A NOTE ON SOBOLEV ISOMETRIC IMMERSIONS BELOW $W^{2,2}$ REGULARITY

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ABSTRACT. This paper aims to investigate the Hessian of second order Sobolev isometric immersions below the natural $W^{2,2}$ setting. We show that the Hessian of each coordinate function of a $W^{2,p}$, $p < 2$, isometric immersion satisfies a low rank property in the almost everywhere sense, in particular, its Gaussian curvature vanishes almost everywhere. Meanwhile, we provide an example of a $W^{2,p}$, $p < 2$, isometric immersion from a bounded domain of $\mathbb{R}^2$ into $\mathbb{R}^3$ that has multiple singularities.

1. INTRODUCTION

An isometric immersion of an n-dimensional domain to $\mathbb{R}^{n+m}$ is a mapping which preserves length of any curve that passes through each point in its domain and angles between any two of them. Precisely, it is defined as a Lipschitz mapping u that satisfies $Du^\top Du = \mathrm{I}_{n\times n}$ almost everywhere. The graph of such a mapping is called a flat surface, meaning it is isometrically equivalent to a flat domain. It is well-known in differential geometry since the end of 19th century that a smooth surface in $\mathbb{R}^3$ which is isometrically equivalent to the plane is developable: following the terminology of [7], it means that passing through any point on the surface there is a line segment lying on that surface. This terminology is used to indicate that it is developed from the plane without any stretching or compressing. In the mid-20th century, striking developments appeared showing that the rigidity or flexibility of isometric immersion relies heavily on the regularity of the surface. Among them there was the surprising work of Nash [14] and Kuiper [9], where they established the existence of a $\mathcal{C}^1$ isometric embedding of any Riemannian manifold into another manifold of one higher dimension, in particular, balls of arbitrarily small radius. As a contrast, Hartman and Nirenberg [5] showed that any $\mathcal{C}^2$ isometric immersion has its image as a developable surface. Even stronger result below $\mathcal{C}^2$ regularity has been obtained by Pogorelov [17], [16], where the key assumption is that the image under the gradient map has vanishing Lebesgue measure.

Key words and phrases. Isometric immersions; degenerate Monge-Ampère equation; Hessian determinant.

In the 21th century, due to its important role in nonlinear elasticity, analysts start looking into isometric immersions of Sobolev regularity. The natural setting are the intermediate classes between $\mathcal{C}^1$ and $\mathcal{C}^2$: the second order Sobolev spaces. A first positive results in this direction was due to Kirchheim [8] who proved that any $W^{2,\infty}$ function f in a bounded domain in $\mathbb{R}^2$ with almost everywhere vanishing Hessian determinant must be 1-developable. Following this method, Pakzad [15] improved $W^{2,\infty}$ to $W^{2,2}$ and applied this result to the rigidity of isometric immersions from a bounded domain of $\mathbb{R}^2$ into $\mathbb{R}^3$. Pakzad's generalization relies on an important observation: such an isometric immersion satisfies the property that the determinant of the Hessian of each coordinate function vanishes in the L^1 sense, or, vanishes as a measure. Liu and Pakzad [11] generalized this result to isometric immersion from a bounded domain in $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$, still under $W^{2,2}$ regularity assumption, based on the same observation that such a map has all the 2×2 minors of the Hessian of each coordinate function vanishing in the L^1 sense, or again, as a measure. Therefore, it is natural to expect that $W^{2,2}$ is the lowest Sobolev regularity for an isometric immersion to be developable and this is indeed the case. Consider, for example, when $n = 2$, the map $u(r\cos\theta, r\sin\theta) := (\frac{r}{2}\cos(2\theta), \frac{r}{2}\sin(2\theta), \frac{\sqrt{3}}{2}r)$, which sends the unit disc to the cone in $\mathbb{R}^3$ with a singularity at the origin: it is an isometric immersion of class $W^{2,p}, p < 2$, but it fails to be affine on any of the line segments passing through the origin. More generally, Jerrard and Pakzad [7] proved that any $W^{2,p}$ isometric immersion of co-dimension $m, m \in \{1, ..., n-1\}$ is weakly $(n-m+1)$-developable whenever $p \geq m$. Roughly speaking, weak developability allows for a point to belong to two m-dimensional hyperplanes, along each of which the gradient of the isometry f is $\mathcal{H}^m$ a.e. constant, but the corresponding values of ∇f do not meet, see [7] for the precise definition. Again, their first step was to prove that such an isometric immersion of co-dimension k has all the $m \times m$ minors of the Hessian of each coordinate function vanishing as an L^1 function. However, it is worth pointing out that in their proof, $W^{2,2}$ regularity was enough to obtain that all these $m \times m$ minors vanish almost everywhere. It is the $W^{2,m}$ assumption that excludes the possibility of them being singular measures.

