

THE HÖRMANDER MULTIPLIER THEOREM I: THE LINEAR CASE REVISITED

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ABSTRACT. We discuss $L^p(\mathbb{R}^n)$ boundedness for Fourier multiplier operators that satisfy the hypotheses of the Hörmander multiplier theorem in terms of an optimal condition that relates the distance $|\frac{1}{p} - \frac{1}{2}|$ to the smoothness s of the associated multiplier measured in some Sobolev norm. We provide new counterexamples to justify the optimality of the condition $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$ and we discuss the endpoint case $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$.

1. INTRODUCTION

To a bounded function σ on \mathbb{R}^n we associate a linear multiplier operator

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) \sigma(\xi) e^{2\pi i x \cdot \xi} d\xi$$

where f is a Schwartz function on \mathbb{R}^n and $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$ is its Fourier transform. The classical theorem of Mihklin [10] states that if the condition

$$(1) \quad |\partial^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}$$

holds for all multi-indices α with size $|\alpha| \leq [n/2] + 1$, then T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$.

Mihklin's theorem was extended by Hörmander [8] to multipliers with fractional derivatives in some L^r space. To precisely describe this extension, let Δ be the Laplacian, let $(I - \Delta)^{s/2}$ denote the operator given on the Fourier transform by multiplication by $(1 + 4\pi^2 |\xi|^2)^{s/2}$ and for $s > 0$, and let L_s^r be the standard Sobolev space of all functions h on \mathbb{R}^n with norm

$$\|h\|_{L_s^r} := \|(I - \Delta)^{s/2} h\|_{L^r} < \infty.$$

Let Ψ be a Schwartz function whose Fourier transform is supported in the annulus of the form $\{\xi : 1/2 < |\xi| < 2\}$ which satisfies $\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j} \xi) = 1$ for all $\xi \neq 0$.

Hörmander's extension of Mihklin's theorem says that if $1 < r \leq 2$ and $s > n/r$, a bounded function σ satisfies

$$(2) \quad \sup_{k \in \mathbb{Z}} \|\widehat{\Psi} \sigma(2^k \cdot)\|_{L_s^r} < \infty,$$

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i.e., σ is uniformly (over all dyadic annuli) in the Sobolev space L_s^r , then T_σ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$, and is also of weak type $(1, 1)$. An endpoint result for this multiplier theorem involving a Besov space was given by Seeger [14]. The least number of derivatives imposed on the multiplier in Hörmander's condition (2) is when $r = 2$. In this case, under the assumption of $n/2 + \varepsilon$ derivatives in L^2 uniformly (over all dyadic annuli), we obtain boundedness of T_σ on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$. It is natural to ask whether L^p boundedness holds for some p if $s < n/2$.

Calderón and Torchinsky [1] used an interpolation technique to prove that if (2) holds, then the multiplier operator T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself whenever p satisfies

$$(3) \quad \left| \frac{1}{p} - \frac{1}{2} \right| < \frac{s}{n}$$

and

$$(4) \quad \left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{1}{r}.$$

It is not hard to verify that if σ satisfies (2) and T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself, then we must necessarily have $rs \geq n$; see Proposition 4.1. Thus $\frac{1}{r} \leq \frac{s}{n}$ and comparing conditions (3) and (4) we notice that (4) restricts (3). On the other hand, if we only have conditions (2) and (3) for some r, s with $rs > n$, $r \in (1, \infty)$, $s \in (0, \infty)$, then one can find an r_o such that $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{r_o} < \frac{s}{n}$ and $r_o < r$. In view of standard embeddings between Sobolev spaces¹ we obtain that

$$(5) \quad \sup_{k \in \mathbb{Z}} \|\widehat{\Psi}\sigma(2^k \cdot)\|_{L_s^{r_o}} \leq C \sup_{k \in \mathbb{Z}} \|\widehat{\Psi}\sigma(2^k \cdot)\|_{L_s^r} < \infty,$$

and thus we can deduce the boundedness of T_σ on $L^p(\mathbb{R}^n)$ by the aforementioned Calderón and Torchinsky [1] result using the space $L_s^{r_o}$. So assumption (4) is not necessary.

In this note we provide a self-contained proof of the L^p boundedness of T_σ only under assumption (3). Moreover, we show that (3) is optimal in the sense that within the class of multipliers σ for which (2) holds, if T_σ is bounded from L^p to itself, then we must necessarily have $|\frac{1}{p} - \frac{1}{2}| \leq \frac{s}{n}$. Theorem 1.1 is mostly folklore, and could be proved via the interpolation result of Connett and Schwartz [2], but here we provide a “bare hands” proof. The counterexamples we supply (Section 4) seem to be new.

Theorem 1.1. *Fix $1 < r < \infty$ and $0 < s \leq \frac{n}{2}$ such that $rs > n$. Assume that (2) holds. Then T_σ maps $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$ such that $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$. Moreover, if T_σ is bounded from $L^p(\mathbb{R}^n)$ to itself for all σ such that (2) holds, then we must have $|\frac{1}{p} - \frac{1}{2}| \leq \frac{s}{n}$.*

We note that the strict inequality in condition $rs > n$ is necessary as there exist unbounded functions in $L_{n/r}^r(\mathbb{R}^n)$, while multipliers are always in L^∞ .

¹This could be proved via the Kato-Ponce inequality $\|FG\|_{L_s^q} \leq C\|F\|_{L_s^{q_1}}\|G\|_{L_s^{q_2}}$, $1/q = 1/q_1 + 1/q_2$ with $q = r_o$ and $q_1 = r$; see [9], [5].

On the critical case $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$, $1 \leq p < 2$, there are positive results for $1 < p < 2$ (see Seeger [13]) and for $p = 1$ by Seeger [14]. In Section 5 we discuss a direct way to relate the results in the cases $p = 1$ and $1 < p < 2$ via direct interpolation that yields the following result as a consequence of the main theorem in [14]:

Proposition 1.2 ([13]). *Given $0 \leq s \leq \frac{n}{2}$, $1 < p < 2$ satisfy $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$, then we have*

$$\|T_\sigma\|_{L^p \rightarrow L^{p,2}} \leq C \sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot) \widehat{\Psi}\|_{B_{\frac{n}{s}}^{s,1}}.$$

Here $L^{p,2}$ denotes the Lorentz space of functions f for which $t^{1/p} f^*(t)$ lies in $L^2((0, \infty), \frac{dt}{t})$, where f^* is the nondecreasing rearrangement of f ; for the definition of the Besov space $B_{\frac{n}{s}}^{s,1}$ see Section 5. Other types of endpoint results involving L^p norms as opposed to $L^{p,2}$ norms were provided by Seeger [15].

2. COMPLEX INTERPOLATION

This section contains an interpolation result proved in a simpler way than that of Calderón and Torchinsky [1]. We denote by S the strip in the complex plane with $0 < \Re(z) < 1$.

