

MODULUS IN BANACH FUNCTION SPACES

VENDULA HONZLOVÁ EXNEROVÁ, JAN MALÝ, AND OLLI MARTIO

ABSTRACT. Moduli of path families are widely used to mark curves which may be neglected for some applications. We introduce ordinary and approximative modulus with respect to Banach function spaces. While these moduli lead to the same result in reflexive spaces, we show that there are important path families (like paths tangent to a given set) which can be labeled as negligible by the approximative modulus with respect to the Lorentz $L^{p,1}$ -space for an appropriate p , in particular, to the ordinary L^1 -space if $p = 1$, but not by the ordinary modulus with respect to the same space.

1. INTRODUCTION

For $p \geq 1$ the M_p -modulus of a curve family Γ in \mathbb{R}^n , $n \geq 1$, is defined as

$$M_p(\Gamma) = \inf \int_{\mathbb{R}^n} \rho^p dx$$

where the infimum is taken over all non-negative Borel functions ρ such that

$$\int_{\gamma} \rho ds \geq 1$$

for every curve $\gamma \in \Gamma$.

The M_p -modulus is used in the theory of function spaces. B. Fuglede [5] showed that a function u in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ is not only absolutely continuous on almost every line segment parallel to a coordinate axis but satisfies

$$(1) \quad |u(\gamma(b)) - u(\gamma(a))| \leq \int_{\gamma} |\nabla u| ds$$

for every curve $\gamma : [a, b] \rightarrow \mathbb{R}^n$ except for a family of M_p -modulus zero. In a metric measure space X , the inequality (1) has been taken as a defining property to create the Newtonian space $N^{1,p}(X)$, which has many properties similar to $W^{1,p}(\mathbb{R}^n)$, see [15], [6] and [2]. Due to the conformal invariance, the M_n -modulus has turned out to be a basic tool to study conformal and quasiconformal mappings in \mathbb{R}^n .

The approximative modulus, AM_p -modulus, is defined as

$$AM_p(\Gamma) = \inf_{(\rho_i)} \liminf_{i \rightarrow \infty} \int_{\mathbb{R}^n} \rho_i^p dx$$

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where the infimum is taken over all sequences (ρ_i) of Borel functions $\rho_i : \mathbb{R}^n \rightarrow [0, \infty]$ such that

$$\liminf_{i \rightarrow \infty} \int_{\gamma} \rho_i ds \geq 1$$

for every $\gamma \in \Gamma$. The AM_1 -modulus was introduced in [13] to study functions of bounded variation in \mathbb{R}^n and in metric measure spaces.

The purpose of the paper is to study an analog of the M_p - and AM_p -modulus in the more general framework of Banach function spaces than in the L^p -spaces. We introduce $M_{\mathcal{F}}$ modulus and $AM_{\mathcal{F}}$ modulus for an arbitrary Banach function space \mathcal{F} . The moduli in Lebesgue scale are obtained as $M_p = M_{L^p}^p$ and $AM_p = AM_{L^p}^p$.

Although we are mostly concerned in distinction between M and AM moduli, we first study when they are equal.

Theorem 1. *Let X be a metric measure space and \mathcal{F} be a reflexive Banach function space on X . Then*

$$(2) \quad M_{\mathcal{F}}(\Gamma) = AM_{\mathcal{F}}(\Gamma).$$

for every curve family Γ in X . In particular,

$$M_p(\Gamma) = AM_p(\Gamma) \quad \text{if } 1 < p < \infty.$$

To find situation in which M and AM moduli differ, we must turn our attention to nonreflexive spaces. Besides L^1 and L^∞ , the most important examples of nonreflexive Banach function spaces are the Lorentz spaces $L^{p,1}(\mathbb{R}^n)$ and their duals. Note that L^1 is just the limiting case $p = 1$, indeed $L^{1,1}(\mathbb{R}^n) = L^1(\mathbb{R}^n)$. We focus our attention to the Lorentz spaces $L^{p,1}(\mathbb{R}^n)$ rather than to their duals, as the spaces $L^{p,1}(\mathbb{R}^n)$ are intimately connected with Hausdorff measures, see [9], [10], [8]. We prove the following theorem:

Theorem 2. *Let $1 \leq p \leq n$. Then there exists a curve family Γ in \mathbb{R}^n such that*

$$(3) \quad AM_{L^{p,1}}(\Gamma) = 0 \text{ but } M_{L^{p,1}}(\Gamma) = \infty.$$

In particular, if $n \geq 1$, then there exists a curve family Γ in \mathbb{R}^n such that

$$(4) \quad AM_1(\Gamma) = 0 \text{ but } M_1(\Gamma) = \infty.$$

In the course of proofs of Theorem 2 and related results, we construct various curve families which are self-interesting. Let $E \subset \mathbb{R}^n$. We denote the family of all curves γ which meet E by $\Gamma(E)$. Already this family can distinguish between AM and M moduli. However, to construct examples in which $AM_{L^{k,1}}(\Gamma) = 0 < M_{L^{k,1}}(\Gamma)$, we need slightly more sophisticated, but important families:

Definition 1. Let X be a metric space and $E \subset X$. We define $\Gamma_i(E)$ as the family of all rectifiable curves γ which meet E infinitely times, this means that the set $\{t: \gamma(t) \in E\}$ is infinite.

Definition 2. If E is an $(n-k)$ -dimensional C^1 surface in \mathbb{R}^n , we say that γ is *right tangential* to E if there exists t such that $\gamma(t) \in E$ and $\gamma'_+(t)$ belongs to the tangent space $T_{\gamma(t)}(E)$. The family of all right tangential curves to E is denoted by $\Gamma_t(E)$.

Then we can state the following theorems:

Theorem 3. Let $1 \leq k \leq n-1$ and $E \subset \mathbb{R}^n$ be an $(n-k)$ -dimensional C^1 surface. Then

$$(5) \quad \begin{aligned} AM_{L^{k,1}}(\Gamma_t(E)) &= AM_{L^{k,1}}(\Gamma_i(E)) = 0, \\ \text{but } M_{L^{k,1}}(\Gamma_t(E)) &= M_{L^{k,1}}(\Gamma_i(E)) = \infty. \end{aligned}$$

In particular, if the dimension of E is $n-1$, then

$$(6) \quad \begin{aligned} AM_1(\Gamma_t(E)) &= AM_1(\Gamma_i(E)) = 0, \\ \text{but } AM_1(\Gamma_t(E)) &= AM_1(\Gamma_i(E)) = \infty. \end{aligned}$$

Note that estimate (6) has applications to the fine setting of the Stokes theorem, see [7].

We prove our results in a broader generality than indicated in Theorems above. Namely, we can formulate most results in the setting of metric measure spaces. Also, there are other versions of tangential behavior and, in particular, we present versions of tangential behavior in the setting of metric spaces.

2. PRELIMINARIES

First, we introduce Banach function spaces as in [1].

Definition 3. Let (X, ν) be a measure space, let \mathcal{M}_+ be the cone of all ν -measurable functions on X with values in $[0, \infty]$. A mapping $\Xi: \mathcal{M}_+ \rightarrow [0, \infty]$ is called a (*Banach*) *function norm* if, for all $f, g, f_n \in \mathcal{M}_+$, for all $a \geq 0$ and for all ν -measurable sets $E \subset X$, it holds

- (1) $\Xi(f) = 0 \Leftrightarrow f = 0$ ν -a. e.; $\Xi(af) = a\Xi(f)$; $\Xi(f+g) \leq \Xi(f) + \Xi(g)$;
- (2) $0 \leq g \leq f$ ν -a. e. $\Rightarrow \Xi(g) \leq \Xi(f)$;
- (3) $0 \leq f_n \uparrow f$ ν -a. e. $\Rightarrow \Xi(f_n) \uparrow \Xi(f)$;
- (4) $\nu(E) < \infty \Rightarrow \Xi(\chi_E) < \infty$;
- (5) $\nu(E) < \infty \Rightarrow \int_E f d\nu \leq C_E \Xi(f)$ for C_E depending on E and ν .