Motivated by the result of [7], Liu and Malý [10] constructed a strictly convex function of Sobolev class $W^{2,p}, p < m$, whose Hessian has rank strictly less than m almost everywhere for any $m \in \{1, ..., n-1\}$. In particular, it is not affine on any line segment, a strong contrast to Jerrard and Pakzad's result. In fact, our original intention was to construct a more "complicated" example of isometric immersion than the cone with a singularity at the origin. However, counterexamples for Sobolev isometric immersion with second order Sobolev differentiability are extremely difficult to construct. In fact it has been wondering since Pakzad's result in 2004 [15] whether there exists an example of $W^{2,p}, p < 2$, isometric immersion of a flat domain in $\mathbb{R}^2$ into $\mathbb{R}^3$ that has more than one singularity (The exact meaning of the notion of singularity in this context will be defined soon). In recent years, using the technique of convex integration, there have been successful constructions of $\mathcal{C}^{1,\alpha}$ isometric immersions for some $\alpha > 0$ into higher dimensional spaces with the same Nash-Kuiper flexibility (in particular, they are nowhere affine). We refer the readers to the pioneering work of Conti, De Lellis and Székelyhidi [2] for $\alpha < 1/3$ and a series of work by De Lellis, Székelyhidi and others for higher and higher α. The convex integration technique, to our knowledge, does not apply in the framework of $W^{2,p}$ for any p due to the fact that it blows up the second order derivative uniformly. For a map to be in $W^{2,p}$ for any p, the blowing up must be concentrated.

The strictly convex Sobolev function constructed in [10], even though not an example of isometric immersions, is a related one, as connected by our first result:

Theorem 1.1. *Let Ω be a domain in $\mathbb{R}^n$ and let $u = (u^1, ..., u^{n+m}) \in W^{2,1}(\Omega, \mathbb{R}^{n+m})$, $m \in \{1, ..., n-1\}$, be an isometric immersion. Then each coordinate function $u^\nu, \nu \in \{1, ..., n+m\}$, satisfies the Hessian rank inequality* $\mathrm{rank} D^2 u^\nu \leq m$ *almost everywhere.*

Differently from [7], where the same Hessian rank inequality was proved for isometric immersion under $W^{2,2}$ regularity, the integrability of the second gradient does not allow us to control the 2×2 minors of the Hessian. Therefore, any computation in the integral sense such as integration by parts cannot be carried on. To overcome the lack of integrability, we borrowed a technique from [13], where they proved a low rank property for Sobolev mapping via slicing and lower dimensional pullback Sobolev differential forms, the same technique was also used in [1] and [12] in the context of weak contact equations.

Let us discuss a bit further regularity questions. For simplicity, we restrict our attention to the planar case. The distributional Hessian of a function $u \in W^{1,2}(\Omega)$ (where $\Omega \subset \mathbb{R}^2$ is open) is defined as

$$\mathrm{Det}\, D^2 u := \frac{1}{2} \mathrm{curl}^\top \mathrm{curl} Du \otimes Du,$$

see [7, 1.3]. If $f \in W^{2,p}(\Omega, \mathbb{R}^3)$ is an isometric immersion, then our result implies that each coordinate function v of f satisfies the degenerate Monge-Ampère equation $\det D^2 v = 0$ pointwise a.e. However, it can violate the equation $\mathrm{Det}\, D^2 v = 0$. For example, recall the map $f(r\cos\theta, r\sin\theta) := (\frac{r}{2}\cos(2\theta), \frac{r}{2}\sin(2\theta), \frac{\sqrt{3}}{2}r)$, which sends the unit disc B to the cone in $\mathbb{R}^3$ with a singularity at the origin; it is an isometric immersion of class $W^{2,p}$, $p < 2$, but the distributional Hessian of $f_3 : x \mapsto \frac{\sqrt{3}}{2}|x|$ is a constant multiple of the Dirac delta measure

δ_0. Indeed, under this regularity the distributional Hessian of f_3 can be computed as the distributional Jacobian of Df_3. It is well known that the distributional Jacobian of $x \mapsto \frac{x}{|x|}$ is $\pi\delta_0$.