Lemma 2.1. *Let $0 < p_0 < p < p_1 < \infty$ be related as in $1/p = (1-\theta)/p_0 + \theta/p_1$ for some $\theta \in (0, 1)$. Given $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$, there exist smooth functions h_j^ε , $j = 1, \dots, N_\varepsilon$, supported in cubes on \mathbb{R}^n with pairwise disjoint interiors, and nonzero complex constants c_j^ε such that the functions*

$$f_z^\varepsilon = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^{\frac{p}{p_0}(1-z) + \frac{p}{p_1}z} h_j^\varepsilon$$

satisfy

$$\|f_\theta^\varepsilon - f\|_{L^2} + \|f_\theta^\varepsilon - f\|_{L^{p_0}}^{\min(1,p_0)} + \|f_\theta^\varepsilon - f\|_{L^{p_1}}^{\min(1,p_1)} < \varepsilon$$

and

$$\|f_{it}^\varepsilon\|_{L^{p_0}}^{p_0} \leq \|f\|_{L^p}^p + \varepsilon', \quad \|f_{1+it}^\varepsilon\|_{L^{p_1}}^{p_1} \leq \|f\|_{L^p}^p + \varepsilon',$$

where ε' depends on ε, p_0, p_1, p , $\|f\|_{L^p}$ and tends to zero as $\varepsilon \rightarrow 0$.

Proof. Given $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$, by uniform continuity there are N_ε cubes Q_j^ε (with disjoint interiors) and nonnegative constants c_j^ε such that

$$\left\| f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \right\|_{L^2} + \left\| f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \right\|_{L^{p_0}}^{\min(1,p_0)} + \left\| f - \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \right\|_{L^{p_1}}^{\min(1,p_1)} < \varepsilon.$$

Find nonnegative smooth functions $g_j^\varepsilon \leq \chi_{Q_j^\varepsilon}$ such that

$$\left\| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon (g_j^\varepsilon - \chi_{Q_j^\varepsilon}) \right\|_{L^2} + \left\| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon (g_j^\varepsilon - \chi_{Q_j^\varepsilon}) \right\|_{L^{p_0}}^{\min(1,p_0)} + \left\| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon (g_j^\varepsilon - \chi_{Q_j^\varepsilon}) \right\|_{L^{p_1}}^{\min(1,p_1)} < \varepsilon$$

and

$$\left(\sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^p \|g_j^\varepsilon - \chi_{Q_j^\varepsilon}\|_{L^{p_0}}^{p_0} \right)^{\frac{1}{p_0}} + \left(\sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon|^p \|g_j^\varepsilon - \chi_{Q_j^\varepsilon}\|_{L^{p_1}}^{p_1} \right)^{\frac{1}{p_1}} < \varepsilon.$$

Let ϕ_j^ε be the argument of the complex number c_j^ε . Set $h_j^\varepsilon = e^{i\phi_j^\varepsilon} g_j^\varepsilon$ and notice that $f_\theta^\varepsilon = \sum_{j=1}^{N_\varepsilon} |c_j^\varepsilon| h_j^\varepsilon = \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon g_j^\varepsilon$ satisfies

$$\|f_\theta^\varepsilon - f\|_{L^2} + \|f_\theta^\varepsilon - f\|_{L^{p_0}}^{\min(1, p_0)} + \|f_\theta^\varepsilon - f\|_{L^{p_1}}^{\min(1, p_1)} < \varepsilon.$$

Moreover, the choice of g_j^ε implies that

$$\|f_{it}\|_{L^{p_0}} \leq \left(B^{\min(1, p_0)} + \varepsilon^{\min(1, p_0)} \right)^{\frac{1}{\min(1, p_0)}},$$

where

$$B = \left\| \sum_{j=1}^{N_\varepsilon} c_j^\varepsilon \chi_{Q_j^\varepsilon} \right\|_{L^p}^{\frac{p}{p_0}} \leq \left(\left(\varepsilon^{\min(1, p)} + \|f\|_{L^p}^{\min(1, p)} \right)^{\frac{1}{\min(1, p)}} \right)^{\frac{p}{p_0}}.$$

An analogous estimate holds for f_{1+it} . Given $a, c > 0$ and $\varepsilon > 0$ set $\varepsilon' = \varepsilon'(\varepsilon, a, c) = (\varepsilon^a + c^a)^{1/a} - c$. Then $(\varepsilon^a + c^a)^{1/a} \leq \varepsilon' + c$ and $\varepsilon' \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then for a suitable ε' that only depends on $\varepsilon, p, p_0, p_1, \|f\|_{L^p}$, the preceding estimates give: $\|f_{it}^\varepsilon\|_{L^{p_0}}^{p_0} \leq \|f\|_{L^p}^p + \varepsilon'$ and $\|f_{1+it}^\varepsilon\|_{L^{p_1}}^{p_1} \leq \|f\|_{L^p}^p + \varepsilon'$, as claimed. \square

Lemma 2.2. *For z in the strip $a < \Re(z) < b$ and $x \in \mathbb{R}^n$, let $H(z, x)$ be analytic in z and smooth in $x \in \mathbb{R}^n$ that satisfies*

$$\left| H(z, x) + \left| \frac{dH}{dz}(z, x) \right| \right| \leq H_*(x), \quad \forall a < \Re(z) < b,$$

where H_* is a measurable function on \mathbb{R}^n . Let f be a complex-valued smooth function on \mathbb{R}^n such that

$$\int_{\mathbb{R}^n} \max \left\{ |f(x)|^a, |f(x)|^b \right\} \left\{ 1 + |\log(|f(x)|)| \right\} H_*(x) dx < \infty.$$

Then the function

$$G(z) = \int_{\mathbb{R}^n} |f(x)|^z e^{i \operatorname{Arg} f(x)} H(z, x) dx$$

is analytic on the strip $a < \Re(z) < b$ and continuous up to the boundary.

Proof. Let $A = \{x : f(x) \neq 0\}$. For $x \in A$ denote

$$F(z, x) = |f(x)|^z e^{i \operatorname{Arg} f(x)} H(z, x).$$

Fix $a < \Re(z_0) < b$ and $x \in A$. Then

$$\begin{aligned} & \lim_{z \rightarrow z_0} \frac{F(z, x) - F(z_0, x)}{z - z_0} \\ &= |f(x)|^{z_0} \log |f(x)| e^{i \operatorname{Arg} f(x)} H(z_0, x) + |f(x)|^{z_0} e^{i \operatorname{Arg} f(x)} \frac{dH}{dz}(z_0, x) \end{aligned}$$

for all $x \in A$. We also have

$$\left| \frac{F(z, x) - F(z_0, x)}{z - z_0} \right| \leq \max \left\{ |f(x)|^a, |f(x)|^b \right\} \left(1 + |\log |f(x)|| \right) H_*(x)$$

for all $x \in A$. By Lebesgue dominated convergence theorem, the function G is analytic and its derivative is

$$G'(z) = \int_{\mathbb{R}^n} \left[|f(x)|^z \log(|f(x)|) e^{i \operatorname{Arg} f(x)} H(z, x) + |f(x)|^z e^{i \operatorname{Arg} f(x)} \frac{dH}{dz}(z, x) \right] dx$$

Also, the function G is also continuous on the boundary $\Re(z) = a$ and $\Re(z) = b$. \square

Lemma 2.3 ([3, 6]). *Let F be analytic on the open strip $S = \{z \in \mathbb{C} : 0 < \Re(z) < 1\}$ and continuous on its closure. Assume that for all $0 \leq \tau \leq 1$ there exist functions A_τ on the real line such that*

$$|F(\tau + it)| \leq A_\tau(t) \quad \text{for all } t \in \mathbb{R},$$

and suppose that there exist constants $A > 0$ and $0 < a < \pi$ such that for all $t \in \mathbb{R}$ we have

$$0 < A_\tau(t) \leq \exp \{ A e^{a|t|} \}.$$

Then for $0 < \theta < 1$ we have

$$|F(\theta)| \leq \exp \left\{ \frac{\sin(\pi\theta)}{2} \int_{-\infty}^{\infty} \left[\frac{\log |A_0(t)|}{\cosh(\pi t) - \cos(\pi\theta)} + \frac{\log |A_1(t)|}{\cosh(\pi t) + \cos(\pi\theta)} \right] dt \right\}.$$

