq \|h\|_{L^2_{\bar{x},\varepsilon}} \tag{3.13}$$

as a direct consequence of approximation by mollifiers and decay of $\phi_{\bar{x},\varepsilon}$.

Existence and uniqueness of strong solutions on bounded domains is a well known result (see e.g. [13]). The finite speed of propagation holds as the source term f^k is Lipschitz. Hence, for every $k \in \mathbb{N}$ we can construct

$$u^k \in W^{1,2}_{loc}([0,\infty); L^2(\mathbb{R}^d)) \cap L^2_{loc}([0,\infty); W^{2,2}(\mathbb{R}^d)), \quad u_t \in L^{\mu+1}_{loc}([0,\infty); L^{\mu+1}(\mathbb{R}^d))$$

which is a global strong solution (the equation (3.14) is satisfied almost everywhere in $(0,T) \times \mathbb{R}^d$) of

$$u^k_{tt} + g(u^k_t) - \Delta u^k + \alpha u^k + f^k(u) = h^k, \qquad t > 0, \quad x \in \mathbb{R}^d \tag{3.14}$$

satisfying the initial conditions

$$u^k(0,x) = u^k_0(x), \quad u^k_t(0,x) = u^k_1(x), \qquad x \in \mathbb{R}^d.$$

Moreover, $u^k(t,\cdot)$ is compactly supported for any $t \in [0,\infty)$ and $(u^k,u^k_t) \in C([0,T]; W^{1,2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \hookrightarrow C([0,T]; \Phi_{\bar{x},\varepsilon})$.

Step 2 - uniform estimates in weighted Lebesgue spaces. Let us multiply both sides of (3.14) by $u_t^k\phi_{\bar{x},\varepsilon}$ and integrate the resulting equality w.r.t. x over $\mathbb{R}^d$. We get

$$\frac{d}{dt}\int_{\mathbb{R}^d}\Big(E[u^k]+F^k(u^k)\Big)\phi_{\bar{x},\varepsilon}\,dx+\int_{\mathbb{R}^d}g(u_t^k)u_t^k\phi_{\bar{x},\varepsilon}\,dx$$
$$=\int_{\mathbb{R}^d}h^k u_t^k\phi_{\bar{x},\varepsilon}\,dx+\int_{\mathbb{R}^d}\nabla u^k u_t^k\nabla\phi_{\bar{x},\varepsilon}\,dx\le\int_{\mathbb{R}^d}E[u^k]\phi_{\bar{x},\varepsilon}\,dx+\frac{1}{2}\int_{\mathbb{R}^d}|h^k|^2\phi_{\bar{x},\varepsilon}\,dx,$$

where F^k is the primitive function of f^k such that $F^k(0)=0$. From the Gronwall lemma and condition (G3), we obtain

$$\|u^k(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}}+\|u_t^k(\tau)\|^2_{L^2_{\bar{x},\varepsilon}}+\gamma_2\int_0^\tau\|u_t^k(t)\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}}\,dt\le e^\tau\Bigg(\|u^k(0)\|^2_{W^{1,2}_{\bar{x},\varepsilon}}+\|u_t^k(0)\|^2_{L^2_{\bar{x},\varepsilon}}$$
$$+\|F^k(u^k(0))\|_{L^1_{\bar{x},\varepsilon}}+\int_0^\tau\frac{1}{2}\|h^k(t)\|^2_{L^2_{\bar{x},\varepsilon}}\,dt+\tau\int_{\mathbb{R}^d}\gamma_3\phi_{\bar{x},\varepsilon}\,dx\Bigg)\tag{3.15}$$

for arbitrary $\tau\in(0,T)$. Therefore,

$$\sup_{\tau\in(0,T)}\Big(\|u^k(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}}+\|u_t^k(\tau)\|^2_{L^2_{\bar{x},\varepsilon}}\Big)+\int_0^T\|u_t^k(t)\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}}\,dt\le C$$

for some $C>0$ depending only on u_0, u_1, h and T. Applying the basic weak compactness arguments and (2.5), there is a subsequence of $\{u^k\}_{n\in\mathbb{N}}$ (not relabelled) and measurable functions u, $\bar{f}$, $\bar{g}$ such that

$$\begin{aligned}
(u^k,u_t^k)&\rightharpoonup^*(u,u_t) && \text{in } L^\infty((0,T);\Phi_{\bar{x},\varepsilon}),\\
u_t^k&\rightharpoonup^* u_t && \text{in } L^{\mu+1}((0,T);L^{\mu+1}_{\bar{x},\varepsilon}),\\
u^k&\rightharpoonup^* u && \text{in } L^\infty\left((0,T);L^{2^*}_{\bar{x},\frac{2^*}{p}\varepsilon}\right),\\
f^k(u^k)&\rightharpoonup^*\bar{f} && \text{in } L^\infty\left((0,T);L^{\frac{2^*}{p}}_{\bar{x},\frac{2^*}{p}\varepsilon}\right), && (3.16)\\
g(u^k)&\rightharpoonup\bar{g} && \text{in } L^{(\mu+1)/\mu}\left((0,T);L^{(\mu+1)/\mu}_{\bar{x},\varepsilon}\right), && (3.17)\\
u^k&\to u && \text{almost everywhere in } (0,T)\times\mathbb{R}^d. && (3.18)
\end{aligned}$$

Using supremum over $\bar{x}\in\mathbb{R}^d$ on both sides of (3.15) gives us (cf. Lemma 2.1)

$$\sup_{t\in(0,T)}\Big(\|u^k(t)\|^2_{W^{1,2}_b}+\|u_t^k(t)\|^2_{L^2_b}\Big)+\sup_{\bar{x}\in\mathbb{R}^d}\int_0^T\|u_t^k(t)\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}}\,dt\le C\tag{3.19}$$

where $C>0$ depends only on $\|(u_0,u_1)\|_{\Phi_b}$, $\|h\|_{L^2_b(0,\infty;L^2)}$ and $T>0$. Thus, using (3.18), (3.16) and assumptions on f^k together with the embedding

$$W^{1,2}_b\hookrightarrow L^{2^*}_b\hookrightarrow L^{2^*}_{\bar{x},\varepsilon},$$

we have $f^k(u^k)$ uniformly bounded in $L^\infty\left((0,T);L^{\frac{2^*}{p}}_{\bar{x},\varepsilon}\right)$, therefore

$$f^k(u^k)\to f(u)\quad\text{in } L^r\left((0,T);L^q_{\bar{x},\varepsilon}\right)\tag{3.20}$$

for any $q\in\left[1,\frac{2^*}{p}\right)$ and $r\in[1,\infty)$, hence $\bar{f}=f(u)$.