Thus, the failure of $\operatorname{Det} D^2 v = 0$ indicates the singularity at the origin. However, there is an example of a $W^{2,1}$ function u such that $\operatorname{Det} D^2 u$ vanishes and u still does not deserve to be called regular; see the discussion around [6, (6.2)]. To give an appropriate definition of singularity, we associate a current G_{Du} with the graph of Du (Definition 5.1) and say that the $W^{2,1}$-solution of the degenerate Monge-Ampère equation is *regular* in an open set U if the boundary of the current G_{Du} in $U \times \mathbb{R}^2$ vanishes. (For the definition of the boundary of a current and related discussion see [4, Section 2.3].) *Singular points* are those that do not belong to the maximal domain of regularity. In case of an isolated singularity then there exists a neighborhood U of x such that the boundary of G_{Du} in $U \times \mathbb{R}^2$ is concentrated in $\{x\} \times \mathbb{R}^2$.

In the regular case, u is a Monge-Ampère function in the sense of [6] (or [3]) and a rigidity result due to Jerrard [6, Theorem 6.1] can be applied. In Proposition 5.2 we present its simplified version taking into account that our function u belongs to $W^{2,1}$, so that we do not need to take care of the singular part fo $D^2 u$. On the other hand, we get a slightly stronger conclusion that in [6].

Our example from [10] is a "nowhere regular" $W^{2,1}$ solution of the (pointwise) null Monge-Ampére equation. In contrast, all known examples of $W^{2,1}$ isometric immersions have been similar to the cone example above, with a single point singularity. Here we present an example with a prescribed finite number of singularities:

Theorem 1.2. *Let m be an positive integer. Then there exists a bounded open set $\Omega \subset \mathbb{R}^2$ and an isometric immersion $f \in W^{2,p}(\Omega, \mathbb{R}^3)$ such that f has m singularities.*

Note that f in this example satisfies the conclusion of Theorem 1.1. On the other hand, by the developability properties of $W^{2,2}$ isometric immersions in [15], its $W^{2,2}$-norm must blow up around any singular point z.

We must note that the domain in our example is non-convex. We do not know whether there is an example of an isometric immersion defined on a convex domain with multiple singularities.

2. Sobolev pullback of a differential form: $n = 2$

In this section we apply an useful tool from [13], which is based on a method developed in [1] and [12]. Let $\Omega \subset \mathbb{R}^2$ be an open set and $f_1, \dots, f_N, g_1, \dots, g_N \in W^{1,1}(\Omega)$. Then the sum

$$f_1 \nabla g_1 + \cdots + f_N \nabla g_N$$

can be interpreted as the pullback of the differential form

$$\omega = x_1\, dy_1 + \cdots + x_N\, dy_N$$

on $\mathbb{R}^{2N}$ under the Sobolev mapping $(f, g) \in W^{1,1}(\Omega, \mathbb{R}^{2N})$. The pullback of the differential

$$d\omega = dx_1\, dy_1 + \cdots + dx_N\, dy_N$$

is the function

$$\det(\nabla f_1, \nabla g_1) + \cdots + \det(\nabla f_N, \nabla g_N).$$

Therefore we can use [13, Theorem 3.2] and deduce immediately the following result.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set and $f_1, \dots, f_N, g_1, \dots, g_N \in W^{1,1}(\Omega)$. Assume that*

$$f_1 \nabla g_1 + \cdots + f_N \nabla g_N = 0.$$

Then

$$\det(\nabla f_1, \nabla g_1) + \cdots + \det(\nabla f_N, \nabla g_N) = 0.$$

3. n-dimensional case

In this section we derive the n-dimensional counterpart of Lemma 2.1 by the slicing method (see also [1], [12]). Here and in the sequel we use the notation

$$\begin{aligned}\partial_k f &= \frac{\partial f}{\partial x_k},\\ \partial_{k,\ell} f &= \frac{\partial^2 f}{\partial x_k \partial x_\ell},\\ \langle f, g\rangle &= f_1 g_1 + \cdots + f_N g_N.\end{aligned}$$