In calculations it is crucial to note that

$$(6) \quad \frac{\sin(\pi\theta)}{2} \int_{-\infty}^{\infty} \frac{dt}{\cosh(\pi t) - \cos(\pi\theta)} = 1 - \theta, \quad \frac{\sin(\pi\theta)}{2} \int_{-\infty}^{\infty} \frac{dt}{\cosh(\pi t) + \cos(\pi\theta)} = \theta.$$

The main result of this section is the following:

Theorem 2.4. *Fix $1 < q_0, q_1, r_0, r_1 < \infty$, $0 < p_0, p_1, s_0, s_1 < \infty$. Suppose that $r_0 s_0 > n$ and $r_1 s_1 > n$. Let $\widehat{\Psi}$ be supported in the annulus $1/2 \leq |\xi| \leq 2$ on \mathbb{R}^n and satisfy*

$$\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

Assume that for $k \in \{0, 1\}$ we have

$$(7) \quad \|T_\sigma(f)\|_{L^{q_k}} \leq K_k \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{L^{r_k}} \|f\|_{L^{p_k}}$$

for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. For $0 < \theta < 1$ let

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad s = (1-\theta)s_0 + \theta s_1.$$

Then there is a constant $C_* = C_*(r_0, r_1, s_0, s_1, n)$ such that for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have

$$(8) \quad \|T_\sigma(f)\|_{L^q(\mathbb{R}^n)} \leq C_* K_0^{1-\theta} K_1^\theta \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{L^r} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Fix $\widehat{\Phi}$ such that $\operatorname{supp}(\widehat{\Phi}) \subset \{\frac{1}{4} \leq |\xi| \leq 4\}$ and $\widehat{\Phi} \equiv 1$ on the support of the function $\widehat{\Psi}$. Denote $\varphi_j = (I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot) \widehat{\Psi}]$ and define

$$(9) \quad \sigma_z(\xi) = \sum_{j \in \mathbb{Z}} (I - \Delta)^{-\frac{s_0(1-z) + s_1 z}{2}} \left[|\varphi_j|^{r(\frac{1-z}{r_0} + \frac{z}{r_1})} e^{i \operatorname{Arg}(\varphi_j)} \right] (2^{-j}\xi) \widehat{\Phi}(2^{-j}\xi).$$

This sum has only finitely many terms and we estimate its L^∞ norm. Fix $\xi \in \mathbb{R}^n$. Then there is a j_0 such that $|\xi| \approx 2^{j_0}$ and there are only two terms in the sum in (9). For these terms we estimate the L^∞ norm of $(I - \Delta)^{-\frac{s_0(1-z)+s_1z}{2}} [|\varphi_j|^{r(\frac{1-z}{r_0} + \frac{z}{r_1})} e^{i\text{Arg}(\varphi_j)}]$. For $z = \tau + it$ with $0 \leq \tau \leq 1$, let $s_\tau = (1 - \tau)s_0 + \tau s_1$ and $1/r_\tau = (1 - \tau)/r_0 + \tau/r_1$. By the Sobolev embedding theorem we have

$$\begin{aligned} & \left\| (I - \Delta)^{-\frac{s_0(1-z)+s_1z}{2}} [|\varphi_j|^{r(\frac{1-z}{r_0} + \frac{z}{r_1})} e^{i\text{Arg}(\varphi_j)}] \right\|_{L^\infty} \\ & \leq C(r_\tau, s_\tau, n) \left\| (I - \Delta)^{-\frac{s_0(1-z)+s_1z}{2}} [|\varphi_j|^{r(\frac{1-z}{r_0} + \frac{z}{r_1})} e^{i\text{Arg}(\varphi_j)}] \right\|_{L^{r_\tau}} \\ & \leq C(r_\tau, s_\tau, n) \left\| (I - \Delta)^{it\frac{s_0-s_1}{2}} [|\varphi_j|^{r(\frac{1-z}{r_0} + \frac{z}{r_1})} e^{i\text{Arg}(\varphi_j)}] \right\|_{L^{r_\tau}} \\ & \leq C'(r_\tau, s_\tau, n)(1 + |s_0 - s_1| |t|)^{n/2+1} \left\| |\varphi_j|^{r(\frac{1-z}{r_0} + \frac{z}{r_1})} e^{i\text{Arg}(\varphi_j)} \right\|_{L^{r_\tau}} \\ & \leq C''(r_0, r_1, s_0, s_1, \tau, n)(1 + |t|)^{n/2+1} \left\| |\varphi_j|^{r(\frac{1-\tau}{r_0} + \frac{\tau}{r_1})} \right\|_{L^{r_\tau}} \\ & = C''(r_0, r_1, s_0, s_1, \tau, n)(1 + |t|)^{n/2+1} \|\varphi_j\|_{L^{r_\tau}}^{\frac{r}{r_\tau}}. \end{aligned}$$

It follows from this that

$$(10) \quad \|\sigma_{\tau+it}\|_{L^\infty} \leq C''(r_0, r_1, s_0, s_1, \tau, n)(1 + |t|)^{n/2+1} \left(\sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{L_s^r} \right)^{\frac{r}{r_\tau}}.$$

Let T_z be the family of operators associated to the multipliers σ_z . Let ε be given. Fix $f, g \in \mathcal{C}_0^\infty$ and $0 < p_0 < p < p_1 < \infty$, $1 < q'_0 < q' < q'_1 < \infty$. Given $\varepsilon > 0$, by Lemma 2.1 there exist functions f_z^ε and g_z^ε such that $\|f_\theta^\varepsilon - f\|_{L^p} < \varepsilon$, $\|g_\theta^\varepsilon - g\|_{L^{q'}} < \varepsilon$, and that

$$\begin{aligned} \|f_{it}^\varepsilon\|_{L^{p_0}} &\leq (\|f\|_{L^p} + \varepsilon)^{\frac{p}{p_0}}, & \|f_{1+it}^\varepsilon\|_{L^{p_1}} &\leq (\|f\|_{L^p} + \varepsilon)^{\frac{p}{p_1}}, \\ \|g_{it}^\varepsilon\|_{L^{q'_0}} &\leq (\|g\|_{L^{q'}} + \varepsilon)^{\frac{q'}{q'_0}}, & \|g_{1+it}^\varepsilon\|_{L^{q'_1}} &\leq (\|g\|_{L^{q'}} + \varepsilon)^{\frac{q'}{q'_1}}. \end{aligned}$$

Define

$$\begin{aligned} F(z) &= \int T_{\sigma_z}(f_z^\varepsilon) g_z^\varepsilon dx \\ &= \int \sigma_z(\xi) \widehat{f}_z^\varepsilon(\xi) \widehat{g}_z^\varepsilon(\xi) d\xi \\ &= \sum_{j \in \mathbb{Z}} \int (I - \Delta)^{-\frac{s_0(1-z)+s_1z}{2}} [|\varphi_j|^{r(\frac{1-z}{r_0} + \frac{z}{r_1})} e^{i\text{Arg}(\varphi_j)}] (2^{-j}\xi) \widehat{\Phi}(2^{-j}\xi) \widehat{f}_z^\varepsilon(\xi) \widehat{g}_z^\varepsilon(\xi) d\xi \\ &= \sum_{j \in \mathbb{Z}} \int [|\varphi_j|^{r(\frac{1-z}{r_0} + \frac{z}{r_1})} e^{i\text{Arg}(\varphi_j)}] (2^{-j}\xi) (I - \Delta)^{-\frac{s_0(1-z)+s_1z}{2}} [\widehat{\Phi}(2^{-j}\cdot) \widehat{f}_z^\varepsilon \widehat{g}_z^\varepsilon] (\xi) d\xi. \end{aligned}$$