Step 3 - stability in $C([0,T];\Phi_{\bar{x},\varepsilon})$ and existence. Let us subtract the equation for u^l from the equation for u^k, multiply the difference by $(u_t^k-u_t^l)\phi_{\bar{x},\varepsilon}$ and integrate over $\mathbb{R}^d$ with respect to x. Using the monotonicity of g and standard estimates, we obtain

$$\|u^k(\tau)-u^l(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}}+\|u_t^k(\tau)-u_t^l(\tau)\|^2_{L^2_{\bar{x},\varepsilon}}\le\|u^k(0)-u^l(0)\|^2_{W^{1,2}_{\bar{x},\varepsilon}}+\|u_t^k(0)-u_t^l(0)\|^2_{L^2_{\bar{x},\varepsilon}}$$
$$+\|f^k(u^k)-f^l(u^l)\|_{L^{(\mu+1)/\mu}\left((0,\tau);L^{(\mu+1)/\mu}_{\bar{x},\varepsilon}\right)}\|u_t^k-v_t^l\|_{L^{\mu+1}\left((0,\tau);L^{\mu+1}_{\bar{x},\varepsilon}\right)}$$
$$+\|h^k-h^l\|^2_{L^2\left((0,\tau);L^2_{\bar{x},\varepsilon}\right)}+C\int_0^\tau\Big(\|u^k(t)-u^l(t)\|^2_{W^{1,2}_{\bar{x},\varepsilon}}+\|u_t^k(t)-u_t^l(t)\|^2_{L^2_{\bar{x},\varepsilon}}\Big)\,dt.$$

for every $\tau \in (0,T)$. From the Gronwall lemma, we infer that

$$\begin{aligned}\sup_{\tau\in(0,T)} &\|u^k(\tau) - u^l(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}} + \|u^k_t(\tau) - u^l_t(\tau)\|^2_{L^2_{\bar{x},\varepsilon}} \\ &\le C(T)\left(\|u^k(0) - u^l(0)\|^2_{W^{1,2}_{\bar{x},\varepsilon}} + \|u^k_t(0) - u^l_t(0)\|^2_{L^2_{\bar{x},\varepsilon}}\right) \\ &+ C(T)\|f^k(u^k) - f^l(u^l)\|_{L^{(\mu+1)/\mu}\left((0,\tau);L^{(\mu+1)/\mu}_{\bar{x},\varepsilon}\right)}\|u^k_t - v^l_t\|_{L^{\mu+1}\left((0,\tau);L^{\mu+1}_{\bar{x},\varepsilon}\right)}. \end{aligned} \tag{3.21}$$

Observe that (3.21) together with (3.12), (3.13), (3.17) and (3.20) gives

$$(u^k,u^k_t) \to (u,u_t) \quad \text{in } C([0,T];\Phi_{\bar{x},\varepsilon}). \tag{3.22}$$

Finally, we conclude that $u^k_t \to u_t$ almost everywhere in $(0,T) \times \mathbb{R}^d$ which in combination with (3.17) implies that $\bar{g} = g(u_t)$. Summing up the results on convergence given above and noting that u^k satisfies (3.5), we can pass to the limit in (3.8).

The energy equality (3.9) holds for u^k and using the convergence results above, in particular (3.22), it follows that it holds also for u. The relation (3.4) follows from Gronwall's lemma and taking supremum over $\bar{x} \in \mathbb{R}^d$ (see also estimates leading to (3.19)).

Step 4 - uniqueness of weak solutions. Let us test the weak formulation for u and v by $(u-v)_t$ keeping in mind that actually we test by $D_\tau[u-v]$ and sending $\tau \to 0$. Subtracting both equalities. we obtain the energy equality in the following form:

$$\begin{aligned}&\int_{\mathbb{R}^d} E(u(\tau) - v(\tau))\phi_{\bar{x},\varepsilon}\,dx + \int_0^\tau \int_{\mathbb{R}^d} (f(u) - f(v))\,(u_t - v_t)\phi_{\bar{x},\varepsilon}\,dx\,dt \\ &+ \int_0^\tau \int_{\mathbb{R}^d} (g(u_t) - g(v_t))\,(u_t - v_t)\phi_{\bar{x},\varepsilon}\,dx\,dt = \int_0^\tau \int_{\mathbb{R}^d} (\nabla u - \nabla v))\,(u_t - v_t)\nabla\phi_{\bar{x},\varepsilon}\,dx\,dt\end{aligned}$$

for any $\tau \in (0,T)$. Using assumptions (F2) with $p < 2^*/2$,(G1) and (3.4), we get

$$\int_{\mathbb{R}^d} E(u(\tau) - v(\tau))\phi_{\bar{x},\varepsilon}\,dx \le C\int_0^\tau \int_{\mathbb{R}^d} E(u(t) - v(t))\phi_{\bar{x},\varepsilon}\,dx\,dt. \tag{3.23}$$

Hence, $u(t) = v(t) \in \Phi_{\bar{x},\varepsilon}$ almost everywhere in $[0,T]$ as a consequence of Gronwall's lemma. □

Theorem 3.2. *The solution operator $S(T) : \Phi_b \to \Phi_b$ defined by*

$$S(T)(u_0,u_1) = (u(T),u_t(T))$$

where $(u(T),u_t(T))$ is the weak solution of (1.1) *with $(u(0),u_t(0)) = (u_0,u_1)$, is locally Lipschitz. Moreover, if $\mathcal{B} \subseteq \Phi_b$ is bounded, then $S(T) : (\mathcal{B}, \|\cdot\|_{\Phi_{\bar{x},\varepsilon}}) \to (\Phi_b, \|\cdot\|_{\Phi_{\bar{x},\varepsilon}})$ is Lipschitz.*

Proof. Assume that (u_0,u_1), $(v_0,v_1) \in \mathcal{B}$. Following the same line as in the proof of uniqueness, we obtain (3.23). Standard application of Gronwall's lemma gives

$$\|(u-v)(T)\|_{\Phi_{\bar{x},\varepsilon}} \le C(T,\mathcal{B})\|(u-v)(0)\|_{\Phi_{\bar{x},\varepsilon}}. \tag{3.24}$$

Finally, applying supremum over $\bar{x} \in \mathbb{R}^d$ on both sides of (3.24), from Lemma 2.1 we infer

$$\|(u-v)(T)\|_{\Phi_b} \le \tilde{C}(T,\mathcal{B})\|(u-v)(0)\|_{\Phi_b}.$$ □

4 Dissipation of energy

In contrast to the bounded domain case, the energy of the solutions does not necessarily decrease over time. This may be attributed to the last element in (3.6) and the absence of the embeddings between the weighted spaces of the same weight. Thus, it seems that an additional assumption has to be made in order to show any dissipation of energy. As we will see below, we can either have linearly bounded g and possibly superlinear function f, or we can ensure the dissipation by connecting the growths of the functions f and g. By a *dissipation assumption*, we understand one of the following:

(D1) $\mu = 1$,

(D2) $\mu \in (1,(d+2)/(d-2))$ and there exists $\kappa \in (0,1)$ and $C > 0$ such that

$$g(r)s \leq \kappa f(s)s + C\,(g(r)r + 1) \qquad \forall r, s \in \mathbb{R}.$$

The assumption (D1), i.e. linearly bounded damping, is well studied in the case of the bounded domain. The assumption (D2) is a variant of an assumption from [5] and allows for example the use of the functions

$$g(r) = r|r|^{\mu-1}, \quad f(s) = |s|^{p-1}s - as, \qquad \text{where } \mu \in [1,3) \text{ and } p \in [\mu, 3) \tag{4.1}$$

with $d = 3$ and $0 < a < \alpha$.

We emphasize that the upper entropy bound established the last section does not depend on the particular choice of the dissipation condition.

In the rest of this section we will also assume that for every $\varepsilon > 0$ there exists $C_h = C_h(\varepsilon) > 0$ such that

$$\sup_{\bar{x}\in\mathbb{R}^d} \int_{t_1}^{t_2} \|h(t)\|^2_{L^2_{\bar{x},\varepsilon}}\,dt \leq C_h(t_2 - t_1) \tag{4.2}$$

holds for all $0 \leq t_1 < t_2$. The estimate (4.2) automatically satisfied in an autonomous case discussed in the following sections.

Lemma 4.1. *Let (4.2) and either of the conditions (D1), (D2) hold. Then there exist $\varepsilon, \zeta > 0, C_0, C_1 > 0$ such that for every weak solution $(u(t), u'(t))$ with initial condition $(u_0, u_1) \in \mathcal{B}$ the estimate*

$$\int_{\mathbb{R}^d} F[u](T)\phi_{\bar{x},\varepsilon}\,dx \leq C_1 e^{-\zeta T} \int_{\mathbb{R}^d} F[u](0)\phi_{\bar{x},\varepsilon}\,dx + C_0 \tag{4.3}$$

holds for all $T > 0$.