Lemma 3.1. *Let $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be an open set and $f, g \in W^{1,1}(\Omega; \mathbb{R}^N)$. Assume that*

$$\langle f, \partial_k g\rangle = 0 \qquad \text{a.e. in } \Omega, \quad k = 1, \dots, n.$$

Then

$$\langle \partial_k f, \partial_\ell g\rangle - \langle \partial_\ell f, \partial_k g\rangle = 0 \qquad \text{a.e. in } \Omega, \quad k, \ell = 1, \dots, n.$$

Proof. For $n = 2$, the claim is trivial if $k = \ell$ and reduces to Lemma 2.1 if $k \neq \ell$. Now let $n \geq 3$. It suffices to consider any n-dimensional cube $Q \subset\subset \Omega$. Let $Q_{k\ell}$ be the projection of Q onto the linear space spanned by $\mathbf{e}_k$ and $\mathbf{e}_\ell$ and $\hat{Q}_{k\ell}$ be the projection of Q onto the space $\{x \in \mathbb{R}^n \colon x_k = 0,\, x_\ell = 0\}$. Fix $z \in \hat{Q}_{k\ell}$ and denote

$$f_z(y) := f(z + y), \quad g_z(y) := g(z + y), \qquad y \in Q_{k\ell}.$$

For a.e. $z \in \hat{Q}_{k\ell}$, Fubini's theorem gives $f_z,\, g_z \in W^{1,1}(Q_{k\ell})$ and moreover,

$$\partial_k f_z = (\partial_k f)_z, \quad \partial_k f_z = (\partial_k f)_z, \qquad k = 1, \dots, n.$$

Then from what has been establishes in 2-dimensional case,

$$\langle (\partial_k f)_z, (\partial_\ell g)_z\rangle - \langle (\partial_\ell f)_z, (\partial_k g)_z\rangle = 0 \qquad \text{a.e. in } Q_{k\ell}.$$

In another word, for a.e. $z \in \hat{Q}_{k\ell}$ and a.e. $y \in Q_{k\ell}$,

$$\langle \partial_k f(z+y), \partial_\ell g(z+y)\rangle - \langle \partial_\ell f(z+y), \partial_k g(z+y)\rangle = 0$$

for all $i, j \in \{1, ..., n\}$. By Fubini's theorem, this implies

$$\langle \partial_k f, \partial_\ell g\rangle - \langle \partial_\ell f, \partial_k g\rangle = 0 \quad \text{a.e. in } Q. \tag{3.1}$$

This concludes the proof. □

4. Hessian of isometric immersion

Proof of Theorem 1.1. Recall that the $W^{2,1}$ Sobolev isometric immersions of co-dimension m are mappings $u \in W^{2,1}(\Omega, \mathbb{R}^{n+m})$ satisfying

$$Du^\top Du = \mathrm{I}_{n\times n} \text{ a.e. in } \Omega \tag{4.1}$$

Note that this condition implies immediately that u is 1-Lipschitz.

Following [7], it suffices to prove the following identity

$$\langle \partial_{i,k} u, \partial_{j,\ell} u\rangle - \langle \partial_{i,\ell} u, \partial_{j,k} u\rangle = 0 \text{ a.e. in } \Omega \tag{4.2}$$

Once this identity is established, Theorem 1.1 follows the same argument as [7, Proposition 2.1].

We can rewrite (4.1) as

$$\langle \partial_i u, \partial_j u\rangle = \delta_{ij} \quad \text{a.e. in } \Omega, \quad i, j \in \{1, ..., n\}. \tag{4.3}$$

Since $Du \in W^{1,1}(\Omega, \mathbb{R}^{n+1}) \cap L^\infty(\Omega, \mathbb{R}^{n+1})$, we can differentiate (4.3) using the product rule to obtain

$$\langle \partial_{i,k} u, \partial_j u\rangle + \langle \partial_i u, \partial_{j,k} u\rangle = 0 \quad \text{a.e. in } \Omega, \quad i, j, k \in \{1, ..., n\}. \tag{4.4}$$