Definition 4. Let \mathcal{M} be the set of all extended scalar-valued ν -measurable functions on X . Let Ξ be a function norm. The collection \mathcal{F} of all functions in \mathcal{M} for which $\Xi(|f|) < \infty$ is called a *Banach function space*. We define a norm on \mathcal{F} by

$$\|f\|_{\mathcal{F}} = \Xi(|f|).$$

Important examples of Banach function spaces are provided by the scale of Lorentz spaces.

Definition 5. Let (X, ν) be a σ -finite measure space and $1 \leq p, q \leq \infty$. The Lorentz space $L^{p,q} = L^{p,q}(X, \nu)$ consists of all ν -measurable functions f with finite values ν -a.e. for which the quantity

$$\|f\|_{L^{p,q}} = \begin{cases} \left\{ \int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t} \right\}^{\frac{1}{q}}, & q \in [1, \infty), \\ \sup_{t \in (0, \infty)} \left\{ t^{\frac{1}{p}} f^*(t) \right\}, & q = \infty, \end{cases}$$

is finite. Here f^* is the decreasing rearrangement

$$f^*(t) = \inf \{s > 0 : \nu(\{x \in X : |f(x)| > s\}) \leq t\}.$$

We present there some basic properties of Lorentz spaces. For details, see [1, Chapter 4.4].

The expression $\|\cdot\|_{L^{p,q}}$ is a genuine norm (after identifying functions which are equal a.e.) if $1 \leq q \leq p$; for $1 < p < q$ it is equivalent to a genuine norm.

The spaces $L^{p,1}(X, \nu)$ and $L^{p',\infty}(X, \nu)$ are in duality, in particular, the Hölder type inequality

$$(7) \quad \int_X fg \, d\nu \leq \|f\|_{L^{p,1}} \|g\|_{L^{p',\infty}}$$

holds.

It is easy to compute that the $L^{p,q}$ -norm of a characteristic function of a measurable set E is $c\nu(E)^{1/p}$ for some constant $c = c(p, q)$, we need that $c(p, 1) = p$.

In typical situations (like in \mathbb{R}^n with the Lebesgue measure), the spaces $L^{p,q}(X, \nu)$ are reflexive if and only if $1 < p, q < \infty$; the reflexivity part follows from the characterization of the dual space in [1, Corollary 4.4.7], the nonreflexivity part can be derived from [1, Corollary 1.4.4] and [14, Theorem 9.5.]

The Lebesgue spaces $L^p(X, \nu)$ are included in the scale of Lorentz spaces as $L^p(X, \nu) = L^{p,p}(X, \nu)$, $p \in [1, \infty]$.

For $1 \leq p \leq \infty$ and $1 \leq q \leq s \leq \infty$, we have the embedding

$$\|f\|_{L^{p,s}} \leq C\|f\|_{L^{p,q}}, \quad f \in L^{p,q}(X, \nu).$$

Definition 6. In what follows, we restrict ourselves to rectifiable curves, which will be called paths. So, a *path* will be a non-constant Lipschitz continuous mapping $\gamma: [a, b] \rightarrow X$. Every path can be parametrized by its arclength and we assume that it is done so if not specified otherwise. The domain of γ will be $[0, \ell]$, where $\ell = \ell(\gamma)$ is the total length of γ .

Note that the curvilinear integral

$$\int_\gamma \rho \, ds = \int_0^{\ell(\gamma)} \rho(\gamma(t)) \, dt$$

is well defined whenever γ is a path and ρ is a non-negative Borel function on X .

Now, we can define a modulus and approximative modulus with respect to a Banach function space \mathcal{F} .

Definition 7. Let (X, ν) be a metric space with a Borel regular measure ν . Let \mathcal{F} be a Banach function space on X and Γ a family of paths in X . A Borel measurable function $\rho : X \rightarrow [0, \infty]$, is called *admissible* for Γ if $\int_{\gamma} \rho ds \geq 1$ for every $\gamma \in \Gamma$. Define the \mathcal{F} -modulus of Γ as

$$M_{\mathcal{F}}(\Gamma) = \inf\{\|\rho\|_{\mathcal{F}} : \rho \text{ is admissible for } \Gamma\}$$

where the infimum is taken over all admissible function for Γ .

Remark 1. In case of $\mathcal{F} = L^p(\mathbb{R}^n)$, $p > 1$, the \mathcal{F} -modulus and the standard p -modulus are almost the same – the only difference is that the p -modulus uses the p -th-power of the L^p -norm, i.e. $M_p(\Gamma) = (M_{L^p}(\Gamma))^p$. In particular, the zero families are the same. For $p = 1$ the moduli are obviously exactly the same.

Definition 8. Let (X, ν) be a metric space with a Borel regular measure ν . Let \mathcal{F} be a Banach function space and Γ a family of paths in X . A sequence of non-negative Borel functions $\rho_i : X \rightarrow [0, \infty]$ is *admissible* for Γ if

$$\liminf_{i \rightarrow \infty} \int_{\gamma} \rho_i ds \geq 1$$

for every $\gamma \in \Gamma$. The *approximative \mathcal{F} -modulus* of Γ is defined as

$$AM_{\mathcal{F}}(\Gamma) = \inf(\liminf_{i \rightarrow \infty} \|\rho_i\|_{\mathcal{F}})$$

where the infimum is taken over all admissible sequences (ρ_i) for Γ .

3. BASIC PROPERTIES

In this section, we show some basic properties of \mathcal{F} -modulus and approximative \mathcal{F} -modulus. The ideas are analogous to [5] and [13].

Remark 2. It holds that $AM_{\mathcal{F}}(\Gamma) \leq M_{\mathcal{F}}(\Gamma)$. It is easy to see this inequality because if ρ is admissible for Γ , then $\rho_i = \rho$, $i = 1, 2, \dots$, is admissible for Γ and

$$AM_{\mathcal{F}}(\Gamma) \leq \|\rho\|_{\mathcal{F}}.$$

Taking infimum over all admissible ρ , the assertion follows.

Theorem 4. *The \mathcal{F} -modulus is an outer measure on the set of paths in X , i. e.*

$$(8) \quad M_{\mathcal{F}}(\emptyset) = 0;$$

$$(9) \quad \Gamma_1 \subset \Gamma_2 \Rightarrow M_{\mathcal{F}}(\Gamma_1) \leq M_{\mathcal{F}}(\Gamma_2);$$

$$(10) \quad M_{\mathcal{F}}\left(\bigcup_{j=1}^{\infty} \Gamma_j\right) \leq \sum_{j=1}^{\infty} M_{\mathcal{F}}(\Gamma_j).$$

Proof. The property (8) is obvious since $\rho = 0$ is admissible for the empty set.

Similarly, (9) follows from the fact that each admissible function for Γ_2 is also admissible for Γ_1 .

The inequality (10) is a bit more complicated. Assume that

$$\sum_{j=1}^{\infty} M_{\mathcal{F}}(\Gamma_j) < \infty.$$

Fix $\varepsilon > 0$. For every j we can choose an admissible function ρ_j such that

$$\|\rho_j\|_{\mathcal{F}} \leq M_{\mathcal{F}}(\Gamma_j) + 2^{-j}\varepsilon.$$

Define $\rho(x) = \sup_j \rho_j(x)$. Obviously, as $\rho_j \leq \rho$, the function ρ is admissible for $\bigcup_{j=1}^{\infty} \Gamma_j$. Moreover, $\rho \leq \sum_{j=1}^{\infty} \rho_j$.

Then, using [1, Theorem 1.6],

$$\begin{aligned} M_{\mathcal{F}}\left(\bigcup_{j=1}^{\infty} \Gamma_j\right) &\leq \|\rho\|_{\mathcal{F}} \leq \left\| \sum_{j=1}^{\infty} \rho_j \right\|_{\mathcal{F}} \leq \sum_{j=1}^{\infty} \|\rho_j\|_{\mathcal{F}} \\ &\leq \sum_{j=1}^{\infty} (M_{\mathcal{F}}(\Gamma_j) + 2^{-j}\varepsilon) = \sum_{j=1}^{\infty} M_{\mathcal{F}}(\Gamma_j) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, (10) follows. \square

Theorem 5. *The approximative \mathcal{F} -modulus is an outer measure on the set of paths in X , i. e.*

$$(11) \quad AM_{\mathcal{F}}(\emptyset) = 0;$$

$$(12) \quad \Gamma_1 \subset \Gamma_2 \Rightarrow AM_{\mathcal{F}}(\Gamma_1) \leq AM_{\mathcal{F}}(\Gamma_2);$$

$$(13) \quad AM_{\mathcal{F}}\left(\bigcup_{j=1}^{\infty} \Gamma_j\right) \leq \sum_{j=1}^{\infty} AM_{\mathcal{F}}(\Gamma_j).$$

Proof. The assertions (11) and (12) are obvious as in the previous proof.