Notice that $(I - \Delta)^{-\frac{s_0(1-z)+s_1z}{2}} [\widehat{\Phi}(2^{-j}\cdot) \widehat{f}_z^\varepsilon \widehat{g}_z^\varepsilon] (\xi)$ is equal to a finite sum of the form

$$\sum_{k,l} |c_k^\varepsilon|^{\frac{p}{p_0} + (\frac{p}{p_1} - \frac{p}{p_0})z} |d_l^\varepsilon|^{\frac{q'}{q'_0} + (\frac{q'}{q'_1} - \frac{q'}{q'_0})z} (I - \Delta)^{-\frac{s_0(1-z)+s_1z}{2}} [\widehat{\Phi}(2^{-j}\cdot) \zeta_{k,l}] (\xi) = H(\xi, z),$$

where $\zeta_{k,l}$ are Schwartz functions, and thus it is an analytic function in z .

Lemma 2.2 guarantees that $F(z)$ is analytic on the strip $0 < \Re(z) < 1$ and continuous up to the boundary. Furthermore, by Hölder's inequality, $|F(it)| \leq \|T_{\sigma_{it}}(f_{it}^\varepsilon)\|_{L^{q_0}} \|g_{it}^\varepsilon\|_{L^{q'_0}}$, and

$$\begin{aligned} \|T_{\sigma_{it}}(f_{it}^\varepsilon)\|_{L^{q_0}} &\leq K_0 \sup_{k \in \mathbb{Z}} \left\| \sigma_{it}(2^k \cdot) \widehat{\Psi} \right\|_{L_{s_0}^{r_0}} \|f_{it}^\varepsilon\|_{L^{p_0}} \\ &\leq C(n, r_0) (1 + |s_1 - s_0| |t|)^{\frac{n}{2}+1} K_0 \sup_{j \in \mathbb{Z}} \|\varphi_j\|_{L^r}^{\frac{r}{r_0}} (\|f\|_{L^p}^p + \varepsilon')^{\frac{1}{p_0}} \\ &= C(n, r_0) (1 + |s_1 - s_0| |t|)^{\frac{n}{2}+1} K_0 \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot) \widehat{\Psi}]\|_{L^r}^{\frac{r}{r_0}} (\|f\|_{L^p}^p + \varepsilon')^{\frac{1}{p_0}}. \end{aligned}$$

Thus, for some constant $C = C(n, r_0, s_0, s_1)$ we have

$$|F(it)| \leq C (1 + |t|)^{\frac{n}{2}+1} K_0 \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot) \widehat{\Psi}]\|_{L^r}^{\frac{r}{r_0}} (\|f\|_{L^p}^p + \varepsilon')^{\frac{1}{p_0}} (\|g\|_{L^{q'}}^{q'} + \varepsilon')^{\frac{1}{q'_0}}.$$

Similarly, for some constant $C = C(n, r_1, s_0, s_1)$ we obtain

$$|F(1 + it)| \leq C (1 + |t|)^{\frac{n}{2}+1} K_1 \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot) \widehat{\Psi}]\|_{L^r}^{\frac{r}{r_1}} (\|f\|_{L^p}^p + \varepsilon')^{\frac{1}{p_1}} (\|g\|_{L^{q'}}^{q'} + \varepsilon')^{\frac{1}{q'_1}}$$

Thus for $z = \tau + it$, $t \in \mathbb{R}$ and $0 \leq \tau \leq 1$ it follows from (10) and from the definition of $F(z)$ that

$$|F(z)| \leq C'' (1 + |t|)^{\frac{n}{2}+1} \left(\sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{L^r_s} \right)^{\frac{r}{r_\tau}} \|f_z^\varepsilon\|_{L^2} \|g_z^\varepsilon\|_{L^2} = A_\tau(t),$$

noting that $\|f_z^\varepsilon\|_{L^2} \|g_z^\varepsilon\|_{L^2}$ is bounded above by constants independent of t and τ . Since $A_\tau(t) \leq \exp(Ae^{a|t|})$ it follows that the hypotheses of Lemma 2.3 are valid.

Applying Lemma 2.3 we obtain

$$|F(\theta)| \leq C K_0^{1-\theta} K_1^\theta \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot) \widehat{\Psi}]\|_{L^r} (\|f\|_{L^p}^p + \varepsilon')^{\frac{1}{p}} (\|g\|_{L^{q'}}^{q'} + \varepsilon')^{\frac{1}{q'}}.$$

But

$$F(\theta) = \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}_\theta^\varepsilon(\xi) \widehat{g}_\theta^\varepsilon(\xi) d\xi.$$

Then

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}_\theta^\varepsilon(\xi) \widehat{g}_\theta^\varepsilon(\xi) d\xi - \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}(\xi) \widehat{g}(\xi) d\xi \right| \\ &= \left| \int_{\mathbb{R}^n} \sigma(\xi) \left[\widehat{f}_\theta^\varepsilon(\xi) (\widehat{g}_\theta^\varepsilon(\xi) - \widehat{g}(\xi)) + \widehat{g}(\xi) (\widehat{f}_\theta^\varepsilon(\xi) - \widehat{f}(\xi)) \right] d\xi \right| \\ &\leq \|\sigma\|_{L^\infty} \left[\|\widehat{f}_\theta^\varepsilon\|_{L^2} \|\widehat{g}_\theta^\varepsilon - \widehat{g}\|_{L^2} + \|\widehat{g}_\theta^\varepsilon\|_{L^2} \|\widehat{f}_\theta^\varepsilon - \widehat{f}\|_{L^2} \right] \\ &\leq \|\sigma\|_{L^r_s} \left[\|f\|_{L^2} \|\widehat{g}_\theta^\varepsilon - \widehat{g}\|_{L^2} + \|g\|_{L^2} \|\widehat{f}_\theta^\varepsilon - \widehat{f}\|_{L^2} \right]. \end{aligned}$$