Proof. Let $T > 0$ and $t_1, t_2 \in [0,T], t_1 < t_2$. We test the equation by $u_t + \delta u$, where $\delta > 0$ will be determined later. We obtain the equality

$$\begin{aligned}
&\int_{\mathbb{R}^d} F[u](t_2)\phi_{\bar{x},\varepsilon}\,dx + \delta\Big(u_t(t_2), u(t_2)\Big)_{\bar{x},\varepsilon} - \int_{\mathbb{R}^d} F[u](t_1)\phi_{\bar{x},\varepsilon}\,dx - \delta\Big(u_t(t_1), u(t_1)\Big)_{\bar{x},\varepsilon} \\
&\quad + \int_{t_1}^{t_2} \Big(g(u_t(t)), u_t(t)\Big)_{\bar{x},\varepsilon}\,dt - \delta\int_{t_1}^{t_2} \|u_t(t)\|^2_{\bar{x},\varepsilon}\,dt + \delta\int_{t_1}^{t_2} \Big(f_1(u(t)), u(t)\Big)_{\bar{x},\varepsilon}\,dt \\
&\quad + \delta\int_{t_1}^{t_2} \|\nabla u(t)\|^2_{\bar{x},\varepsilon} + \alpha\|u(t)\|^2_{\bar{x},\varepsilon}\,dt = \int_{t_1}^{t_2} \Big(h(t) - f_2(u(t)), u_t(t) + \delta u(t)\Big)_{\bar{x},\varepsilon}\,dt \\
&\quad - \delta\int_{t_1}^{t_2} \Big(g(u_t(t)), u(t)\Big)_{\bar{x},\varepsilon}\,dt - \int_{t_1}^{t_2} \Big(\nabla u(t), (u_t(t) + \delta u(t))\nabla\phi_{\bar{x},\varepsilon}\Big)\,dt.
\end{aligned} \tag{4.4}$$

For $\delta_1 \in (0,1)$, we use (3.3) to get

$$\int_{t_1}^{t_2} \Big(f_1(u(t)), u(t)\Big)_{\bar{x},\varepsilon}\,dt \geq \delta_1 \int_{t_1}^{t_2}\int_{\mathbb{R}^d} F_1(u(t))\phi_{\bar{x},\varepsilon}\,dx\,dt - \frac{\delta_1\beta}{2}\int_{t_1}^{t_2} \|u(t)\|^2_{\bar{x},\varepsilon}\,dt. \tag{4.5}$$

Also for $\delta_2 > 0$, by (4.2) we have

$$\int_{t_1}^{t_2} \Big(h - f_2(u(t)), u_t(t) + \delta u(t)\Big)_{\bar{x},\varepsilon}\,dt \leq \frac{1}{\delta_2}(t_2 - t_1)\left(C_h + \gamma_5^2 C_\varepsilon\right) + \delta_2\int_{t_1}^{t_2} \|u_t(t)\|^2_{\bar{x},\varepsilon}\,dt. \tag{4.6}$$

Other elementary estimates and (2.3) give

$$\begin{aligned}
&\int_{t_1}^{t_2} \Big(\nabla u(t), (u_t(t) + \delta u(t))\nabla\phi_{\bar{x},\varepsilon}\Big)\,dt \\
&\qquad\qquad \leq C\varepsilon\int_{t_1}^{t_2} \|\nabla u(t)\|^2_{\bar{x},\varepsilon} + \|u_t(t)\|^2_{\bar{x},\varepsilon} + \delta^2\|u(t)\|^2_{\bar{x},\varepsilon}\,dt
\end{aligned} \tag{4.7}$$

and

$$-\delta\delta_3\int_{t_1}^{t_2} \Big(u_t(t), u(t)\Big)_{\bar{x},\varepsilon}\,dt \geq -\frac{\delta\delta_3}{2}\left(\int_{t_1}^{t_2} \|u_t(t)\|^2_{\bar{x},\varepsilon}\,dt + \int_{t_1}^{t_2} \|u\|^2_{\bar{x},\varepsilon}\,dt\right). \tag{4.8}$$

Assume that (D1) holds. From (G3) we obtain

$$\int_{t_1}^{t_2} \Big(g(u_t(t)), u_t(t)\Big)_{\bar{x},\varepsilon} dt \geq \gamma_2 \int_{t_1}^{t_2} \|u_t(t)\|^2_{\bar{x},\varepsilon}\, dt - \gamma_3(t_2 - t_1)C_\varepsilon, \tag{4.9}$$

and

$$\begin{aligned} &- \delta \int_{t_1}^{t_2} \Big(g(u_t(t)), u(t)\Big)_{\bar{x},\varepsilon} dt \\ &\qquad \leq \frac{\gamma_4^2 \delta}{\alpha} \int_{t_1}^{t_2} \|u_t(t)\|^2_{\bar{x},\varepsilon}\, dt + \frac{\delta\alpha}{2} \int_{t_1}^{t_2} \|u(t)\|^2_{\bar{x},\varepsilon}\, dt + \frac{\gamma_4^2 \delta}{\alpha} C_\varepsilon (t_2 - t_1). \end{aligned} \tag{4.10}$$

The assertion the follows by inserting the estimates (4.5)–(4.10) into (4.4) and finishing the argument by choosing the constants $\delta_1, \delta, \delta_2, \varepsilon, \delta_3$ (possibly in this order) sufficiently small and by Gronwall's lemma applied to

$$\eta(t) = \int_{\mathbb{R}^d} F[u](t)\phi_{\bar{x},\varepsilon}\, dx + \delta\Big(u_t(t), u(t)\Big)_{\bar{x},\varepsilon}.$$

Under the assumption (D2), we have

$$\begin{aligned} &- \delta \int_{t_1}^{t_2} \Big(g(u_t(t)), u(t)\Big)_{\bar{x},\varepsilon} dt \\ &\qquad \leq \delta \left(\kappa \int_{t_1}^{t_2} \Big(f(u(t)), u(t)\Big)_{\bar{x},\varepsilon} dt + \int_{t_1}^{t_2} \Big(g(u_t(t)), u_t(t))\Big)_{\bar{x},\varepsilon} dt + CC_\varepsilon\right) \end{aligned}$$

and we use (3.2) to get

$$(1-\delta) \int_{t_1}^{t_2} \Big(g(u_t(t)), u_t(t)\Big)_{\bar{x},\varepsilon} dt \geq (1-\delta)\left(\gamma_2 \int_{t_1}^{t_2} \|u_t(t)\|^2_{\bar{x},\varepsilon}\, dt - \gamma_3 C_\varepsilon (t_2 - t_1)\right). \tag{4.11}$$

The conclusion is then reached similarly as in the case (D1). □

Theorem 4.2. *Let the assumptions of Lemma 4.1 hold. Then there exists a closed positively invariant absorbing set $\mathcal{B} \subseteq \Phi_b$ bounded in Φ_b.*

Proof. Let $\varepsilon, \zeta, C_0, C_1 > 0$ be as in Lemma 4.1. Using the embedding $W_b^{1,2}(\mathbb{R}^d) \hookrightarrow L_b^{p+1}(\mathbb{R}^d)$ and the equivalence of weighted and locally uniform norms in Lemma 2.1 we have

$$\int_{\mathbb{R}^d} F_1(u_0)\phi_{\bar{x},\varepsilon}\, dx \leq C\|u_0\|^{p+1}_{W_b^{1,2}(\mathbb{R}^d)} + CC_\varepsilon.$$