To prove (13), assume that $\sum_{j=1}^{\infty} AM_{\mathcal{F}}(\Gamma_j) < \infty$. Then fix $\varepsilon > 0$ and for each $j = 1, 2, \dots$ pick an admissible sequence $(\rho_i^j)_{i=1}^{\infty}$ for Γ_j such that

$$(14) \quad \|\rho_i^j\|_{\mathcal{F}} \leq AM_{\mathcal{F}}(\Gamma_j) + 2^{-j}\varepsilon$$

for every $i = 1, 2, \dots$.

Set $\rho_i = \sum_{j=1}^{\infty} \rho_i^j$. Then ρ_i is admissible for $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_j$ as if $\gamma \in \Gamma$, then there exists j_0 such that $\gamma \in \Gamma_{j_0}$ and

$$\liminf_{i \rightarrow \infty} \int_{\gamma} \rho_i ds = \liminf_{i \rightarrow \infty} \int_{\gamma} \sum_{j=1}^{\infty} \rho_i^j ds \geq \liminf_{i \rightarrow \infty} \int_{\gamma} \rho_i^{j_0} ds \geq 1.$$

Now, using [1, Theorem 1.6] we can estimate

$$\begin{aligned} AM_{\mathcal{F}}(\Gamma) &\leq \liminf_{i \rightarrow \infty} \|\rho_i\|_{\mathcal{F}} = \liminf_{i \rightarrow \infty} \left\| \sum_{j=1}^{\infty} \rho_i^j \right\|_{\mathcal{F}} \leq \liminf_{i \rightarrow \infty} \sum_{j=1}^{\infty} \|\rho_i^j\|_{\mathcal{F}} \leq \\ &\leq \sum_{j=1}^{\infty} (AM_{\mathcal{F}}(\Gamma_j) + 2^{-j}\varepsilon) = \sum_{j=1}^{\infty} AM_{\mathcal{F}}(\Gamma_j) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we obtain (13). \square

Definition 9. A set of paths Γ_2 is *minorised* by Γ_1 if for every path $\gamma \in \Gamma_2$ there exists a subpath of γ in Γ_1 . It is denoted as $\Gamma_1 < \Gamma_2$.

Proposition 1. If $\Gamma_1 < \Gamma_2$, then $M_{\mathcal{F}}(\Gamma_2) \leq M_{\mathcal{F}}(\Gamma_1)$.

Proposition 2. If $\Gamma_1 < \Gamma_2$, then $AM_{\mathcal{F}}(\Gamma_2) \leq AM_{\mathcal{F}}(\Gamma_1)$.

Proof. For both theorems, the proofs are easy and similar. If $\rho \in \mathcal{F}$ ($\rho_i \in \mathcal{F}$ respectively) is admissible for Γ_1 , it is admissible for Γ_2 too. Thus a set of admissible functions for Γ_2 is the same or larger then for Γ_1 and the infimum over a larger set is smaller or the same. \square

Proposition 3. Let Γ be a family of paths in X . Then $M_{\mathcal{F}}(\Gamma) = 0$ if and only if there is an admissible sequence (ρ_i) for Γ such that

$$(15) \quad \liminf_{i \rightarrow \infty} \|\rho_i\|_{\mathcal{F}} = 0.$$

Proof. If $M_{\mathcal{F}}(\Gamma) = 0$, then there exist admissible functions ρ_i such that $\|\rho_i\|_{\mathcal{F}} \leq \frac{1}{i}$ for each $i = 1, 2, \dots$ and this is the required sequence.

For the other direction, let (ρ_i) be as in (15). We can choose a subsequence (ω_i) of (ρ_i) such that $\|\omega_i\|_{\mathcal{F}} \leq 2^{-i-1}\varepsilon$ for $i = 1, 2, \dots$.

Now define $\bar{\rho} = \sum_{i=1}^{\infty} \omega_i$. Since (ω_i) is admissible for Γ , there exists $k \in \mathbb{N}$ such that for all $i \geq k$ it holds that $\int_{\gamma} \omega_i ds \geq \frac{1}{2}$. We can use the Lebesgue monotone convergence theorem to infer that

$$\int_{\gamma} \bar{\rho} ds = \sum_{i=1}^{\infty} \int_{\gamma} \omega_i ds = \infty > 1$$

and thus $\bar{\rho}$ is admissible for Γ .

Using again [1, Theorem 1.6], we get

$$M_{\mathcal{F}}(\Gamma) \leq \|\bar{\rho}\|_{\mathcal{F}} \leq \sum_{i=1}^{\infty} \|\omega_i\|_{\mathcal{F}} \leq \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, $M_{\mathcal{F}}(\Gamma)$ is zero as desired. \square

Remark 3. The zero set of $AM_{\mathcal{F}}$ -modulus can be a set of positive $M_{\mathcal{F}}$ -modulus only if (15) fails for any admissible sequence (ρ_i) , i.e. there exists no minimizing sequence (ρ_i) .

The following lemma is a version of Fuglede's theorem [5, Theorem 3(f)] adapted to the setting of Banach functions spaces.

Lemma 1. *If a sequence $\rho_j \in \mathcal{F}$ converges strongly in \mathcal{F} to ρ , then there is a subsequence ρ_j such that for almost every path*

$$\lim_{j \rightarrow \infty} \int_{\gamma} |\rho - \rho_j| ds = 0$$

except for a set of paths of a zero \mathcal{F} -modulus.

Remark 4. Since the approximative \mathcal{F} -modulus is smaller, the lemma holds for approximative \mathcal{F} -modulus, too.

Proof. We can choose a subsequence ρ_j such that $\|\rho - \rho_j\|_{\mathcal{F}} < 2^{-2j}$. Denote

$$\Gamma_j = \left\{ \gamma : \int_{\gamma} |\rho - \rho_j| ds > 2^{-j} \right\},$$

$$\Psi_k = \bigcup_{j>k} \Gamma_j, \quad \Psi = \bigcap_k \Psi_k.$$

The function $2^j |\rho - \rho_j|$ is admissible for Γ_j , thus

$$M_{\mathcal{F}}(\Gamma_j) \leq 2^j \|\rho - \rho_j\|_{\mathcal{F}} < 2^{-j}.$$

By Theorem 4, for every $k \in \mathbb{N}$, we have

$$M_{\mathcal{F}}(\Psi) \leq M_{\mathcal{F}}(\Psi_k) \leq \sum_{j>k} M_{\mathcal{F}}(\Gamma_j) < 2^{-k}.$$

Since the previous inequality holds for every k , $M_{\mathcal{F}}(\Psi) = 0$. For every $\gamma \notin \Psi$ there exists an index k such that $\gamma \notin \Psi_k$, i.e. $\int_{\gamma} |\rho - \rho_j| ds \leq 2^{-j}$ for every $j > k$. Hence

$$\lim_{j \rightarrow \infty} \int_{\gamma} |\rho - \rho_j| ds = 0.$$

□

The next theorem is also due to Fuglede [5, Theorem 2]. We omit the proof; it is easy to observe that Fuglede's argument holds also in the setting of Banach function spaces. The idea is also clear from the (more complicated) proof of Theorem 7.

Theorem 6. *Let Γ be a family of paths in X . Then $M_{\mathcal{F}}(\Gamma) = 0$ if and only if there is a non-negative Borel function $\rho \in \mathcal{F}(X)$ such that*

$$(16) \quad \int_{\gamma} \rho ds = \infty$$

for each $\gamma \in \Gamma$.