But the sequences $\widehat{f}_\theta^\varepsilon - \widehat{f}$ and $\widehat{g}_\theta^\varepsilon - \widehat{g}$ converge to zero in L^2 . Letting $\varepsilon \rightarrow 0$, these observations imply that

$$\left| \int_{\mathbb{R}^n} \sigma(\xi) \widehat{f}(\xi) \widehat{g}(\xi) d\xi \right| \leq C K_0^{1-\theta} K_1^\theta \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot) \widehat{\Psi}]\|_{L^r} \|f\|_{L^p} \|g\|_{L^{q'}}$$

and taking the supremum over all functions $g \in L^{q'}$ with $\|g\|_{L^{q'}} \leq 1$ we obtain

$$\|T_\sigma(f)\|_{L^q} \leq C_* K_0^{1-\theta} K_1^\theta \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [\sigma(2^j \cdot) \widehat{\Psi}]\|_{L^r} \|f\|_{L^p},$$

where $C_* = C_*(n, r_1, r_2, s_0, s_1)$. □

3. PROOF OF BOUNDEDNESS IN THEOREM 1.1

To prove Theorem 1.1 we use Theorem 2.4 applied as follows: fix $p \in (1, 2)$ such that $\frac{1}{p} - \frac{1}{2} < \frac{s}{n}$. Pick $p_0 = 1 + \delta$ with δ small such that $1 < p_0 < p$ and set $s_0 = n/2 + \varepsilon$ and $r_0 = 2$ where ε is small. Also set $p_1 = 2$, $s_1 = \varepsilon + \varepsilon^2$, and $r_1 = n/\varepsilon$. We have that

$$(11) \quad \|T_\sigma(f)\|_{L^{p_0}} \leq C(n, p_0, r_0, s_0) \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{L^{r_0}} \|f\|_{L^{p_0}}$$

and

$$(12) \quad \|T_\sigma(f)\|_{L^2} \leq C(n, p_0, r_1, s_1) \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{L^{r_1}} \|f\|_{L^2}$$

The conditions

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad s = (1-\theta)s_0 + \theta s_1.$$

translate into

$$\frac{1}{p} - \frac{1}{2} = \frac{1}{1+\delta} - \frac{1}{2} + \theta \left(\frac{1}{2} - \frac{1}{1+\delta} \right), \quad \frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta\varepsilon}{n}, \quad s = (1-\theta)\frac{n}{2} + (1+\theta)(\varepsilon + \varepsilon^2)$$

or

$$\begin{aligned} \frac{1}{p} - \frac{1}{2} &= \left(\frac{s}{n} - (1+\theta)\frac{\varepsilon + \varepsilon^2}{n} \right) \left(1 - \frac{2\delta}{1+\delta} \right) \\ &= \frac{s}{n} - \left((1+\theta)\frac{\varepsilon + \varepsilon^2}{n} + \frac{2\delta}{1+\delta} \frac{s}{n} - (1+\theta)\frac{\varepsilon + \varepsilon^2}{n} \frac{2\delta}{1+\delta} \right) < \frac{s}{n}. \end{aligned}$$

Since δ and ε are very small it follows that $\frac{1}{p} - \frac{1}{2}$ can be arbitrarily close to $\frac{s}{n}$. Note that once s is fixed for a given p , the optimal r is close to $\frac{n}{s}$ (i.e., $\frac{1}{r} = \frac{s}{n} - \frac{\varepsilon'}{n}$). Interpolating between (11) and (12), via Theorem 2.4, yields the required assertion.

4. NECESSARY CONDITIONS

In this section we discuss examples that reinforce the minimality of the conditions on the indices in Theorem 1.1. One way to see this is to use the multiplier $m_{a,b}(\xi) = \psi(\xi)|\xi|^{-b}e^{i|\xi|^a}$ where $a > 0$, $a \neq 1$, $b > 0$, and ψ is a smooth function which vanishes in a neighborhood of the origin and is equal to 1 for large ξ . One can verify that $m_{a,b}$ satisfies (5) for $s = b/a$ and $r > n/s$. But it is known that $T_{m_{a,b}}$ is bounded in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $|\frac{1}{p} - \frac{1}{2}| \leq \frac{b/a}{n}$ (see Hirschman [7, comments after Theorem 3c], Wainger [16, Part II], and Miyachi [11, Theorem 3]). Alternative examples were given in Miyachi and Tomita [12, Section 7].

In this section we provide yet new examples to indicate the necessity of the indices in Theorem 1.1. We are not sure as to whether boundedness into L^p , or even weak L^p , is valid in general under assumption (2) exactly on the critical line $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$.

Proposition 4.1. *If for all $\sigma \in L^\infty(\mathbb{R}^n)$ such that $\sup_k \|\sigma(2^k \cdot) \widehat{\Psi}\|_{L^r_s(\mathbb{R}^n)} < \infty$ we have*

$$(13) \quad \|T_\sigma\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq C_p \sup_k \|\sigma(2^k \cdot) \widehat{\Psi}\|_{L^r_s(\mathbb{R}^n)} < \infty,$$

then we must necessarily have $rs \geq n$ and $|\frac{1}{p} - \frac{1}{2}| \leq \frac{s}{n}$.

Proof. First we prove the necessary condition $rs \geq n$. Let $\widehat{\zeta}$ be a smooth function supported in the ball $B(0, 1/10)$ in \mathbb{R}^n and let $\widehat{\phi}$ be supported in the ball $B(0, 1/2)$ equal to 1 on $B(0, 1/5)$. Define $\widehat{f}(\xi) = \widehat{\zeta}(N(\xi - a))$ with $|a| = 1$, and $\sigma(\xi) = \widehat{\phi}(N(\xi - a))$, then a direct calculation gives $\|f\|_{L^p(\mathbb{R}^n)} \approx N^{-n+n/p}$ and $\|\sigma\|_{L^r_s(\mathbb{R}^n)} \leq CN^s N^{-n/r}$; for the last estimate see Lemma 4.2. Moreover, $T_\sigma(f)(x) = N^{-n}\zeta(x/N)e^{2\pi i x \cdot a}$. We thus obtain that $\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \approx N^{-n+n/p}$. Then (13) yields the inequality $N^{-n+n/p} \leq CN^s N^{-n/r} N^{-n+n/p}$, which forces $s - n/r \geq 0$ by letting N go to infinity.

We now turn to the other necessary condition $|\frac{1}{p} - \frac{1}{2}| \leq \frac{s}{n}$. By duality it suffices to prove the case when $1 < p \leq 2$. We will prove our result by constructing an example. We consider the case $n = 1$ first while the higher dimensional case will be an easy generalization.

Let $\widehat{\psi}, \widehat{\varphi} \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $0 \leq \widehat{\varphi} \leq \chi_{[-1/100, 1/100]}$ and $\chi_{[-1/10, 1/10]} \leq \widehat{\psi} \leq \chi_{[-1/2, 1/2]}$. Therefore $\widehat{\psi}\widehat{\varphi} = \widehat{\varphi}$. For a fixed large positive integer N , we define

$$(14) \quad \widehat{f}_N(\xi) = \sum_{j=-N}^N \widehat{\varphi}(N\xi - j), \quad \sigma_{N,t}(\xi) = \sum_{j \in J_N} a_j(t) \widehat{\psi}(N\xi - j),$$

where $J_N = \{j \in \mathbb{Z} : \frac{N}{2} \leq |j| \leq 2N\}$ and $t \in [0, 1]$. Here $\{a_j\}_{j=-\infty}^\infty$ is the sequence of Rademacher functions indexed by all integers.