Inserting into (4.3) we obtain

$$\int_{\mathbb{R}^d} F[u](T)\phi_{\bar{x},\varepsilon}\, dx \leq e^{-\zeta T} Q\left(\|u_0\|_{W_b^{1,2}(\mathbb{R}^d)}, \|u_1\|_{L_b^2(\mathbb{R}^d)}\right) + C$$

which leads to

$$\sup_{\bar{x}\in\mathbb{R}^d} \int_{\mathbb{R}^d} E[u](T)\phi_{\bar{x},\varepsilon}\, dx \leq e^{-\zeta T} Q\left(\|u_0\|_{W_b^{1,2}(\mathbb{R}^d)}, \|u_1\|_{L_b^2(\mathbb{R}^d)}\right) + \tilde{C}$$

Set $\tilde{\mathcal{B}} = B(0, \tilde{C}) \subseteq \Phi_b$ and find $t_0 > 0$ such that $S(t)\tilde{\mathcal{B}} \subseteq \tilde{\mathcal{B}}$. We define

$$\mathcal{B} = \overline{\bigcup_{t\geq t_0} S(t)\tilde{\mathcal{B}}}^{\Phi_b}$$

and observe that $\mathcal{B}$ is positively invariant, cf. Theorem 3.2. □

5 Locally uniform squeezing property

In this section we introduce the trajectory setting and prove that the solution operator in the space of trajectories satisfies a local variant of so-called squeezing property (cf. [6]), which will in turn lead to the asymptotic compactness and an upper bound on Kolmogorov's ε-entropy. To achieve this, we require additional assumptions on μ and the damping nonlinearity g. We note that one can obtain the asymptotic compactness required for the existence of a locally compact attractor also without these additional assumptions by means of a standard decomposition argument.

From now on, let $h \equiv 0$ and for simplicity we assume $d = 3$. In addition, we require

$$\mu \in [1, 7/3), \qquad p \in [0, 3), \tag{5.1}$$

$$C(1+|r|)^{\mu-1} \le g'(r) \le C(1+|r|)^{\mu-1} \qquad \forall r \in \mathbb{R}. \tag{5.2}$$

These assumptions and the properties of f lead to the estimates

$$\begin{aligned}
|f(r) - f(s)| &\le C(1 + (|r| + |s|)^{p-1})|r - s|,\\
(g(r) - g(s))(r - s) &\ge C|r - s|^2,\\
(g(r) - g(s))(r - s) &\ge C \int_0^1 (1 + |tr + (1-t)s|^{\mu-1})|r - s|^2 \, dt,\\
|g(r) - g(s)| &\le C \int_0^1 (1 + |tr + (1-t)s|^{\mu-1})|r - s| \, dt,\\
|g(r) - g(s)| &\le (1 + (|r| + |s|)^{\mu-1})|r - s|.
\end{aligned} \tag{5.3}$$

Let $\ell > 1$ and $v > 1$ be fixed and let ϕ be an admissible weight function. We define the space of trajectories by

$$\mathcal{E}^{\ell,v}_{b,\phi} = \{(\chi, \chi_t) : \chi : (0, \ell) \times \mathbb{R}^3 \to \mathbb{R} \text{ and } \|u\|^2_{\mathcal{E}^{\ell,v}_{b,\phi}} := \sup_{k\in\mathbb{N}} \phi(x_k) \int_0^\ell \int_{Z_k(t)} \mathcal{E}[u] \, dx \, dt < \infty\},$$

$$\mathcal{B}_\ell = \{(\chi, \chi_t) \in \mathcal{E}^{\ell,v}_{b,\phi}; \chi \text{ solves the equation (1.1) in } [0, \ell] \text{ with } (\chi(0), \chi_t(0)) \in \mathcal{B}\}.$$

where we denote

$$\begin{aligned}
Z_k(t) &= B(x_k, v(2\ell - t)),\ t \in (0, 2\ell), & K(x_k) &= \{(t, x) \in (0, \ell) \times \mathbb{R}^d : x \in Z_k(t)\},\\
\widetilde{Z}_k(t) &= B(x_k, v(3\ell - t)),\ t \in (0, 3\ell), & \widetilde{K}(x_k) &= \{(t, x) \in (0, 2\ell) \times \mathbb{R}^d : x \in \widetilde{Z}_k(t)\}.
\end{aligned}$$

Note that the half-cone $\{(t, x) \in \widetilde{K}(0); 0 < t < \ell\}$ can be covered by a finite number of cones $K(x_j)$, $j \in \mathcal{N}$, $x_j \in B(0, 3v\ell)$. We emphasize that the size of $\mathcal{N}$ is independent of ℓ.

We define the operators $e : \mathcal{B}_\ell \to \Phi_b$ and $L(t) : \mathcal{B}_\ell \to \mathcal{B}_\ell$ by

$$\begin{aligned}
e((\chi, \chi_t)) &= (\chi(\ell), \chi_t(\ell)),\\
[L(t)(\chi, \chi_t)](s) &= S(t + s)(\chi(0), \chi_t(0)), \qquad s \in (0, \ell).
\end{aligned}$$

Let $\mathcal{O} \subseteq \mathbb{R}^3$ and let ϕ be an admissible weight function. We define

$$\begin{aligned}
\|u\|^2_{\Phi_{b,\phi}(\mathcal{O})} &= \sup_{k\in\mathbb{I}(\mathcal{O})} \phi(x_k) \int_{C_k} |u|^2 + |\nabla u|^2 + |u_t|^2 \, dx,\\
\|u\|^2_{\mathcal{E}^{\ell,v}_{b,\phi}(\mathcal{O})} &= \sup_{k\in\mathbb{I}(\mathcal{O})} \phi(x_k) \int_0^\ell \int_{Z_k(t)} E[u] \, dx \, dt.
\end{aligned}$$

Again, if $\phi \equiv 1$ we write $\Phi_b(\mathcal{O})$ instead of $\Phi_{b,1}(\mathcal{O})$.

Lemma 5.1. *Let $\ell, v > 1$ and ϕ be arbitrary weight function. The following holds:*

1. *$L = L(\ell) : \mathcal{B}_\ell \to \mathcal{B}_\ell$ is Lipschitz continuous,*
2. *$e : \mathcal{B}_\ell \to \Phi_b$ is Lipschitz continuous,*
3. *$\mathcal{B}_\ell$ is positively invariant under L, i.e. $L(t)\mathcal{B}_\ell \subseteq \mathcal{B}_\ell$ for every $t \ge 0$.*

Proof. The proof follows from the finite speed of propagation and is similar to [16, Lemma 2.1]. Let $\chi_1, \chi_2 \in \mathcal{B}_\ell$ and let u^1, u^2 be the respective solutions. Set $w = u^1 - u^2$ and let $0 < t_1 < t_2 < 2\ell$. We test the equation by w_t and using mollification and a similar argument as in the proof of existence of solutions, namely approximating by more regular data and stability, we get to the equation

$$\int_{\widetilde{Z}_k(t_2)} E[w](t_2)\,dx - \int_{\widetilde{Z}_k(t_1)} E[w](t)\,dx + \int_{t_1}^{t_2}\int_{\widetilde{Z}_k(t)} \big(g(u_t^1) - g(u_t^2)\big)\, w_t\,dx\,dt$$
$$= -\int_{t_1}^{t_2}\int_{\widetilde{Z}_k(t)} \big(f(u^1) - f(u^2)\big)\, w_t\,dx\,dt + \int_{t_1}^{t_2}\int_{\partial\widetilde{Z}_k(t)} w_t \nabla w\cdot n - vE[w]\,dS_x\,dt.$$