For the $AM_{\mathcal{F}}$ -modulus, there is also a corresponding result.

Theorem 7. *Let Γ be a family of paths in X . Then $AM_{\mathcal{F}}(\Gamma) = 0$ if and only if there is a sequence $(\tilde{\rho}_i)$ of non-negative Borel functions $\tilde{\rho}_i$ such that*

$$(17) \quad \liminf_{i \rightarrow \infty} \|\tilde{\rho}_i\|_{\mathcal{F}} < \infty$$

and for each $\gamma \in \Gamma$

$$(18) \quad \lim_{i \rightarrow \infty} \int_{\gamma} \tilde{\rho}_i ds = \infty.$$

Proof. The conditions (17) and (18) are clearly sufficient to show that $AM_{\mathcal{F}}(\Gamma) = 0$. For the converse, suppose that $AM_{\mathcal{F}}(\Gamma) = 0$ and let $\varepsilon > 0$. For each $j \in \mathbf{N}$, we can choose a sequence $(\rho_i^j)_i$ such that $(\rho_i^j)_i$ is admissible for Γ and

$$(19) \quad \|\rho_i^j\|_{\mathcal{F}} \leq 2^{-j}$$

for each i and j . For every i set

$$\tilde{\rho}_i = \sum_{j=1}^{\infty} \rho_i^j$$

Then [1, Theorem 1.6] and (19) yield

$$\|\tilde{\rho}_i\|_{\mathcal{F}} \leq \sum_{j=1}^{\infty} \|\rho_i^j\|_{\mathcal{F}} \leq \sum_{j=1}^{\infty} 2^{-j} = 1$$

and this gives (17).

It remains to show that $(\tilde{\rho}_i)$ satisfies (18). Fix a path $\gamma \in \Gamma$ and let $k \in \mathbf{N}$. Since each sequence $(\rho_i^j)_i$ is admissible for Γ , there is i_0 such that for $i \geq i_0$

$$\int_{\gamma} \rho_i^j ds > 1/2$$

for $j = 1, 2, \dots, k$. Now

$$\int_{\gamma} \tilde{\rho}_i ds \geq \sum_{j=1}^k \int_{\gamma} \rho_i^j ds \geq k/2$$

for $i \geq i_0$ and hence

$$\liminf_{i \rightarrow \infty} \int_{\gamma} \tilde{\rho}_i ds \geq k/2.$$

Letting $k \rightarrow \infty$ we obtain

$$\lim_{i \rightarrow \infty} \int_{\gamma} \tilde{\rho}_i ds = \infty$$

as required. \square

Remark 5. The left hand part of (17) can be made arbitrarily small.

4. TRUNCATION

Here we consider a phenomenon which clarifies the difference between the $M_{\mathcal{F}}$ - and $AM_{\mathcal{F}}$ -modulus.

Let Γ be a family of paths in X . We say that a family Γ_T is a *truncated family associated with* Γ if $\Gamma \subset \Gamma_T$ and for each $\gamma \in \Gamma$, Γ_T contains some family of arbitrary short subpaths of γ . Note that there are many truncated families associated with Γ .

Theorem 8. *If Γ is a family of paths in X with $M_{\mathcal{F}}(\Gamma) > 0$, then $M_{\mathcal{F}}(\Gamma_T) = \infty$ for every truncated family Γ_T associated with Γ .*

Proof. Assume that $M_{\mathcal{F}}(\Gamma) < \infty$. Let $\rho \in \mathcal{F}(X)$ be admissible for Γ_T . Then ρ is admissible for Γ as well and hence there is a path $\gamma \in \Gamma$ such that

$$(20) \quad \int_{\gamma} \rho ds < \infty$$

because if

$$\int_{\gamma} \rho ds = \infty$$

for every $\gamma \in \Gamma$, then $M_{\mathcal{F}}(\Gamma) = 0$ by the Fuglede Theorem 6 which is a contradiction. Now the absolute continuity of integral shows that ρ is not admissible for Γ_T since the path γ in (20) contains paths of arbitrary small length and we can find a subpath $\tilde{\gamma}$ of γ such that $\int_{\tilde{\gamma}} \rho ds < 1$. The theorem follows. \square

Remark 6. Note that Theorem 8 is not true for the $AM_{\mathcal{F}}$ -modulus in general. This is an important difference between these two concepts.

Remark 7. As the following section shows, the $M_{\mathcal{F}}$ - and $AM_{\mathcal{F}}$ -modulus are the same for every reflexive space \mathcal{F} , and thus the property of Theorem 8 transmits to the $AM_{\mathcal{F}}$ modulus in reflexive spaces. In particular, it holds for the AM_p -modulus, $1 < p < \infty$.

5. EQUIVALENCE OF MODULI IN REFLEXIVE SPACES

In this section, we will prove that \mathcal{F} -modulus and approximative \mathcal{F} -modulus are the same if the space \mathcal{F} is reflexive. In particular, Theorem 1 shows that M_p -modulus and AM_p -modulus are the same for $1 < p < \infty$.

Proof of Theorem 1. Since $AM_{\mathcal{F}}(\Gamma) \leq M_{\mathcal{F}}(\Gamma)$ by Remark 2, it is enough to prove the reverse inequality. \blacksquare

We may assume that $AM_{\mathcal{F}}(\Gamma) < \infty$.

Let $\delta > 0$. Find an admissible sequence (ρ_i) for Γ such that

$$(21) \quad \liminf_{i \rightarrow \infty} \|\rho_i\|_{\mathcal{F}} \leq AM_{\mathcal{F}}(\Gamma) + \delta.$$

Since the space \mathcal{F} is reflexive and the sequence (ρ_i) is bounded, by [16, Section V.2, Theorem 1], there exist $\rho \in \mathcal{F}$ and a subsequence of (ρ_i) ,

denoted again by (ρ_i) , such that $\rho_i \rightarrow \rho$ weakly in \mathcal{F} and satisfies (21). This subsequence is still admissible for Γ . Since $\rho_i \rightarrow \rho$ weakly in \mathcal{F} , by the Mazur lemma [16, Section V.1, Theorem 2], for each $k \in \mathbb{N}$, there exists a convex combination ν_k of $\rho_k, \rho_{k+1}, \dots$ such that

$$\|\nu_k - \rho\|_{\mathcal{F}} < 1/k.$$

By Lemma 1, a subsequence, denoted again by (ν_k) , satisfies

$$\int_{\gamma} \nu_k ds \rightarrow \int_{\gamma} \rho ds$$

for every $\gamma \in \bar{\Gamma} \subset \Gamma$, where $M_{\mathcal{F}}(\Gamma \setminus \bar{\Gamma}) = 0$. Let $\gamma \in \bar{\Gamma}$. Then by convexity,

$$\int_{\gamma} \nu_k ds \geq \inf_{i \geq k} \int_{\gamma} \rho_i ds, \quad k = 1, 2, \dots,$$

so that

$$\int_{\gamma} \rho ds = \lim_{k \rightarrow \infty} \int_{\gamma} \nu_k ds \geq \liminf_{i \rightarrow \infty} \int_{\gamma} \rho_i ds = 1.$$

It follows that ρ is admissible for $\bar{\Gamma}$. Then by Fatou Lemma [1, Lemma 1.5],

$$M_{\mathcal{F}}(\bar{\Gamma}) \leq \|\rho\|_{\mathcal{F}} \leq \liminf_{i \rightarrow \infty} \|\rho_i\|_{\mathcal{F}} \leq AM_{\mathcal{F}}(\Gamma) + \delta.$$

Letting $\delta \rightarrow 0$, we obtain $M_{\mathcal{F}}(\bar{\Gamma}) \leq AM_{\mathcal{F}}(\Gamma)$. Since $M_{\mathcal{F}}(\Gamma \setminus \bar{\Gamma}) = 0$, we conclude that

$$M_{\mathcal{F}}(\Gamma) \leq AM_{\mathcal{F}}(\Gamma).$$

□

6. ESTIMATES OF APPROXIMATIVE MODULUS

In this section we assume that the measure ν is doubling.