Permutation of indices i, j, k yields,

$$\langle \partial_{i,j} u, \partial_k u\rangle + \langle \partial_i u, \partial_{k,j} u\rangle = 0 \quad \text{a.e. in } \Omega, \quad i, j, k \in \{1, ..., n\}. \tag{4.5}$$

$$\langle \partial_{k,i} u, \partial_j u\rangle + \langle \partial_k u, \partial_{j,i} u\rangle = 0 \quad \text{a.e. in } \Omega, \quad i, j, k \in \{1, ..., n\}. \tag{4.6}$$

Using the fact that $\partial_{i,j} u = \partial_{j,i} u$ for all i, j, we add (4.4) and (4.5), then subtract (4.6) to obtain,

$$\langle \partial_i u, \partial_{j,k} u\rangle = 0 \quad \text{a.e. in } \Omega, \quad i, j, k \in \{1, ..., n\}. \tag{4.7}$$

Now, that (4.7) implies (4.2) indeed follows from Lemma 3.1 if we set $f = \partial_i u$, $g = \partial_j u$. □

5. Regularity in dimension 2

To study singularity in a broader setting we need to define non-convex Monge-Ampère functions. We refer to [6] for terminology and notation concerning currents and, in particular, Cartesian currents, see also [4]. For simplicity we restrict our attention to a $W^{2,1}$-isometric immersion f of a planar domain Ω to $\mathbb{R}^3$. Let u be one of coordinate components of f. In view of our Theorem 1.1, u is a pointwise solution of the null Monge-Ampère equation, so that, in particular the Hessian determinant of D^2u is integrable, whereas D^2u itself is integrable as well by the $W^{2,1}$ assumption. Now, we define the corresponding Cartesian current.

Definition 5.1. Let $\Omega \subset \mathbb{R}^2$ be an open set and $u \in W^{2,1}(\Omega)$. Suppose that $\det D^2u \in L^1(\Omega)$. The current G_{Du} of integration over the graph of Du is defined by

$$\begin{aligned} G_{Du}(\phi) = \int_\Omega \Big(& \phi^{(1,2),\emptyset}(x, Du(x)) - \phi^{1,1}(x, Du(x)) D_{21}u(x) \\ & + \phi^{2,1}(x, Du(x)) D_{11}u(x) - \phi^{1,2}(x, Du(x)) D_{22}u(x) \\ & + \phi^{2,2}(x, Du(x)) D_{12}u(x) \\ & + \phi^{\emptyset,(1,2)}(x, Du(x)) \det D^2u(x) \Big)\, dx, \quad \phi \in C_c^\infty(\Omega). \end{aligned} \tag{5.1}$$

In fact, the last term vanishes in our case by the null Monge-Ampère equation. Recall that u is regular if the boundary of G_{Du} in $\Omega \times \mathbb{R}^2$ vanishes. The following result is a simplified version of [6, Theorem 6.1]. Note, however, that in the original version, instead of continuous differentiability it is only claimed that all points of Ω are Lebesgue points for Du.

Proposition 5.2. *Let u be a regular solution of the null Monge-Ampère equation in $\Omega \subset \mathbb{R}^2$. Then u is continuously differentiable and for each $x \in \Omega$ at least one of the following statements holds:*

(a) *Du is constant on a neighborhood of x,*

(b) *there exists a line segment ℓ_x passing through x with each of end-points either at infinity or on $\partial\Omega$, such that Du is constant along ℓ_x.*

Proof. It is only the continuity of the derivative which remains to be proved. Consider a point $z \in \Omega$. There is nothing to prove if Du is constant on the neighborhood of z. In the case (b), there is a line segment ℓ passing through z such that Du is constant on ℓ. We may assume that $z = 0$ and ℓ is a part of the x_1-axis. Also, there is $r > 0$ such that $(-r, r)^2 \subset \Omega$. Choose $0 < \delta < r$ and $x \in (-\delta, \delta)^2$. Then there is a line segment ℓ_x passing through x such that Du is constant on ℓ_x. If $Du(x) \neq Du(0)$ (which is the only case that matters), ℓ_x does not cross ℓ inside $[-r, r]^2$. Hence there exists a linear polynomial