Since $v > 1$, the boundary integral is non-positive and using (G1) and a similar estimate on the first element on the right-hand side of the previous equation as in the proof of uniqueness, we arrive to

$$\int_{\widetilde{Z}_k(t_2)} E[w](t_2)\,dx \le \int_{\widetilde{Z}_k(t_1)} E[w](t)\,dx + C\int_{t_1}^{t_2}\int_{\widetilde{Z}_k(t)} E[w]\,dx\,dt.$$

Invoking Gronwall's lemma we get

$$\int_{\widetilde{Z}_k(t)} E[w](t)\,dx \le \Big(1 + C(t-s)e^{C(t-s)}\Big)\int_{\widetilde{Z}_k(s)} E[w](s)\,dx$$

for $0 < s < t < 2\ell$. Integrating from 0 to ℓ by s and from ℓ to 2ℓ by t leads to

$$\int_\ell^{2\ell}\int_{\widetilde{Z}_k(t)} E[w]\,dx\,dt \le C\sum_{j\in\mathcal{N}}\int_0^\ell\int_{x_k+Z_j(t)} E[w]\,dx\,dt.$$

We multiply the equation by $\phi(x_k)$ and use the property (2.1) to get

$$\phi(x_k)\int_0^\ell\int_{Z_k(t)} E[Lu^1 - Lu^2]\,dx\,dt \le C\#\mathcal{N}\max_{j\in\mathcal{N}}\phi(x_j)\int_0^\ell\int_{x_k+Z_j(t)} E[u^1-u^2]\,dx\,dt. \quad (5.4)$$

The Lipschitz continuity of L in $\mathcal{E}_{b,\phi}^{\ell,v}$ follows by taking supremum over $k \in \mathbb{N}$ and estimating the maximum on the right-hand side by the supremum over $j \in \mathbb{N}$. The Lipschitz continuity of e can be obtained in a similar manner.

The positive invariance of $\mathcal{B}_\ell$ follows immediately from the definitions. □

Definition. *We say that the mapping $L : \mathcal{B}_\ell \to \mathcal{B}_\ell$ has a* locally uniform squeezing property *(LUSP) for an admissible weight function ϕ if for every $\theta > 0$ there exists $\ell > 1$, $v > 1$, $\kappa > 0$ and $\mathcal{N} \subseteq \mathbb{N}$ such that $x_j \in B(0, 3v\ell) \subseteq \mathbb{R}^d$ for every $j \in \mathcal{N}$ and for every $k \in \mathbb{N}$ and $\chi_1, \chi_2 \in \mathcal{B}_\ell$ and the respective solutions u_1, u_2 we have*

$$\phi(x_k)\int_0^\ell\int_{Z_k(t)} E[Lu_1 - Lu_2]\,dx\,dt \le \theta\sum_{j\in\mathcal{N}(k)}\phi(x_j)\int_0^\ell\int_{Z_j(t)} E[u_1-u_2]\,dx\,dt$$
$$+\,\kappa\left(\phi(x_k)\int_0^\ell\int_{Z_k(t)} |Lu_1 - Lu_2|^2\,dx\,dt + \sum_{j\in\mathcal{N}(k)}\phi(x_j)\int_0^\ell\int_{Z_j(t)} |u_1-u_2|^2\,dx\,dt\right), \quad (5.5)$$

where

$$\mathcal{N}(k) = \{j \in \mathbb{N}; x_j = x_i + x_k \text{ for some } i \in \mathcal{N}\}.$$

The above definition contains a slight abuse of terminology as one has to first choose $\theta > 0$ and only then find suitable ℓ and v to get the squeezing property of $L = L(\ell) : \mathcal{B}_\ell \to \mathcal{B}_\ell$. However, this will not be of any concern in any of the later uses as $\theta > 0$ will be chosen only once.

Lemma 5.2. *The operator $L = L(\ell)$ has (LUSP) for every admissible weight function.*

Proof. The proof in similar to [16, Lemma 3.1]. Let us restrict ourselves to the case $\mu \in (1, 7/3)$ and $p \in (1, 3)$ since the remaining cases are similar or easier.

Let $\tau \in (0, \ell)$, $\chi_1, \chi_2 \in \mathcal{B}_\ell$ with the respective solutions u^1, u^2 and denote $w = u^1 - u^2$. Similarly as in the proof of Lemma 5.1 we get

$$\begin{aligned}\int_{\widetilde{Z}_k(2\ell)} E[w](2\ell)\,dx\,dt - \int_{\widetilde{Z}_k(\tau)} E[w](\tau)\,dx + \int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)} (g(u_t^1) - g(u_t^2))w_t\,dx\,dt \\ + \int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)} (f(u^1) - f(u^2))w_t\,dx\,dt = \int_\tau^{2\ell}\int_{\partial\widetilde{Z}_k(t)} w_t\nabla w\cdot n - vE[w]\,dS_x\,dt,\end{aligned} \tag{5.6}$$

$$\begin{aligned}\int_{\widetilde{Z}_k(2\ell)} ww_t\,dx\,dt + \int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)} |\nabla w|^2 + \alpha|w|^2\,dx\,dt + \int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)} (f(u^1) - f(u^2))w\,dx\,dt \\ = \int_{\widetilde{Z}_k(\tau)} ww_t\,dx + \int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)} |w_t|^2 - (g(u^1) - g(u^2))w\,dx\,dt \\ + \int_\tau^{2\ell}\int_{\partial\widetilde{Z}_k(t)} w\nabla w\cdot n - vw_tw\,dS_x\,dt\end{aligned} \tag{5.7}$$

Using the estimates (5.3) in (5.6) we have

$$\begin{aligned}\int_{\widetilde{Z}_k(2\ell)} E[w](2\ell) - \int_{\widetilde{Z}_k(\tau)} E[w](\tau)\,dx + C_1\int_\tau^{2\ell} \|w_t\|^2_{L^2(\widetilde{Z}_k(t))}\,dt + C_1\int_\tau^{2\ell} \mathcal{J}(t)\,dt \\ \le C\int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)} (1 + (|u^1| + |u^2|)^{p-1}|w||w_t|\,dx\,dt + \int_\tau^{2\ell}\int_{\partial\widetilde{Z}_k(t)} w_t\nabla w\cdot n - vE[w]\,dS_x\,dt,\end{aligned} \tag{5.8}$$

where

$$\mathcal{J}(t) = \int_{\widetilde{Z}_k(t)}\int_0^1 (1 + |su_t^1 + (1-s)u_t^2|^\mu)|w_t|^2\,ds\,dx.$$

We estimate the first element on the right-hand side using the dissipation of energy by

$$\begin{aligned}C\int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)} (1 + (|u^1| + |u^2|)^{p-1}|w||w_t|\,dx\,dt \\ \le C\int_\tau^{2\ell} \|1 + |u^1| + |u^2|\|^{p-1}_{L^{(p-1)r_1}(\widetilde{Z}_k(t))}\|w_t\|_{L^2(\widetilde{Z}_k(t))}\|w\|_{L^{r_2}(\widetilde{Z}_k(t))}\,dt \\ \le \int_\tau^{2\ell} C_1/2\|w_t\|^2_{L^2(\widetilde{Z}_k(t))} + C\|w\|^2_{L^{r_2}(\widetilde{Z}_k(t))}\,dt,\end{aligned} \tag{5.9}$$

where we put $r_1 = 6/(p-1)$ and $1/r_1 + 1/r_2 = 1/2$, therefore $r_2 \in (2, 6)$. Combining (5.8) and (5.9) we arrive to

$$\begin{aligned}\int_{\widetilde{Z}_k(2\ell)} \mathcal{E}[w](2\ell)\,dx - \int_{\widetilde{Z}_k(\tau)} \mathcal{E}[w](\tau)\,dx + \frac{C_1}{2}\int_\tau^{2\ell} \|w_t\|^2_{L^2(\widetilde{Z}_k(t))}\,dt + C_1\int_\tau^{2\ell} \mathcal{J}(t)\,dt \\ \le C\int_\tau^{2\ell} \|w\|^2_{L^{r_2}(\widetilde{Z}_k(t))}dt + \int_\tau^{2\ell}\int_{\partial\widetilde{Z}_k(t)} w_t\nabla w\cdot n - vE[w]\,dS_x\,dt.\end{aligned} \tag{5.10}$$