Definition 10. For $0 \leq q$ let

$$co\underline{M}^q(E) = \liminf_{t \rightarrow 0} \frac{\nu(\{x \in X : d(x, E) < t\})}{t^q}$$

denote the lower Minkowski content of codimension q of the set $E \subset X$. By $coH^q(E)$ we denote the Hausdorff measure of codimension q of E defined as

$$coH^q(E) = \sup_{\delta > 0} coH_{\delta}^q(E)$$

where for $\delta > 0$

$$coH_{\delta}^q(E) = \inf \left\{ \sum_{j=1}^{\infty} \frac{\nu(B(x_j, r_j))}{r_j^q} : E \subset \bigcup_{j=1}^{\infty} B(x_j, r_j), \sup_j r_j < \delta \right\}$$

is the δ -content associated with $coH^q(E)$.

It easily follows from the 5-covering lemma, see e.g. [2, Lemma 1.7], that $coH^q(E) \leq c_0 co\underline{M}^q(E)$ where c_0 depends only on the doubling constant of ν .

Definition 11. We say that a path $\gamma : [0, \ell(\gamma)] \rightarrow X$ has a *meeting of order* $\alpha > 0$ with a set $E \subset X$ if

$$\lim_{\delta \rightarrow 0} \frac{|\{s \in [0, \ell(\gamma)] : d(\gamma(s), E) < \delta\}|}{\delta^{1/\alpha}} = \infty.$$

Here $|A|$ stands for the Lebesgue measure of $A \subset \mathbf{R}$.

Theorem 9. *Suppose that $\text{co}\underline{M}^q(E) < \infty$ and that Γ is a family of paths in X such that each $\gamma \in \Gamma$ has a meeting of order α with E . Then $AM_{L^{p,1}}(\Gamma) = 0$ provided that $1 \leq p \leq \alpha q$.*

Proof. Pick first a sequence $1 \geq \delta_1 > \delta_2 > \dots$ such that $\lim_{i \rightarrow \infty} \delta_i = 0$ and for some $M < \infty$

$$\frac{\nu(\{x \in X : d(x, E) < \delta_i\})}{\delta_i^q} \leq M.$$

Let $\varepsilon > 0$ and for $i = 1, 2, \dots$ let

$$\rho_i = \varepsilon \delta_i^{-1/\alpha} \chi_{E(\delta_i)}$$

where $E(t) = \{x \in X : d(x, E) < t\}$. Then

$$\|\rho_i\|_{L^{p,1}(X)} = \varepsilon \delta_i^{-1/\alpha} p (\nu(E(\delta_i)))^{1/p}.$$

We show that the sequence (ρ_i) is admissible for Γ .

To this end, fix $\gamma \in \Gamma$. Since γ has a meeting of order α with E , there is i_0 such that

$$\varepsilon |\{s \in [0, \ell(\gamma)] : d(\gamma(s), E) < \delta_i\}| \geq \delta_i^{1/\alpha}.$$

for $i \geq i_0$. Now for $i \geq i_0$

$$\int_{\gamma} \rho_i ds = \varepsilon \delta_i^{-1/\alpha} |\{s \in [0, \ell(\gamma)] : d(\gamma(s), E) < \delta_i\}| \geq 1,$$

and so the sequence (ρ_i) is admissible for Γ .

Since $p/\alpha \leq q$ and since (ρ_i) is admissible for Γ , we can use the sequence (ρ_i) to obtain

$$\begin{aligned} AM_{L^{p,1}}(\Gamma) &\leq \liminf_{i \rightarrow \infty} \|\rho_i\|_{L^{p,1}(X)} \leq \varepsilon \liminf_{i \rightarrow \infty} \frac{p \nu(E(\delta_i))^{1/p}}{\delta_i^{1/\alpha}} \\ &\leq \varepsilon \liminf_{i \rightarrow \infty} \left(\frac{\nu(E(\delta_i))}{\delta_i^q} \right)^{1/p} \leq M^{1/p} \varepsilon. \end{aligned}$$

Since M is independent of ε , we have $AM_{L^{p,1}}(\Gamma) = 0$ as required. \square

Remark 8. The above theorem is not true if the the lower Minkowski content of codimension q is replaced by the Hausdorff measure of codimension q . Easy examples can be constructed by adding a countable set to the set E . This has no effect on $\text{co}H^q(E)$ but the meeting property holds for a much larger family Γ of paths than before and so the condition $AM_p(\Gamma) = 0$ need not hold for $p \leq \alpha q$.

Corollary 1. *Let $E \subset \mathbb{R}^n$ be an $(n-k)$ -dimensional C^1 surface, $k \in \{1, 2, \dots, n\}$, and Γ be a family of paths such that each $\gamma \in \Gamma$ has a meeting of order 1 with E . Then $AM_{L^{k,1}}(\Gamma) = 0$.*

Proof. Obviously, for each compact part K of E , $co\underline{M}^k(K) < \infty$. Now, we can use the countable subadditivity of the approximative modulus (Theorem 5). \square

Proposition 4. *Let $E \subset X$ and $\Gamma_i(E)$ be as in Definition 1. Then each $\gamma \in \Gamma_i(E)$ has a meeting of order 1 with E .*

Proof. Choose $m \in \mathbb{N}$ and pick distinct points $t_1, \dots, t_m \in (0, \ell(\gamma))$ such that $\gamma(t_i) \in E$, $i = 1, \dots, m$. Find $\delta > 0$ such that the intervals $(t_i - \delta, t_i + \delta)$, $i = 1, \dots, m$, are pairwise disjoint and contained in $(0, \ell(\gamma))$. Let $0 < s < \delta$. Then for each $i = 1, \dots, m$ and $t \in (t_i - s, t_i + s)$ we have $d(\gamma(t), E) < s$ due to the 1-Lipschitz property of γ . Hence

$$|\{t \in [0, \ell(\gamma)]: d(\gamma(t), E) < s\}| \geq 2ms.$$

Letting $m \rightarrow \infty$ we obtain the assertion. \square

Corollary 2. *Let $E \subset X$ and $co\underline{M}^p(E) < \infty$. Then $AM_{L^{p,1}}(\Gamma_i(E)) = 0$.*

Remark 9. If we replace the assumption $co\underline{M}^p(E) < \infty$ by $coH^p(E) < \infty$, it can be proved that $AM_{L^p}(\Gamma_i(E)) = 0$, see [7]. Thus, for $p = 1$ the conclusion of Corollary 2 can be obtained under a weaker assumption, whereas for $p > 1$ the conclusion is also weaker (Lebesgue spaces instead of Lorentz spaces).

Definition 12. For $E \subset X$ we define

$$\Gamma_\tau(E) = \left\{ \gamma \in \Gamma(E) : \lim_{t \rightarrow 0_+} \frac{d(\gamma(t), E)}{t} = 0 \right\}.$$

Remark 10. If γ is right tangential in the sense of Definition 2, then there is a subpath of γ in $\Gamma_\tau(E)$.

Proposition 5. *Let $E \subset X$ and $\gamma \in \Gamma_\tau(E)$. Then γ has a meeting of order 1 with E .*

Proof. Choose $m \in \mathbb{N}$ and find $\delta > 0$ such that

$$0 < t < \delta \implies d(\gamma(t), E) \leq \frac{t}{m}.$$

Let $s < \frac{\delta}{m}$. Then for each $t \in (0, ms)$ we have $t < \delta$ and $d(\gamma(t), E) \leq s$. Thus,

$$|\{t \in [0, \ell(\gamma)]: d(\gamma(t), E) < s\}| \geq ms.$$

Letting $m \rightarrow \infty$ we obtain the assertion. \square

The converse of Proposition 5 is not true. For example, consider the set $E = (-\infty, \infty) \times \{0\}$ in \mathbf{R}^2 and the curve γ with locus

$$[0, 1] \times \{0\} \cup \bigcup_{i=1}^{\infty} \{2^{-i}\} \times [0, 2^{-i}].$$

Corollary 3. *Let $\text{co}\underline{M}^q(E) < \infty$. Then $AM_{L^{p,1}}(\Gamma_\tau(E)) = 0$ provided that $1 \leq p \leq q$.*

Corollary 4. *Let $k \in \{1, \dots, n-1\}$ and E be an $(n-k)$ -dimensional C^1 surface in \mathbb{R}^n . Then*

$$AM_{L^{k,1}}(\Gamma_i(E)) = AM_{L^{k,1}}(\Gamma_t(E)) = AM_{L^{k,1}}(\Gamma_\tau(E)) = 0.$$

7. DENSITY-TANGENTIAL CURVES

In this section, X will be a metric space with a doubling measure ν . We derive an estimate which will be needed in applications (see [7]). We are able to handle sets of finite Hausdorff measure, but the result does not seem to extend easily to Lorentz spaces.