One can verify that $T_{N,t}(f_N) =: (\sigma_{N,t} f_N)^\vee = (\sum_{j \in J_N} a_j(t) \widehat{\varphi}(N\xi - j))^\vee$. Recall that Rademacher functions satisfy for any $p \in (0, \infty)$

$$c_p \left\| \sum_j a_j(t) A_j \right\|_{L^p([0,1])} \leq \left(\sum_j |A_j|^2 \right)^{1/2} \leq C_p \left\| \sum_j a_j(t) A_j \right\|_{L^p([0,1])},$$

where c_p and C_p are constants. Therefore

$$\begin{aligned} \left(\int_0^1 \|T_{N,t}(f_N)\|_{L^p(\mathbb{R})}^p dt \right)^{1/p} &= \left(\int_0^1 \int_{\mathbb{R}} \left| \sum_{j \in J_N} a_j(t) N^{-1} \varphi(N^{-1}x) e^{2\pi i x j/N} \right|^p dx dt \right)^{1/p} \\ &\approx \left(\int_{\mathbb{R}} \left(\sum_{j \in J_N} |N^{-1} \varphi(N^{-1}x) e^{2\pi i x j/N}|^2 \right)^{p/2} dx \right)^{1/p} \\ &\approx N^{-1} \left(\int_{\mathbb{R}} |N^{1/2} \varphi(N^{-1}x)|^p dx \right)^{1/p} \\ &\approx N^{1/p-1/2}. \end{aligned}$$

The Sobolev norm of $\sigma_{N,t}$ is given by the following lemma, proved in all dimensions. For $n \geq 1$ and $\vec{t} = (t_1, \dots, t_n) \in [0, 1]^n$ we define a function on \mathbb{R}^n by

$$\sigma_{N,\vec{t}}(x_1, \dots, x_n) = \sum_{\vec{j} \in J_N} a_{j_1}(t_1) \cdots a_{j_n}(t_n) \widehat{\varphi}(N\xi_1 - j_1) \cdots \widehat{\varphi}(N\xi_n - j_n),$$

where $J_N = \{\vec{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n : \frac{N}{2} \leq |\vec{j}| \leq 2N\}$. This $\sigma_{N,\vec{t}}$ coincides with $\sigma_{N,t}$ when $n = 1$.

Lemma 4.2. *We have that $\|\sigma_{N,\vec{t}}\|_{L^s_s(\mathbb{R}^n)} \leq CN^s$.*

We postpone the proof of the lemma and continue with the proof of Proposition 4.1 when $n = 1$. We note that \widehat{f}_N has L^q norm bounded by a constant independent of N , which implies by the Young's inequality that $\|f_N\|_{L^q} \leq C$ with C independent of N when $2 \leq q \leq \infty$. We show in the following lemma that this property is valid for all $q \in (1, \infty]$.

Lemma 4.3. *Let f_N be as in (14) and let $p \in (1, \infty]$. Then there is a constant C_p independent of N such that $\|f_N\|_{L^p} \leq C_p$.*

Proof. We note that $f_N = \sum_{j=-N}^N \frac{1}{N} \varphi(x/N) e^{2\pi i x j/N} = \frac{1}{N} \varphi(x/N) D_N(x/N)$, where D_N is the Dirichlet kernel, whose L^p -norm over $[0, 1]$ is comparable to $N^{1/p'}$ when $p > 1$; see for example [3, Exercise 3.1.6]. Using this fact and that φ is a Schwartz function we obtain

$$\begin{aligned} \|f_N\|_{L^p(\mathbb{R})} &= \left\| \frac{1}{N} \varphi\left(\frac{\cdot}{N}\right) D_N\left(\frac{\cdot}{N}\right) \right\|_{L^p(\mathbb{R})} \\ &= \frac{1}{N} N^{1/p} \|\varphi D_N\|_{L^p(\mathbb{R})} \\ &= N^{-1/p'} \left(\sum_{j=-\infty}^{\infty} \int_{j-1}^j |\varphi(x) D_N(x)|^p dx \right)^{1/p} \\ &\leq CN^{-1/p'} \left(\sum_{j=-\infty}^{\infty} \frac{1}{(1+|j|)^M} \int_{j-1}^j |D_N(x)|^p dx \right)^{1/p} \\ &\leq C_p N^{-1/p'} N^{1/p'} = C_p. \end{aligned}$$

This proves the claim. \square

In view of Lemma 4.3 we obtain the following inequalities

$$N^{\frac{1}{p}-\frac{1}{2}} \leq C \left(\int_0^1 \|T_{N,t}(f_N)\|_{L^p(\mathbb{R})}^p dt \right)^{\frac{1}{p}} \leq C A \|f\|_p \left(\int_0^1 \|\sigma_{N,t}\|_{L^s_s}^p dt \right)^{\frac{1}{p}} \leq C C_p A N^s.$$

Letting N go to infinity forces $1/p - 1/2 \leq s$.

We now consider the higher dimensional case. Let $F_N(\vec{x}) = f_N(x_1) \cdots f_N(x_n)$, where f_N is as in (14). It follows from Lemma 4.2 and 4.3 that $\|F_N\|_{L^p} \leq C$ and $\|\sigma_N\|_{L^s_s} \leq CN^s$. A calculation similar to the one dimensional case shows that $\|T_N(F_N)\|_{L^p} \approx N^{(1/p-1/2)n}$, thus letting $N \rightarrow \infty$ we obtain that $|1/p - 1/2| \leq s/n$. \square

We now prove Lemma 4.2.

Proof of Lemma 4.2. It is easy to verify that $\|\sigma_{N,\bar{t}}\|_{L^r} \leq C$ and $\|\sigma_{N,\bar{t}}\|_{L^r_2} \leq CN^2$. Define for $z = u + iv$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ the function

$$F(z) = \int_{\mathbb{R}^n} (I - \Delta)^z \sigma_{N,\bar{t}}(x) \phi(x) dx$$

for z in the closed unit strip. Then F is analytic on the open strip and continuous on its closure. We can also show that by the Mihlin multiplier theorem that

$$|F(z)| \leq P(|v|) \|\sigma_{N,\bar{t}}\|_{L^r_{2u}} \|\phi\|_{L^{r'}} \leq P(|v|) N^{2u+n/r'} \|\varphi\|_{L^r_{2u}} \|\phi\|_{L^{r'}},$$

where $P(t)$ is a polynomial in t which is not necessary to be the same at all occurrences.

We have then $\log |F(z)| \leq \log(N^{2u+n/r'} \|\varphi\|_{L^r_{2u}} \|\phi\|_{L^{r'}}) + C \log |v| \leq C e^{\tau_0 |v|}$ for some $\tau_0 \in (0, 1)$. Applying Lemma 2.3 we obtain for $0 < s < 1$ that

$$(15) \quad \log |F(s)| \leq \frac{\sin(\pi s)}{2} \int_{-\infty}^{\infty} \left[\frac{M_0(t)}{\cosh(\pi t) - \cos(\pi s)} + \frac{M_1(t)}{\cosh(\pi t) + \cos(\pi s)} \right] dt,$$

where $\log |F(it)| \leq M_0(t) = c \log |t| + \log \|\varphi\|_{L^{r'}}$ and $\log |F(1+it)| \leq M_1(t) = 2 \log N + c \log |t| + \log \|\varphi\|_{L^{r'}}$.

We show that (15) is controlled by $2s \log N + C(s) + \log \|\varphi\|_{L^{r'}}$, where $C(s)$ is a finite constant depending on s and independent on N . Then

$$|F(s)| \leq e^{2s \log N} e^{C(s)} e^{\log \|\varphi\|_{L^{r'}}} \leq C(s) N^{2s} \|\varphi\|_{L^{r'}},$$

i.e. $|F(s/2)| \leq C(s) N^s \|\phi\|_{L^{r'}}$ for all $\phi \in \mathcal{S}$, hence $\|(I - \Delta)^{s/2} m\|_{L^r} \leq C(s) N^s$ for $s \in (0, 2)$. Note that the original estimate $\|\sigma_{N,\bar{t}}\|_{L^r_m} \leq CN^m$ is valid for any positive integer m , so a similar argument gives the estimate $\|\sigma_{N,\bar{t}}\|_{L^r_s} \leq CN^s$ for all $s \geq 0$.