Returning to (5.7), by the estimates (5.3) we have

$$\begin{aligned}\int_\tau^{2\ell} \|\nabla w\|^2_{L^2(\widetilde{Z}_k(t))} + \alpha\|w\|^2_{L^2(\widetilde{Z}_k(t))}\,dt \\ \le \int_\tau^{2\ell} \|w_t\|^2_{L^2(\widetilde{Z}_k(t))}\,dt + C\int_{\widetilde{Z}_k(2\ell)} \mathcal{E}[w](2\ell)\,dx + C\int_{B_k(\tau)} \mathcal{E}[w](\tau)\,dx \\ + C\int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)}\int_0^1 (1 + |su_t^1 + (1-s)u_t^2|^\mu)|w_t||w|\phi_{\bar{x},\varepsilon}\,ds\,dx\,dt \\ + C\int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)} (1 + (|u^1| + |u^2|)^{p-1})|w|^2\phi_{\bar{x},\varepsilon}\,dx\,dt \\ + \int_\tau^{2\ell}\int_{\partial\widetilde{Z}_k(t)} w\nabla w\cdot n - vw_tw\,dS_x\,dt.\end{aligned} \tag{5.11}$$

Similarly as in (5.9) we estimate the fourth element on the right-hand side of (5.11) as

$$C\int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)}\int_0^1 (1+|su_t^1+(1-s)u_t^2|^\mu)|w_t||w|\phi_{\bar{x},\varepsilon}\,ds\,dx\,dt \\ \leq C\int_\tau^{2\ell}\mathcal{J}\,dt + C\int_\tau^{2\ell}\|w\|^2_{L^{2s_2}(\widetilde{Z}_k(t))}\,dt, \quad (5.12)$$

where we use the dissipation of energy and set $s_1 = 2/(\mu-1)$ and $1/s_1+1/s_2 = 1$, therefore $2s_2 \in (2,6)$. Similarly the fifth element (5.11) by

$$C\int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)}(1+(|u^1|+|u^2|)^{p-1})|w|^2\,dx\,dt \leq C\int_\tau^{2\ell}\|w\|^2_{L^{2z_2}(\widetilde{Z}_k(t))}\,dt, \quad (5.13)$$

where we again used the dissipation estimate and set $z_1 = 6/(p-1)$ and $1/z_1+1/z_2 = 1$, therefore $2z_2 \in (2,3)$. Set $s = \max(2s_2, 2z_2)$. Combining the estimates (5.11–5.13) we obtain

$$\int_\tau^{2\ell}\|\nabla w\|^2_{L^2(\widetilde{Z}_k(t))} + \alpha\|w\|^2_{L^2(\widetilde{Z}_k(t))}\,dt \\ \leq \int_\tau^{2\ell}\|w_t\|^2_{L^2(\widetilde{Z}_k(t))}\,dt + C\left(\int_{\widetilde{Z}_k(2\ell)}\mathcal{E}[w](2\ell)\,dx + \int_{\widetilde{Z}_k(\tau)}\mathcal{E}[w](\tau)\,dx\right) + C\int_\tau^{2\ell}\mathcal{J}\,dt \\ + C\int_\tau^{2\ell}\|w\|^2_{L^s(\widetilde{Z}_k(t))}\,dt + \int_\tau^{2\ell}\int_{\partial\widetilde{Z}_k(t)} w\nabla w\cdot n - vw_tw\,dS_x\,dt. \quad (5.14)$$

Define $r = \max(s, r_2)$. Multiply (5.14) by $\delta > 0$, add it to (5.10) and choose $v \geq (1+\delta)/(1-\delta)$ and $\delta > 0$ small enough to get

$$\zeta\int_\ell^{2\ell}\int_{\widetilde{Z}_k(t)}\mathcal{E}[w](t)\,dx\,dt \leq C\int_0^{2\ell}\|w\|^2_{L^r(\widetilde{Z}_k(t))}\,dt + 2\int_{\widetilde{Z}_k(\tau)}\mathcal{E}[w](\tau)\,dx$$

for some $\zeta > 0$ and integrate by τ from 0 to ℓ to obtain

$$\zeta\ell\int_\ell^{2\ell}\int_{\widetilde{Z}_k(t)}\mathcal{E}[w](t)\,dx\,dt \leq C\ell\int_0^{2\ell}\|w\|^2_{L^r(\widetilde{Z}_k(t))}\,dt + 2\int_0^\ell\int_{\widetilde{Z}_k(t)}\mathcal{E}[w](t)\phi_{\bar{x},\varepsilon}\,dx\,dt. \quad (5.15)$$

Now split the integral

$$\int_0^{2\ell}\|w\|^2_{L^r(\widetilde{Z}_k(t))}\,dt = \int_0^\ell\|w\|^2_{L^r(\widetilde{Z}_k(t))}\,dt + \int_\ell^{2\ell}\|w\|^2_{L^r(\widetilde{Z}_k(t))}\,dt$$

and divide the equation (5.15) by $\zeta\ell$. Next we employ Ehrling's lemma, namely

$$\|w\|^2_{L^r(\Omega)} \leq \gamma\|w\|^2_{W^{1,2}(\Omega)} + C\|w\|^2_{L^2(\Omega)}$$

for $\Omega = B(x,R) \subseteq \mathbb{R}^d$ with $x \in \mathbb{R}^d$, $R > 0$, $\gamma > 0$ arbitrary and $C = C(\gamma, R)$, on the arguments of the split integrals. Indeed, this is possible since the diameters of the domains in question, i.e. $\widetilde{Z}_k(t)$ for $t \in (0, 2\ell)$, are bounded. Combining these estimates with (5.15) we obtain

$$\left(1-\frac{C\gamma}{\zeta}\right)\int_0^\ell\int_{Z_k(t)}E[Lw]\,dx\,dt \leq \left(\frac{2}{\zeta\ell}+\frac{C\gamma}{\zeta}\right)\sum_{j\in\mathcal{N}(k)}\int_0^\ell\int_{Z_j(t)}E[w]\,dx\,dt \\ + \frac{C}{\zeta}\left(\int_0^\ell\int_{Z_k(t)}|Lw|^2\,dx\,dt + \sum_{j\in\mathcal{N}(k)}\int_0^\ell\int_{Z_j(t)}|w|^2\,dx\,dt\right), \quad (5.16)$$

where $\mathcal{N}(k) \subseteq \mathbb{N}$ is a finite set of size N such that the union of cones $K(x_j)$ over $j \in \mathcal{N}$ covers the cone $\widetilde{K}(x_k)$. Now let $\tilde{\theta} > 0$ be such that $\tilde{\theta}C_\phi\exp(\nu 3v\ell) < \theta$, where $\nu > 0$ is the growth of the admissible function ϕ. By choosing ℓ sufficiently large and γ sufficiently small we get

$$\int_0^\ell\int_{Z_k(t)}E[Lw]\,dx\,dt \leq \tilde{\theta}\sum_{j\in\mathcal{N}(k)}\int_0^\ell\int_{Z_j(t)}E[w]\,dx\,dt \\ + C\left(\int_0^\ell\int_{Z_k(t)}|Lw|^2\,dx\,dt + \sum_{j\in\mathcal{N}(k)}\int_0^\ell\int_{Z_j(t)}|w|^2\,dx\,dt\right). \quad (5.17)$$