Definition 13. Let μ be a finite Borel measure on X , $x \in X$ and $\tau > 0$. We denote

$$R_\tau(x, \mu) = \inf\{r > 0: r\mu(B(x, r)) \geq \tau\nu(B(x, r))\}.$$

We say that γ is τ -density tangential to μ if

$$\lim_{t \rightarrow 0^+} \frac{R_\tau(\gamma(t), \mu)}{d(\gamma(t), \gamma(0))} = 0.$$

Remark 11. Typically μ is the Hausdorff measure of codimension 1 restricted to some set E .

Theorem 10. *Let $\Gamma_{\mu,\tau}$ be the family of all paths which are τ -density tangential to μ . Then $AM_1(\Gamma_{\mu,\tau}) = 0$.*

Proof. Consider the family of all balls $B(x, r)$ with the property that $r\mu(B(x, r)) \geq \tau\nu(B(x, r))$ and $0 < r \leq \frac{1}{k}$. Using the Vitali type theorem, there exists a disjointed subsystem $\{B(x_i^k, r_i^k)\}_i$ of this family such that

$$F_k := \left\{x \in X: R_\tau(x, \mu) \leq \frac{1}{k}\right\} \subset \bigcup_i B(x_i^k, 5r_i^k).$$

Set

$$\rho_k(x) = \sum_i \frac{\chi_{B(x_i^k, 6r_i^k)}}{r_i^k}.$$

Using the doubling property of ν we estimate

$$\begin{aligned} \int_X \rho_k d\nu &= \sum_i \frac{\nu(B(x_i^k, 6r_i^k))}{r_i^k} \leq C \sum_i \frac{\nu(B(x_i^k, r_i^k))}{r_i^k} \\ &\leq C \sum_i \mu(E \cap B(x_i^k, r_i^k)) \leq C\mu(X). \end{aligned}$$

Pick $\gamma \in \Gamma_{\mu,\tau}$. Choose $m \in \mathbb{N}$ and find $\delta > 0$ such that

$$(22) \quad 0 < t < \delta \implies \frac{R_\tau(\gamma(t), \mu)}{d(\gamma(t), \gamma(0))} < \frac{1}{20m}.$$

Choose $T < \delta$ such that $d(\gamma(T), \gamma(0)) > 0$. Find k_m such that $\frac{20m}{k_m} < d(\gamma(T), \gamma(0))$. Let $k \geq k_m$. Find $0 < t_1 < \dots < t_m$ such that

$$d(\gamma(t_j), \gamma(0)) = \frac{20j}{k}, \quad j = 1, \dots, m.$$

By (22),

$$R_\tau(\gamma(t_j), \mu) < \frac{1}{20m} d(\gamma(t_j), \gamma(0)) = \frac{20j}{20mk} \leq \frac{1}{k}.$$

Thus, for each j it holds that $\gamma(t_j) \in F_k$ and there exists $i(j)$ such that

$$\gamma(t_j) \in B(x_{i(j)}^k, 5r_{i(j)}^k).$$

Since $r_i^k \leq \frac{1}{k}$ and the mutual distance of $\gamma(t_j)$ is estimated below by $\frac{20}{k}$, the indices $i(j)$ are distinct. The path γ travels in each $B(x_{i(j)}^k, 6r_{i(j)}^k)$ at least distance $r_{i(j)}^k$ and thus

$$\int_\gamma \rho_k \geq m, \quad k \geq k_m.$$

By Theorem 7 it follows that $AM_1(\Gamma_{\mu,\tau}) = 0$. \square

8. ESTIMATES OF M -MODULUS

In this section we assume that X is locally compact and ν is doubling.

We first recall some notions from analysis on metric measure spaces, see [2], [6].

A Borel measurable function $\rho: X \rightarrow [0, \infty]$ is said to be an *upper gradient* to $u: X \rightarrow \mathbb{R}$ if

$$|u(y) - u(x)| \leq \int_\gamma \rho ds$$

for each $x, y \in X$ and each path γ connecting x to y .

Let $B = B(z, R)$ be a ball and $E \subset B$. We say that a function $u: X \rightarrow \mathbb{R}$ is a *cap-competitor* for (E, B) if $u \geq 1$ on E and $u = 0$ on $X \setminus B$. The Newtonian-Lorentz $NL^{p,1}$ norm of a ν -measurable function $u: X \rightarrow \mathbb{R}$ is defined by

$$\|u\|_{NL^{p,1}} := \|u\|_{L^{p,1}} + \inf \{ \|\rho\|_{L^{p,1}} : \rho \text{ is an upper gradient to } u \}.$$

The *Newtonian-Lorentz $NL^{p,1}$ -capacity* of a set $E \subset B$ is defined as

$$\text{cap}_{NL^{p,1}}(E; B) = \inf \left\{ \|u\|_{NL^{p,1}}^p : u \text{ is a cap-competitor for } (E, B) \right\}.$$

Theorem 11. *Let $p \geq 1$ and let E be a subset of a relatively compact ball $B \subset X$ with $\text{cap}_{NL^{p,1}}(E, B) > 0$. Then $M_{L^{p,1}(X)}(\Gamma_i(E) \cap \Gamma_\tau(E)) = \infty$.*

Proof. Let $\rho \in L^{p,1}(B)$, $\rho \geq 0$, be a lower semicontinuous function. We may assume that ρ is bounded away from 0. For $x \in B$, let Γ_x be the family of all paths $\gamma: [0, \ell] \rightarrow \overline{B}$ such that $\gamma(0) = x$, $\gamma(\ell) \notin B$ and $\gamma((0, \ell)) \subset B$. Set

$$u(x) = \inf \left\{ \int_{\gamma} \rho ds : \gamma \in \Gamma_x \right\}.$$

Let $x, x_i \in \overline{B}$ be such that $x_i \rightarrow x$ and $\sup_i u(x_i) < \infty$. Consider a sequence (γ_i) such that $\gamma_i \in \Gamma_{x_i}$ and

$$\int_{\gamma_i} \rho ds < u(x_i) + 2^{-i}.$$

Then, as ρ is bounded away from zero, the lengths of γ_i are bounded and we can reparametrize these paths to be defined on the same interval with a bounded Lipschitz constant. Then by the Arzela-Ascoli argument, we can find a subsequence converging to a limit path γ such that (as ρ is lower semicontinuous)

$$\int_{\gamma} \rho ds \leq \liminf_i \int_{\gamma_i} \rho ds \leq \liminf_i u(x_i).$$

Hence the function u is lower semicontinuous. Also we easily verify that ρ is an upper gradient of u . Assume first that $u = \infty$ on $E \cap B(z, R)$. Then all the functions u/j serve as cap-competitors for (E, B) and it follows that $\text{cap}_{NL^{p,1}}(E, B) = 0$, a contradiction. Thus, there exists $y \in E$ such that, under the notation above, $u(y) < \infty$. We can find a path parametrized by its arc length $\gamma: [0, \ell] \rightarrow X$, $\gamma(0) = y$ such that $\int_{\gamma} \rho ds < \infty$.

Now, we can construct a path $\tilde{\gamma} \in \Gamma_i(E) \cap \Gamma_{\tau}(E)$. The path $\tilde{\gamma}$ has the same locus as the path γ but goes to $\gamma(0)$ infinitely often and has, at the same time, the required tangential property.