$p\colon \mathbb{R} \to \mathbb{R}$ such that $\ell_x = \{(t, p(t))\colon t \in (-r, r)\}$ and (as a result of an elementary geometric observation)

$$|p(t)| \le \frac{2r\delta}{r-\delta}, \qquad t \in (-r, r).$$

For simplicity assume $\delta < r/2$, then the estimate $|p(t)| \le 4\delta$ follows. For a.e. $t \in (-r, r)$ we have

$$|Du(x) - Du(0)| = |Du(t, p(t)) - Du(t, 0)| \le \int_{-4\delta}^{4\delta} |D^2u(t, s)|\, ds,$$

and by Fubini theorem

$$2r|Du(x) - Du(0)| \le \int_{(-4\delta,4\delta)\times(-r,r)} |D^2u(y)|\, dy$$

which tends to 0 as $\delta \to 0_+$. It follows that Du is continuous at 0. $\square$

6. Example of Sobolev isometric immersion with multiple singularities

In what follows, $\mathbb{S}$ will be the unit sphere in $\mathbb{R}^3$, whereas $B(z, R)$ will denote 2-dimensional balls (discs) centered at z with radius R.

Lemma 6.1. *Let $\gamma\colon \mathbb{R} \to \mathbb{S}$ be a 2π-periodic $\mathcal{C}^2$ curve parametrized by its arclength. Define the mapping $f\colon B(0, R) \to \mathbb{R}^3$ as*

$$f(r\cos\alpha,\ r\sin\alpha) = r\gamma(\alpha), \qquad 0 \le r < R,\ -\pi \le \alpha \le \pi.$$

Then $f \in W^{2,p}(B(0, R), \mathbb{R}^3)$ for each $1 \le p < 2$ and it is an isometric immersion.

Proof. We easily verify that $f \in W^{2,p}(B(0, R), \mathbb{R}^3)$. We compute

$$\begin{aligned}
\gamma(\alpha) &= \frac{\partial}{\partial r}\,(r\gamma(\alpha)) \\
&= \frac{\partial f}{\partial x_1}(r\cos\alpha,\ r\sin\alpha)\ \cos\alpha + \frac{\partial f}{\partial x_2}(r\cos\alpha,\ r\sin\alpha)\ \sin\alpha, \\
r\gamma'(\alpha) &= \frac{\partial}{\partial\alpha}\,(r\gamma(\alpha)) \\
&= -\frac{\partial f}{\partial x_1}(r\cos\alpha,\ r\sin\alpha)\, r\sin\alpha + \frac{\partial f}{\partial x_2}(r\cos\alpha,\ r\sin\alpha)\, r\cos\alpha.
\end{aligned} \tag{6.1}$$

We use these equations to express partial derivatives of f at $x \ne 0$. Using invariance with respect to rotation, we may assume that $x = (r, 0)$ with $r > 0$. Then we solve (6.1) as

$$\begin{aligned}
\frac{\partial f}{\partial x_1}(r, 0) &= \gamma(0), \\
\frac{\partial f}{\partial x_2}(r, 0) &= \gamma'(0).
\end{aligned}$$

Then

$$\nabla f(r,0)^\top \nabla f(r,0) = \begin{pmatrix} |\gamma(0)|^2, & \gamma(0)\cdot\gamma'(0) \\ \gamma(0)\cdot\gamma'(0), & |\gamma'(0)|^2 \end{pmatrix}.$$

Now, we verify the following:

$|\gamma(0)| = 1$, as $\gamma(0) \in \mathbb{S}$.

$|\gamma'(0)| = 1$, as γ is parametrized by its arclength.

Finally, $\gamma(0)\cdot\gamma'(0) = 0$, as $\gamma(0)$ is a normal vector with respect to $\mathbb{S}$ whereas $\gamma'(0)$ is tangential. Therefore, $\nabla f(r,0)^\top \nabla f(r,0)$ is the identity matrix.