It remains to insert the weight function with sufficiently small growth which is easily done by multiplying (5.17) by $\psi(x_k)$, invoking (2.1) and using the restriction on $\tilde{\theta}$. □

6 Locally compact attractor and entropy estimate

Let $\mathcal{O} \subseteq \mathbb{R}^d$. We denote

$$\mathbb{I}(\mathcal{O}) = \{k \in \mathbb{N}; C_k \cap \mathcal{O} \neq \emptyset\}.$$

Let M be a metric space and $K \subseteq M$ be relatively compact. Let $N_\varepsilon(K, M)$ denote the smallest number of balls of radii ε that cover K in M. We define the *Kolmogorov's ε-entropy* by

$$H_\varepsilon(K, M) = \ln N_\varepsilon(K, M).$$

A number of typical examples of upper and lower bounds on the Kolmogorov's ε-entropy in various situations can be found e.g. in [20].

The following lemma is crucial for the estimate of Kolmogorov's ε-entropy and considerably simplifies the proof of asymptotic compactness. We note that an estimate of this kind may be used to establish an infinite dimensional exponential attractor. We postpone this issue to a subsequent paper together with an abstract criterion and a applications to other equations.

Lemma 6.1. *Let $\mathcal{O} \subseteq \mathbb{R}^3$ be bounded and satisfy*

$$\#\mathbb{I}(\mathcal{O}) \leq C_0 \operatorname{vol}(\mathcal{O}). \tag{6.1}$$

Let $\varepsilon > 0$, $\delta \in (0, 1)$ and $(x_0, x_1) \in \mathcal{B}$. Also let ϕ be an admissible weight function. Then there exist $\ell, v > 1$ such that

$$H_{\delta\varepsilon}\left((LB)\big|_{\mathcal{O}}, \mathcal{E}^{\ell,v}_{b,\phi}(\mathcal{O})\right) \leq C_1 \operatorname{vol}(\mathcal{O}), \tag{6.2}$$

where $B = B_\lambda((\chi_0, \chi_1); \mathcal{E}^{\ell,v}_{b,\phi}) \cap \mathcal{B}_\ell$ is a ball centered around the ℓ-trajectory (χ_0, χ_1) starting from (x_0, x_1). The constant C_1 depends only on C_0, ℓ and δ and is independent of (x_0, x_1), ε and $\mathcal{O}$ as long as (6.1) is satisfied.

Proof. The proof adapts the techniques from [16, Lemma 4.1] and [9, Lemma 2.6], the main difference being working with hyperbolic trajectories space instead of parabolic ones.

Without loss of generality, assume that $0 \in \mathcal{N}$. First find $\ell, v > 1$ such that (5.5) holds for $\theta > 0$ such that $4\theta\#\mathcal{N} < \delta^2$ and fix $\lambda > 0$ such that

$$4\theta\#\mathcal{N} + \kappa\lambda^2(\#\mathcal{N} + \operatorname{Lip}(L)) < \delta^2$$

Let $k \in \mathbb{I}(\mathcal{O})$. Define

$$\begin{aligned} P(\chi, \chi_t) &= \left(\phi(x_j)\chi|_{K(x_j)}\right)_{j\in\mathcal{N}}, \qquad (\chi, \chi_t) \in \mathcal{B}_\ell, \\ X &= \{P(\chi, \chi_t);\ (\chi, \chi_t) \in B\}. \end{aligned}$$

We equip the space X with the norm

$$\|y\|_X^2 = \max_{j\in\mathcal{N}} \int_0^\ell \int_{Z_j(t)} |y_j|^2, \qquad y = (y_j)_{j\in\mathcal{N}} \in X.$$

Since $\mathcal{B}_\ell$ (and thus B) is uniformly bounded on every cone $K(x_i)$, $i \in \mathbb{N}$, by the Aubin-Lions lemma there exists $N \in \mathbb{N}$ and $(\chi^i, \chi^i_t) \in B$, $i = 1, \ldots, N$, such that

$$X \subseteq \bigcup_{i=1}^N B_{\lambda\varepsilon}\left(P(\chi^i, \chi^i_t); X\right).$$

It is important to note that N is independent of k and ε, which follows from the estimate

$$\|y\|_X^2 \leq C\varepsilon^2$$

holding uniformly for $\varepsilon > 0$ and $k \in \mathbb{I}(\mathcal{O})$.

Choose $(\chi,\chi_t)\in B$. Then $P(\chi,\chi_t)\in B_{\varepsilon\lambda}(P(\chi^i,\chi^i_t);X)$ for some $1\le i\le N$. Let u and u_i be the respective solution for χ and χ^i and let $w=u-u_i$. Using (LUSP) we may estimate

$$\begin{aligned}
\phi(x_k)&\int_0^\ell\int_{Z_k(t)} E[Lw]\,dx\,dt\\
&\le \theta\sum_{j\in\mathcal{N}(k)}\int_0^\ell\int_{Z_j(t)} E[w]\,dx\,dt+\kappa\sum_{j\in\mathcal{N}(k)}\phi(x_j)\int_0^\ell\int_{Z_j(t)}|w|^2\,dx\,dt\\
&\quad+\kappa\phi(x_k)\int_0^\ell\int_{Z_k(t))}|Lw|^2\,dx\,dt\\
&\le 4\theta\varepsilon^2\#\mathcal{N}+\kappa\varepsilon^2\lambda^2(\#\mathcal{N}+\mathrm{Lip}(L))<\delta^2\varepsilon^2,
\end{aligned}$$

therefore we have

$$H_{\delta\varepsilon}\left((LB)|_{C_k},\mathcal{E}^{\ell,v}_{b,\phi}(C_k)\right)\le\ln N$$

uniformly for every $k\in\mathbb{I}(\mathcal{O})$.

The final estimate follows directly from (6.1) since for covering in $\mathcal{E}^{\ell,v}_{b,\phi}(\mathcal{O})$, one needs to consider the product of all the coverings in $\mathcal{E}^{\ell,v}_{b,\phi}(C_k)$, $k\in\mathbb{I}(\mathcal{O})$. ☐

Proposition 6.2. *The dynamical system $(S(t),\Phi_b)$ is asymptotically compact in the local topology Φ_{loc}.*

Proof. Let $\{x_n\}\subseteq\Phi_b$ be bounded, let $t_n\to\infty$ and let $K\subseteq\mathbb{R}^d$ be compact. Without loss of generality we may assume $x_n\in\mathcal{B}$. Find $\ell,v>1$ such that (6.2) holds for $\phi\equiv1$ and $\theta=1/2$. Let $B\subseteq\mathbb{R}^d$ be a sufficiently large ball such that $K\subseteq B$ and $\mathcal{N}(k)\subseteq B$ for every $k\in\mathbb{I}(K)$.

Passing to a subsequence we may find $\chi_n\in\mathcal{B}_\ell$ such that $S(t_n)x_n=e(L^n\chi_n)$. Using (6.2) we are able to recurrently find a Cauchy subsequence $\{L^n\chi_n\}$ in $\mathcal{E}^{\ell,v}_{b,\phi}(B)$. The proof will be finished once we show that the sequence $e(L\chi_n)$ is Cauchy in $\Phi_b(K)$ and this immediately follows from (5.4) by taking supremum over $k\in\mathbb{I}(K)$. ☐

Using the dissipation of energy and the local asymptotic compactness we are able to show the existence of a locally compact attractor. The proof of the following theorem follows exactly as in [7] or [20] and will be omitted here.