Find a sequence $\delta_m \searrow 0$ such that

$$\delta_1 = \ell,$$

and

$$(23) \quad \sum_{m=1}^{\infty} m \int_0^{\delta_m} \rho(\gamma(t)) dt < \infty.$$

Let q_m be positive integers such that

$$(24) \quad q_m \geq \frac{\delta_m}{\delta_{m+1}}.$$

Set

$$t_m = \sum_{n>m} (2n+2)\delta_n, \quad m = 0, 1, 2, \dots,$$

$$h_m = \frac{\delta_m}{q_m}, \quad m = 1, 2, \dots$$

Our plan is to find the new curve $\tilde{\gamma}$ as

$$\tilde{\gamma}(t) = \gamma(\xi(t)), \quad t \in [0, \tilde{\ell}],$$

where

$$\tilde{\ell} = t_0 = \sum_{n=1}^{\infty} (2n+2)\delta_n$$

and $\xi: [0, \tilde{\ell}] \rightarrow [0, \ell]$ is defined as follows. Fix $m \in \mathbb{N}$ and set

$$\xi(t_{m-1} - s) = s, \quad 0 \leq s \leq \delta_m.$$

Now, we define ξ on $[t_m, t_{m-1} - \delta_m]$ to be linear on each

$$[t_m + (i-1)h_m, t_m + ih_m], \quad i = 1, \dots, (2m+1)q_m$$

and attaining the values

$$\left. \begin{aligned} \xi(t_m + ((k-1)(2m+1) + 2j)h_m) &= (k-1)h_m, \\ \xi(t_m + ((k-1)(2m+1) + (2j+1))h_m) &= kh_m, \end{aligned} \right\} \begin{array}{l} j = 1, \dots, m, \\ k = 1, \dots, q_m. \end{array}$$

For completeness we set $\xi(0) = 0$. Observe that $|\xi'| = 1$ at all points except for the partition ones, so that $\tilde{\gamma}$ is parametrized by its arclength.

Since $\tilde{\gamma}(t_m) = \gamma(0) = y$ for each $m \in \mathbb{N}$, we verify that $\tilde{\gamma} \in \Gamma_i(E)$.

Next, we want to show that $\tilde{\gamma}$ is tangential according to the definition of $\Gamma_\tau(E)$. This reduces to the property

$$(25) \quad \lim_{t \rightarrow 0^+} \frac{\xi(t)}{t} = 0.$$

Let $t \in [t_m, t_{m-1}]$ Then (using (24))

$$\frac{\xi(t)}{t} \leq \frac{\frac{t-t_m}{2m+1} + h_m}{t - t_m + (2m+3)\delta_{m+1}} \leq \frac{1}{2m+1}$$

which tends to 0 as $m \rightarrow \infty$ and (25) is verified.

By (23)

$$\int_{\tilde{\gamma}} \rho ds = \sum_{m=1}^{\infty} (2m+2) \int_0^{\delta_m} \rho(\gamma(t)) dt < \infty.$$

Making $\tilde{\gamma}$ shorter, we can achieve that $\int_{\tilde{\gamma}} \rho ds < 1$ and still $\tilde{\gamma} \in \Gamma_i(E) \cap \Gamma_\tau(E)$, so that ρ is not admissible for $\Gamma_i(E) \cap \Gamma_\tau(E)$. We have shown that there does not exist any lower semicontinuous admissible function for $\Gamma_i(E) \cap \Gamma_\tau(E)$ in $L^{p,1}(X)$. Since admissible functions for a curve family can be approximated by lower semicontinuous admissible functions, it follows that

$$M_{L^{p,1}(X)}(\Gamma_i(E) \cap \Gamma_\tau(E)) = \infty.$$

□

Corollary 5. *Let $k \in \{1, \dots, n\}$ and $E \subset \mathbb{R}^n$ be a $(n-k)$ -dimensional C^1 surface. Then $M_{L^{k,1}}(\Gamma_t(E) \cap \Gamma_i(E)) = \infty$.*

Remark 12. In Euclidean spaces, the Newtonian-Lorentz capacity $\text{cap}_{NL^{p,1}}$ is just the Sobolev-Lorentz capacity. By [10, Theorem 8.19 and Corollary 9.6],

$$(26) \quad H_\infty^{n-p}(E) \leq C \text{cap}_{NL^{p,1}}(E), \quad E \subset \mathbb{R}^n.$$

A similar estimate holds also in metric measure spaces satisfying the $(1, 1)$ -Poincaré inequality [8], see also [12]. For the Newtonian-Lorentz capacity see also [3] and [11].

9. ESTIMATES OF MODULUS: SMOOTH TANGENTIAL PATHS

In this section we provide an elementary proof of Corollary 5. In fact, we prove a stronger assertion, as the paths considered in the proof of Theorem 11 are not smooth. Since null sets for the modulus in consideration are obviously invariant with respect to smooth deformations, we may consider that our surface is flat.

Definition 14. Let $E \subset \mathbb{R}^n$ be a C^1 surface and $T_x(E)$ denote the tangent space to E at a point $x \in E$. Then $\Gamma_s(E)$ is the family of all paths $\gamma: [0, \ell] \rightarrow \mathbb{R}^n$ which are C^1 -smooth in $[0, \ell]$ and satisfy

$$\gamma'_+(0) \in T_{\gamma(0)}(E).$$

Definition 15. Let $k \in \{1, \dots, n\}$ and

$$\mathbb{H}_{n-k} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_{n-k+1} = \dots = x_n = 0\}.$$

Define $D_k = \mathbb{H}_{n-k} \cap B_n(0, 1)$.

Theorem 12. Let $k \in \{1, \dots, n-1\}$. Then $M_{L^{k,1}(\mathbb{R}^n)}(\Gamma_s(D_k)) = \infty$.

Proof. Write $x \in \mathbb{R}^n$ as $x = (y, z)$, where $y \in \mathbb{R}^{n-k}$ and $z \in \mathbb{R}^k$. Denote

$$\begin{aligned} B_k &= B_k(0, 1), & S_{k-1} &= \partial B_k, \\ B_{n-k} &= B_{n-k}(0, 1), & P &= B_{n-k} \times B_k. \end{aligned}$$

Then $D_k = B_{n-k} \times \{0_k\}$,

$$|z| = d((y, z), D_k) \quad \text{in } P$$

and a routine calculation shows that the function

$$(y, z) \mapsto |z|^{1-k}$$

belongs to $L^{k',\infty}(P) = (L^{k,1}(P))^*$. Pick $\rho \in L^{k,1}(\mathbb{R}^n)$. Then the duality argument (see (7)) yields

$$\iint_P \rho(y, z) |z|^{1-k} dy dz < \infty.$$

Find a continuous function $f: [0, \infty) \rightarrow \mathbb{R}$ such that

$$(27) \quad \lim_{t \rightarrow 0^+} \frac{f(t)}{t^{k-1}} = 0,$$

and still

$$(28) \quad \iint_P \frac{\rho(y, z)}{f(|z|)} dy dz < \infty.$$

Further assume that

$$(29) \quad \int_0^1 \frac{s^{k-1}}{f(s)} ds < \infty$$

and

$$(30) \quad 0 \leq f(t) \leq t^{k-1}, \quad t \geq 0.$$

Let $\beta: [0, 1] \rightarrow \mathbb{R}$ be the positive solution of the initial value problem

$$(31) \quad \begin{aligned} \beta'(t) &= (\beta(t))^{1-k} f(\beta(t)), \\ \beta(0) &= 0. \end{aligned}$$

The existence of such a solution is guaranteed by (29). By (30), we have $0 \leq \beta'(t) \leq 1$. Thus we may set

$$(32) \quad \alpha(t) = \int_0^t \sqrt{1 - (\beta'(s))^2} ds, \quad t \in [0, 1].$$

Given $y \in B_{n-k}$ and $\zeta \in S_{k-1}$, set

$$\gamma_{y, \zeta}(t) = (y + \alpha(t)\mathbf{e}_1, \beta(t)\zeta), \quad t \in [0, 1],$$

where

$$\mathbf{e}_1 = (1, 0, \dots, 0).$$

By (32), each $\gamma_{y, \zeta}$ is parametrized by its arclength and from (27) and (31) it follows that $\gamma_{y, \zeta} \in \Gamma_s(D_k)$. We consider the transformation of variables

$$(\tilde{y}, \tilde{z}) = \Phi(y, z) = \gamma_{y, \frac{z}{|z|}}(|z|),$$

so that

$$\Phi(y, t\zeta) = (y + \alpha(t)\mathbf{e}_1, \beta(t)\zeta), \quad y \in B_{n-k}, \zeta \in S_{k-1}, t \in (0, 1).$$