□

Lemma 6.2. *For any $0 < \alpha < \pi/2$ there exists a 2π-periodic $\mathcal{C}^2$ closed curve $\gamma : \mathbb{R} \to \mathbb{S}$ parametrized by its arclength such that $\gamma(t) = (\cos t, \sin t, 0)$ for $t \in (-\alpha, \alpha)$, and $\gamma_3(\pi) > 0$.*

Proof. Let $0 < s < 1$ be fixed and $\eta\colon \mathbb{R} \to \mathbb{R}$ be an infinitely smooth function such that $\eta(\tau) = 0$ for $\tau \le 0$, $\eta(\tau) = 1$ for $\tau \ge 2s$ and $0 < \eta'(\tau) < 1/s$ if $0 < \tau < 2s$. Given $0 < \beta < \pi$, define

$$\begin{aligned} \psi^1_\beta(t) &= \cos(s\eta(t-\beta))\,\cos\big(t + \eta(t-\beta)(2\beta + 2s - 2t)\big), \\ \psi^2_\beta(t) &= \cos(s\eta(t-\beta))\,\sin\big(t + \eta(t-\beta)(2\beta + 2s - 2t)\big), \\ \psi^3_\beta(t) &= \sin(s\eta(t-\beta)), \qquad t \in [0,\ 2\beta + 2s], \end{aligned}$$

and extend ψ_β to a $(4\beta + 4s)$-periodic function on $\mathbb{R}$ such that ψ^1_β and ψ^3_β are even whereas ψ^2_β is odd. Then ψ_β is a $\mathcal{C}^2$ curve. Further, we claim that ψ'_β nowhere vanishes; this property guarantees that after a reparametrization by the arclength the $\mathcal{C}^2$ smoothness is not lost. Indeed, if $\eta = 0$ or $\eta = 1$, then the equations $\psi^{1,2}_\beta$ describe an uniform motion along a circle, whereas obviously $(\psi^3_\beta)' \ne 0$ if $0 < \eta < 1$. An easy computation shows that $|\psi'_\beta| \le C$ with constant C independent of s, and $|\psi'_\beta(t)| \le 1$ if $|t| \le \beta$ or $\beta + 2s \le |t| \le 2\beta + 2s$. It follows that the length ℓ_β of ψ_β over $[-2\beta - 2s,\ 2\beta + 2s]$ is estimated by $4\beta + 4Cs$. Now, we may assume that s is so small that $4\alpha + 4Cs < 2\pi$. Then $\ell_\alpha < 2\pi$. On the other hand, $\psi_{\pi/2}$ connects antipodal points on $(-\pi/2, \pi/2)$ and on $(\pi/2, 3\pi/2 + 4s)$ and thus $\ell_{\pi/2} > 2\pi$. Therefore there exists an intermediate β such that $\ell_\beta = 2\pi$. If we reparametrize this ψ_β by its arclength, we obtain a curve γ with the desired property. □

Lemma 6.3. *Let K be a convex cone in $\mathbb{R}^2$ (not a halfplane) with vertex at z, $R > 0$ and $1 \le p < 2$. Then there exists an isometric immersion $f \in W^{2,p}(B(z,R), \mathbb{R}^3)$ such that f is singular at z and*

$$x \in K \cap B(z,R) \implies f(x) = (x_1, x_2, 0).$$

Proof. Without loss of generality we may assume that $z = 0$ and

$$K = \{x \in \mathbb{R}^2 \colon x\cdot\mathbf{e}_1 < |x|\cos\alpha\}$$

for some $\alpha \in (0, \pi/2)$. By Lemma 6.1, the desired property is satisfied by the mapping

$$f(r\cos\alpha,\ r\sin\alpha) = r\gamma(\alpha), \qquad 0 \le r < R,\ -\pi \le \alpha \le \pi,$$

where γ is as in Lemma 6.2. Appealing to Proposition 5.2, we infer that the regularity of f fails at 0 as Df is not continuous at 0. □

Proof of Theorem 1.2. Let $Q \subset \mathbb{R}^2$ be a convex m-angle with vertices $z_1, \dots, z_m$. Let $R > 0$ be a radius which makes the discs $\overline{B}(z_j, R)$, $j = 1, \dots, m$ pairwise disjoint. Using Lemma 6.3, we construct isometric immersions $f_j \in W^{2,p}(B(z_j, R), \mathbb{R}^3)$ such that $f_j(x) = (x_1, x_2, 0)$ if $x \in Q$ and each f_j is singular at z_j. Now, we merge the constructed mappings to

$$f(x) = \begin{cases} f_j(x), & x \in B(z_j, R), \\ (x_1, x_2, 0), & x \in Q. \end{cases}$$

This defines f on $\Omega := Q \cup \bigcup_{j=1}^m B(z_j, R)$. □

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