It remains to control (15), for which we recall (6). So

$$\frac{\sin(\pi s)}{2} \int_{-\infty}^{\infty} \left[\frac{\log \|\varphi\|_{L^{r'}}}{\cosh(\pi t) - \cos(\pi s)} + \frac{\log \|\varphi\|_{L^{r'}} + 2 \log N}{\cosh(\pi t) + \cos(\pi s)} \right] dt = \log \|\varphi\|_{L^{r'}} + 2s \log N.$$

So matters reduce to showing that for $0 < s < 1$ we have

$$\int_{-\infty}^{\infty} \frac{\log |t|}{\cosh(\pi t) - \cos(\pi s)} dt + \int_{-\infty}^{\infty} \frac{\log |t|}{\cosh(\pi t) + \cos(\pi s)} dt < \infty$$

which is a straightforward calculation. \square

5. THE ENDPOINT CASE $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$

As another application of the interpolation technique of this paper, we discuss an interpolation theorem applicable in the critical case $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$. We introduce the Besov space norm

$$\|h\|_{B^s_{p,q}} := \left(\sum_{j \geq 1} \|2^{js} \Delta_j h\|_{L^p}^q \right)^{\frac{1}{q}} + \|S_0 h\|_{L^p}$$

where Δ_j are the Littlewood-Paley operators and S_0 is an averaging operator that satisfy $S_0 + \sum_{j=1}^{\infty} \Delta_j = I$. We assume that for $j \geq 1$, Δ_j have spectra supported in the annuli $2^j \leq |\xi| \leq 2^{j+2}$, while S_0 has spectrum inside the ball $B(0, 2)$.

We recall the following result of Seeger [14]

$$(16) \quad \|T_\sigma\|_{H^1 \rightarrow L^{1,2}} \leq C \sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot) \widehat{\Psi}\|_{B_2^{\frac{n}{2},1}}$$

concerning the endpoint case $p = 1$. We also have the trivial estimate

$$(17) \quad \|T_\sigma\|_{L^2 \rightarrow L^2} = \|T_\sigma\|_{L^2 \rightarrow L^{2,2}} \leq C \sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot) \widehat{\Psi}\|_{B_\infty^{0,1}}.$$

In this section, we derive the intermediate estimate contained in Seeger [13]:

$$(18) \quad \|T_\sigma\|_{L^p \rightarrow L^{p,2}} \leq C \sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot) \widehat{\Psi}\|_{B_{\frac{n}{s}}^{s,1}}$$

for $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$, $1 < p < 2$, and $0 \leq s \leq \frac{n}{2}$. We deduce estimate (18) from the following theorem.

Theorem 5.1. *Fix $1 < r_0, r_1 \leq \infty$, $1 < p_0, p_1 < \infty$, $0 \leq s_0, s_1 < \infty$. Let $\widehat{\Psi}$ be supported in the annulus $1/2 \leq |\xi| \leq 2$ on \mathbb{R}^n and satisfy*

$$\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

Assume that for $k \in \{0, 1\}$ we have

$$(19) \quad \|T_\sigma(f)\|_{L^{p_k,2}} \leq K_k \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{B_{r_k}^{s_k,1}} \|f\|_{L^{p_k}}$$

for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and σ which make the right hand side finite. For $0 < \theta < 1$ define

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad s = (1-\theta)s_0 + \theta s_1.$$

Then there is a constant $C_* = C_*(r_0, r_1, s_0, s_1, p_0, p_1, p, n)$ such that for all f in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ we have

$$(20) \quad \|T_\sigma(f)\|_{L^{p,2}(\mathbb{R}^n)} \leq C_* K_0^{1-\theta} K_1^\theta \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{B_r^{s,1}} \|f\|_{L^p(\mathbb{R}^n)}.$$

Moreover, conclusion (20) also holds under the assumption that $p_0 = 1$ and (19) is substituted (only for $k = 0$) by

$$(21) \quad \|T_\sigma(f)\|_{L^{1,2}} \leq K_0 \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\Psi}\|_{B_{r_0}^{s_0,1}} \|f\|_{H^1}$$

for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with vanishing integral.

Proof. Let $\widehat{\Phi}(\xi) = \sum_{j \leq 0} \widehat{\Psi}(2^{-j}\xi)$ and $\widehat{\Phi}(0) = 1$; then $\widehat{\Phi}$ is supported in $|\xi| \leq 2$. Fix a bounded function σ . For an integer k define the dilation of σ^k by setting $\sigma^k(\xi) = \sigma(2^k \xi)$. For z in the closed unit strip we introduce linear functions

$$L(z) = \frac{r}{r_0}(1-z) + \frac{r}{r_1}z, \quad M(z) = s - (1-z)s_0 - zs_1$$

and when $j \geq 1$ introduce Littlewood-Paley operators $\Delta_j(g) = g * \Psi_{2^{-j}}$, $\widetilde{\Delta}_j(g) = g * \widetilde{\Psi}_{2^{-j}}$, where $\widetilde{\Psi}$ is a Schwartz function whose Fourier transform is supported in an annulus only slightly larger than $1/2 \leq |\xi| \leq 2$ and equals 1 on the support of $\widehat{\Psi}$. We also define

$\Delta_0(g) = g * \Phi$ and $\tilde{\Delta}_0(g) = g * \tilde{\Phi}$, where the Fourier transform of $\tilde{\Phi}$ is supported in $|\xi| \leq 4$ and equals 1 on the support of $\hat{\Phi}$. Then define:

$$\sigma_z = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{jM(z)} (c_j^k)^{1-L(z)} \tilde{\Delta}_j \left(|\Delta_j(\sigma^k \hat{\Psi})|^{L(z)} e^{i \operatorname{Arg}(\Delta_j(\sigma^k \hat{\Psi}))} \right) (2^{-k} \cdot) \hat{\Psi}(2^{-k} \cdot)$$

where

$$c_j^k = \frac{\|\Delta_j(\sigma^k \hat{\Psi})\|_{L^r}}{\sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} 2^{ls} \|\Delta_l(\sigma^\mu \hat{\Psi})\|_{L^r}}.$$

Next, we estimate

$$(22) \quad \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} 2^{ls_0} \|\Delta_l(\sigma_{it}^\mu \hat{\Psi})\|_{L^{r_0}}.$$

We notice that for a given $\mu \in \mathbb{Z}$, in the sum defining σ_{it}^μ , only finitely many terms in k appear, the ones with $k = \mu, \mu + 1, \mu - 1$. For simplicity we only consider the term with $k = \mu$, since the other ones are similar. This part of (22) is estimated by

$$(23) \quad \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} \sum_{j=0}^N 2^{ls_0} 2^{j(s-s_0)} |c_j^\mu|^{1-\frac{r}{r_0}} \left\| \tilde{\Delta}_l \left(\tilde{\Delta}_j \left(|\Delta_j(\sigma^\mu \hat{\Psi})|^{L(it)} e^{i \operatorname{Arg}(\Delta_j(\sigma^\mu \hat{\Psi}))} \right) \hat{\Psi} \hat{\Psi} \right) \right\|_{L^{r_0}}.$$