Then

$$\Phi'(y, z) = \begin{pmatrix} I, & F \\ 0, & A \end{pmatrix},$$

where I is the $(n-k) \times (n-k)$ unit matrix, F is a $(n-k) \times k$ matrix and A is the derivative of the radial deformation

$$z \mapsto \beta(|z|) \frac{z}{|z|}.$$

Now the Jacobian determinant $J\Phi$ of Φ has the form

$$J\Phi(y, z) = \det(A) = \left(\frac{\beta(|z|)}{|z|} \right)^{k-1} \beta'(|z|)$$

Using (31) we obtain

$$|z|^{1-k} = \frac{J\Phi(y, z)}{f(|\beta(z)|)}.$$

It follows

$$\begin{aligned}
& \int_{B_{n-k}} \left(\int_{S_{k-1}} \left(\int_{\gamma_{y,\zeta}} \rho ds \right) d\mathcal{H}^{k-1}(\zeta) \right) dy \\
&= \int_{B_{n-k}} \left(\int_{S_{k-1}} \left(\int_0^1 \rho(y + \alpha(t)\mathbf{e}_1, \beta(t)\zeta) dt \right) d\mathcal{H}^{k-1}(\zeta) \right) dy \\
&= \int_{B_{n-k}} \left(\int_0^1 \left(\int_{S_{k-1}} \rho(y + \alpha(t)\mathbf{e}_1, \beta(t)\zeta) d\mathcal{H}^{k-1}(\zeta) \right) dt \right) dy \\
&= \int_{B_{n-k}} \left(\int_0^1 \left(\int_{\partial B_k} \rho(\Phi(y, t\zeta)) d\mathcal{H}^{k-1}(\zeta) \right) dt \right) dy \\
&= \int_{B_{n-k}} \left(\int_0^1 \left(t^{1-k} \int_{\partial B_k(0,t)} \rho(\Phi(y, \xi)) d\mathcal{H}^{k-1}(\xi) \right) dt \right) dy \\
&= \int_{B_{n-k}} \left(\int_{B_k} |z|^{1-k} \rho(\Phi(y, z)) dz \right) dy \\
&= \int_{B_{n-k}} \left(\int_{B_k} \rho(\Phi(y, z)) \frac{J\Phi(y, z)}{f(|\beta(z)|)} dz \right) dy \\
&= \iint_{B_{n-k} \times B_k} \frac{\rho(\tilde{y}, \tilde{z})}{f(|\tilde{z}|)} d\tilde{y} d\tilde{z} < \infty,
\end{aligned}$$

where the last integral is finite by (28). Therefore there must exist $(y, \frac{z}{|z|}) \in B_{n-k} \times S_{k-1}$ such that

$$\int_{\gamma_{y, \frac{z}{|z|}}} \rho ds < \infty.$$

If we truncate the domain of $\gamma_{y, \frac{z}{|z|}}$, we obtain a subpath $\tilde{\gamma}_{y, \frac{z}{|z|}} \in \Gamma_s(D_k)$ such that

$$\int_{\tilde{\gamma}_{y, \frac{z}{|z|}}} \rho ds < 1.$$

Therefore there is no admissible function for $\Gamma_s(D_k)$ in $L^{k,1}(P)$. We conclude that

$$M_{L^{k,1}(\mathbb{R}^n)}(\Gamma_s(D_k)) = \infty.$$

□

Remark 13. In contrast to Corollary 5, this result does not hold for $k = n$.

10. COMPARISON RESULTS

In this section, we show that in some instances, AM -modulus gives other results than the corresponding M -modulus. We prove Theorems 2 and 3 from the introduction.

Proof of Theorem 3. The assertion is a mere combination of Corollary 4 and Corollary 5. □

To show examples for p noninteger, we seek for a fractal set E in \mathbb{R}^n such that

$$0 < \text{cap}_{NL^{p,1}}(E) \quad \text{and} \quad \text{co}\underline{M}^p(E) < \infty.$$

By (26), the capacity inequality is verified whenever $H^{n-p}(E) > 0$. (See Remark 12 which clarifies the situation.) On the other hand, for the simplest examples of sets with

$$H^{n-p}(E) < \infty$$

it also holds that $\text{co}\underline{M}^p(E) < \infty$, but it is not easy to find a proof in literature. Thus, we look for a fractal set K with

$$0 < H^{n-p}(K) < \infty$$

but the upper Hausdorff measure estimate must be refined to an estimate of the Minkowski content.

Definition 16. Let $0 < \lambda < 1$. Let K_0 be the unit interval $[0, 1]$. If K_m is a disjointed union of 2^m intervals $I_1^m, \dots, I_{2^m}^m$ of length $2r_m = (\frac{1}{2}(1-\lambda))^m$, we produce K_{m+1} by removing a concentric open interval of length $2\lambda r_m$ from each I_i^m . The resulting fractal

$$K = \bigcap_m K_m$$

is called the *middle λ Cantor set*.

Lemma 2. [4, Example 4.5, Proposition 7.1] *Let K be the middle λ Cantor set, K^n be the Cartesian product of n copies of K and*

$$(33) \quad s = \frac{\log 2}{\log(\frac{2}{1-\lambda})}.$$

Then $H^{ns}(K^n) > 0$.

Lemma 3. *Let K be the middle λ Cantor set and s be as in (33). Then $\text{co}\underline{M}^{n-s}(K^n) < \infty$.*

Proof. Let $0 < r < 1$ and find m such that $r_{m+1} \leq r < r_m$. If x_i^m are the centers of I_i^m , then

$$\{x \in \mathbb{R}^n : d(x, K^n) < r\} \subset \left(\bigcup_{i=1}^{2^m} (x_i^m - r_m - r, x_i^m + r_m + r) \right)^n$$

and thus

$$|\{x \in \mathbb{R}^n : d(x, K^n) < r\}| \leq (2^{m+1}(r_m + r))^n \leq (2^{m+2}r_m)^n \lesssim (1-\lambda)^{nm},$$

whereas

$$r^{n-s} \geq r_{m+1}^{n-s} \gtrsim \left(\frac{1}{2}(1-\lambda)\right)^{nm(1-s)} = (1-\lambda)^{nm}.$$

Here symbols \lesssim, \gtrsim mean inequalities up to a positive multiplicative constant independent of m . Now

$$\frac{|\{x \in \mathbb{R}^n : d(x, K^n) < r\}|}{r^{n-s}} \leq c$$

where the constant c is independent of r and letting $r \rightarrow 0$ we see that $\text{co}\underline{M}^{n-ns}(K^n) < \infty$ as required. \square

Proof of Theorem 2. Let K be the middle λ Cantor set and s be given by (33). If λ is chosen so that $ns = n - p$ and $E = K^n$, then $\text{co}\underline{M}^p(E) < \infty$ and $\text{co}H^p(E) > 0$. By Corollary 2, (26) and Theorem 11,

$$AM_{L^{p,1}}(\Gamma_i(E)) = 0 \text{ but } M_{L^{p,1}}(\Gamma_i(E)) = \infty.$$

\square

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DEPARTMENT OF MATHEMATICAL ANALYSIS, FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY, SOKOLOVSKÁ 83, PRAGUE 8, 186 75 CZECH REPUBLIC

E-mail address: honzlova@karlin.mff.cuni.cz

DEPARTMENT OF MATHEMATICAL ANALYSIS, FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY, SOKOLOVSKÁ 83, PRAGUE 8, 186 75 CZECH REPUBLIC

E-mail address: maly@karlin.mff.cuni.cz

DEPARTMENT OF MATHEMATICS AND STATISTICS, FI-00014 UNIVERSITY OF HELSINKI, FINLAND

E-mail address: olli.martio@helsinki.fi