Theorem 6.3. *There exists a unique set $\mathcal{A}\subseteq\Phi_b$ invariant under $S(t)$ and compact in Φ_{loc} such that $\mathcal{A}$ attracts sets bounded in Φ_b in the local topology Φ_{loc}, i.e. for every $B\subseteq\Phi_b$ bounded*

$$\lim_{t\to\infty}\mathrm{dist}_{\Phi_{loc}}(S(t)B,\mathcal{A})=0.$$

We denote

$$\mathcal{A}_\ell=\{\chi\in\mathcal{E}^{\ell,v}_{b,\phi};\chi\text{ solves the equation in }[0,\ell]\text{ with }(\chi(0),\chi_t(0))\in\mathcal{A}\}.$$

It is clear that $e(\mathcal{A}_\ell)=\mathcal{A}$ and $L(\mathcal{A}_\ell)=\mathcal{A}_\ell$.

Before we proceed to the entropy estimate, we define an auxiliary weight function in the spirit of [20]. Let $\bar{x}\in\mathbb{R}^d$, $R>0$ and $\nu>0$ be fixed. We define an auxiliary weight function $\psi(x_0,R)$ by

$$\psi(x_0,R)=\psi(x_0,R)(x)=\begin{cases}1, & |x-x_0|\le R+\sqrt{d},\\ \exp\left(\nu\left(R+\sqrt{d}-|x-x_0|\right)\right), & \text{otherwise.}\end{cases}$$

Clearly $\psi(x_0,R)$ is an admissible weight function with growth ν and one has

$$H_\varepsilon\left(\mathcal{A},\Phi_b(B(x_0,R))\right)\le H_\varepsilon\left(\mathcal{A},\Phi_{b,\psi(x_0,R)}(\mathbb{R}^d)\right). \tag{6.3}$$

The statement of the following lemma is formally the same as in [9]. However, we should keep in mind that we are working with a different trajectories norm, even if the proof of the lemma runs exactly in the same way as in the original proof.

Lemma 6.4 ([9, Lemma 5.4])**.** *For every $\varepsilon_0 > 0$ there exist $C > 0$ such that for every $x_0 \in \mathbb{R}^3$, $R \geq 1$, $\varepsilon \in (0, \varepsilon_0)$ and $\chi_1, \chi_2 \in \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}$ it holds that*

$$\|\chi_1 - \chi_2\|_{\mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}} \leq \max\left\{\|\chi_1 - \chi_2\|_{\mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}(B(x_0,R_\varepsilon))}, \varepsilon\right\},$$

where

$$R(\varepsilon) = R + C\left(1 + \ln\frac{1}{\varepsilon}\right). \tag{6.4}$$

Theorem 6.5. *There exists $C_0, C_1, \varepsilon_0 > 0$ such that for every $x_0 \in \mathbb{R}^3$, $R \geq 1$ and $\varepsilon \in (0, \varepsilon_0)$ one has the bound*

$$H_\varepsilon\left(\mathcal{A}|_{B(x_0,R)}, \Phi_b\left(B\left(x_0, R\right)\right)\right) \leq C_0\left(R + C_1 \ln\frac{1}{\varepsilon}\right)^3 \ln\frac{1}{\varepsilon}.$$

Proof. The proof uses a similar technique to [9, Theorem 5.1] and is standard. Let $\ell, v > 1$ and let $\psi(x_0, R)$ have sufficiently small growth such that Lemma 6.1 holds with $\delta = 1/2$ and for $\psi(x_0, R)$. By (6.3), the Lipschitz continuity of e shown in Lemma 5.1 and the fact that $C_k \subseteq B_k(\ell)$ allows us to estimate

$$H_\varepsilon\left(\mathcal{A}, \Phi_b\left(B\left(x_0, R\right)\right)\right) \leq H_\varepsilon\left(\mathcal{A}, \Phi_{b,\psi(x_0,R)}\right) \leq H_{\varepsilon/\operatorname{Lip}(e)}\left(\mathcal{A}_\ell, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\right).$$

We find $\varepsilon_0 > 0$ and $\chi \in \mathcal{A}_\ell$ such that $\mathcal{A}_\ell \subseteq B_{\varepsilon_0}(\chi; \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)})$, in other words

$$H_{\varepsilon_0}\left(\mathcal{A}_\ell, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\right) = 0.$$

The proof will be finished once we establish the bound

$$H_{\varepsilon_0 2^{-k}}\left(\mathcal{A}_\ell, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\right) \leq kC_0\left(R + C(1 + \ln\frac{2^k}{\varepsilon_0})\right)^3 \tag{6.5}$$

since then for $\varepsilon \in (0, \varepsilon_0)$ we find $k \in \mathbb{N}$ such that $2^{-k}\varepsilon_0 \leq \varepsilon < 2^{-k+1}\varepsilon_0$ and the desired entropy bound follows from $k < C \ln 1/\varepsilon$ holding for ε sufficiently small.

To prove the recurrent estimate (6.5) we use induction. Let first $k = 1$. Then from Lemma 6.1 we have

$$H_{\varepsilon_0/2}\left(\mathcal{A}_\ell|_{B(x_0,R(\varepsilon_0/2))}, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}(B(x_0, R(\varepsilon_0/2)))\right) \leq C_0\left(R + C(1 + \ln\frac{2^k}{\varepsilon_0})\right)^3.$$

By Lemma 6.4 the $\varepsilon_0/2$-covering in the space $\mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}(B(x_0, R(\varepsilon_0/2)))$ is also a $\varepsilon_0/2$-covering in $\mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}$.

Now let the bound (6.5) hold for $k > 1$, i.e.

$$\mathcal{A}_\ell \subseteq \bigcup_{i=1}^N B_{\varepsilon_0 2^{-k}}\left(\chi_i; \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\right) \tag{6.6}$$

for some $\chi_i \in \mathcal{A}_\ell$. Apply the mapping L to (6.6) to get

$$L(\mathcal{A}_\ell) = \mathcal{A}_\ell \subseteq \bigcup_{i=1}^N B_{\operatorname{Lip}(L)\varepsilon_0 2^{-k}}\left(L\chi_i; \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\right), \tag{6.7}$$

where we used the invariance of $\mathcal{A}_\ell$ under L from Lemma 5.1. By Lemma 6.1 each of the balls on the right-hand side of (6.7) can be covered by balls with radii $\varepsilon_0/2^{-(k+1)}$in $\mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}(B(x_0, R(\varepsilon_0 2^{-k+1})))$ so that

$$\begin{aligned}
&H_{\varepsilon_0 2^{-(k+1)}}\left(\mathcal{A}_\ell|_{B(x_0,R_0(\varepsilon_0/2^{-(k+1)}))}, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\left(B\left(x_0, R_0(\varepsilon_0/2^{-(k+1)})\right)\right)\right) \\
&\quad\leq H_{\varepsilon_0 2^{-k}}\left(\mathcal{A}_\ell, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\right) + C\left(R_0 + C(1 + \ln\frac{2^{k+1}}{\varepsilon_0})\right)^3 \\
&\quad\leq (k+1)C\left(R_0 + C(1 + \ln\frac{2^{k+1}}{\varepsilon_0})\right)^3.
\end{aligned}$$

The proof is finished by another use of Lemma 6.4 as in the step $k = 1$. $\square$

Acknowledgements

The research of J.S. was supported by the Charles University in Prague, project GA UK No. 200716. The research of M.M. leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

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