Using Lemma 5.2 (stated and proved below) we obtain that (23) is bounded by

$$(24) \quad \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} \sum_{j=0}^N 2^{ls_0} 2^{j(s-s_0)} |c_j^\mu|^{1-\frac{r}{r_0}} C_M 2^{-2|1-\frac{1}{r_0}| \max(j,l)M} \|\Delta_j(\sigma^\mu \hat{\Psi})\|_{L^{r_0}}^{\frac{r}{r_0}}.$$

But the sum over l in (24) is bounded by $C_M 2^{js_0} 2^{-j2|1-\frac{1}{r_0}|M} \leq C_M 2^{js_0}$ for M sufficiently large, and consequently (24) is bounded by

$$(25) \quad C_M \sup_{\mu \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{j(s-s_0)} 2^{js_0} |c_j^\mu|^{1-\frac{r}{r_0}} \|\Delta_j(\sigma^\mu \hat{\Psi})\|_{L^r}^{\frac{r}{r_0}} \leq C_M \left(\sup_{\mu \in \mathbb{Z}} \sum_{j \geq 0} 2^{js} \|\Delta_j(\sigma^\mu \hat{\Psi})\|_{L^r} \right)^{\frac{r}{r_0}}$$

by the choice of c_j^μ . Likewise we obtain a similar estimate for the point $1 + it$. We summarize these two estimates as follows:

$$(26) \quad \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} 2^{ls_m} \|\Delta_l(\sigma_z^\mu \hat{\Psi})\|_{L^{r_m}} \leq C_M \left(\sup_{k \in \mathbb{Z}} \sum_{j \geq 0} 2^{js} \|\Delta_j(\sigma^k \hat{\Psi})\|_{L^r} \right)^{\frac{r}{r_m}}$$

where $m = 0, 1$ and $\Re z = m$.

Now consider an analytic family of operators T_z associated with the multipliers σ_z defined by $f \mapsto T_{\sigma_z}(f)$. We have that when $\operatorname{Re} z = 0$, T_z maps $L^{p_0, 2}$ to L^{p_0} if $p_0 > 1$ and H^1 to $L^{1, 2}$ if $p_0 = 1$ with constant

$$B_0 = C_M K_0 \left(\sup_{k \in \mathbb{Z}} \sum_{j \geq 0} 2^{js} \|\Delta_j(\sigma^k \hat{\Psi})\|_{L^r} \right)^{\frac{r}{r_0}}$$

and when $\operatorname{Re} z = 1$, T_z maps $L^{p_1, 2}$ to L^{p_1} with constant

$$B_1 = C_M K_1 \left(\sup_{k \in \mathbb{Z}} \sum_{j \geq 0} 2^{js} \|\Delta_j(\sigma^k \widehat{\Psi})\|_{L^r} \right)^{\frac{r}{r_1}}.$$

We now interpolate using Theorem 1.1 (with $m = 1$) in [4]. We obtain

$$\|T_{\sigma_\theta}(f)\|_{(L^{p_0, 2})^{1-\theta}(L^{p_1, 2})^\theta} \leq C(p_0, p_1, p) B_0^{1-\theta} B_1^\theta \|f\|_{(L^{p_0}, L^{p_1})_\theta}.$$

Noting that $(L^{p_0, 2})^{1-\theta}(L^{p_1, 2})^\theta = L^{p, 2}$ and $(L^{p_0}, L^{p_1})_\theta = L^p$ (even when $p_0 = 1$, in which case L^{p_0} is replaced by H^1), we obtain the claimed assertion. \square

Lemma 5.2. *Using the notation of Theorem 5.1, for any $M > 0$ there is a constant C_M (also depending on the dimension n , on Ψ , and $\widehat{\Psi}$) such that for any $1 \leq q \leq \infty$ we have*

$$(27) \quad \left\| \widetilde{\Delta}_l(\widetilde{\Delta}_j(g) \widehat{\Psi} \widehat{\Psi}) \right\|_{L^q} \leq C_M 2^{-2(1-\frac{1}{q})\max(j, l)M} \|g\|_{L^q}$$

for all $l, j > 0$. We also have that for any $M > n$ there is a constant C_M such that

$$(28) \quad \left\| \widetilde{\Delta}_l(\widetilde{\Delta}_j(g) \widehat{\Psi} \widehat{\Psi}) \right\|_{L^1} \leq C_M 2^{-\max(j, l)(M-n)} \|g\|_{H^1}$$

Proof. The claimed estimate is obviously true when $q = 1$. So we prove it for $q = 2$ and derive (27) as a consequence of classical Riesz-Thorin interpolation theorem. Examining the Fourier transform of the operator in (27), matters reduce to computing the L^∞ norm of the function

$$(29) \quad \widehat{\Psi}(2^{-j}\xi) \int_{\mathbb{R}^n} \widehat{\Psi}(2^{-l}(\xi - \eta)) \phi(\eta) d\eta$$

where $\phi(\eta) = \Psi * \widetilde{\Psi}$ is a Schwartz function. Since the integral is over the set $|\xi - \cdot| \approx 2^l$, we estimate the absolute value of the expression in (29) by

$$C_M \left[\sup \left\{ \frac{1}{(1 + |\eta|)^M} : |\xi - \eta| \approx 2^l \right\} \right] \int_{\mathbb{R}^n} (1 + |\eta|)^{-M} d\eta$$

where $|\xi| \approx 2^j$. Notice that if $l > j + 10$, then $|\eta| \approx 2^l$, while if $j > l + 10$, then $|\eta| \approx 2^j$. These estimates yield the proof of (27).

We now turn our attention to (28). Using Fourier inversion, we write

$$\widetilde{\Delta}_l(\widetilde{\Delta}_j(g) \widehat{\phi})(x) = \int_{\mathbb{R}^n} \widehat{g}(\eta) \widehat{\Psi}(2^{-l}\eta) \int_{\mathbb{R}^n} \widehat{\Psi}(2^{-j}\xi) \phi(\xi - \eta) e^{2\pi i x \cdot \xi} d\xi d\eta.$$

We integrate by parts in the inner integral with respect to the operator $(I - \Delta_\xi)^N$ to obtain that the preceding expression is equal to

$$\sum_{\beta + \gamma = 2N} \frac{C_{\beta, \gamma}}{(1 + 4\pi^2 |x|^2)^N} \int_{\mathbb{R}^n} \widehat{g}(\eta) \widehat{\Psi}(2^{-l}\eta) \int_{\mathbb{R}^n} 2^{-j|\beta|} (\partial^\beta \widehat{\Psi})(2^{-j}\xi) (\partial^\gamma \phi)(\xi - \eta) e^{2\pi i x \cdot \xi} d\xi d\eta.$$

Since for $g \in H^1$ we have $|\widehat{g}(\xi)| \leq c \|g\|_{H^1}$ for all ξ and we deduce the estimate

$$|\widetilde{\Delta}_l(\widetilde{\Delta}_j(g) \widehat{\phi})(x)| \leq \frac{C_M \|g\|_{H^1}}{(1 + 4\pi^2 |x|^2)^N} 2^{ln} \sup_{|\eta| \approx 2^l} \int_{|\xi| \approx 2^j} \frac{d\xi}{(1 + |\xi - \eta|)^{2M}}$$

for $M > n$. We easily derive from this estimate the validity of (28). Note that in the case $j = 0$ the notation $|\xi| \approx 2^j$ should be interpreted as $|\xi| \lesssim 2$; likewise when $l = 0$. \square

Note that Proposition 1.2 is a consequence of Theorem 5.1 with initial estimates (16) and (17).

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