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Reduction theorems for Sobolev embeddings into the spaces of Hölder, Morrey and Campanato type

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Let X be a rearrangement-invariant Banach function space on Q where Q is a cube in $\mathbb{R}^n$ and let $V^1X(Q)$ be the Sobolev space of real-valued weakly differentiable functions f satisfying $|\nabla f| \in X(Q)$. We establish a reduction theorem for an embedding of the Sobolev space $V^1X(Q)$ into spaces of Campanato, Morrey and Hölder type. As a result we obtain a new characterization of such embeddings in terms of boundedness of a certain one-dimensional integral operator on representation spaces.

1 Introduction

Throughout the paper we assume that Q and Q' are cubes in $\mathbb{R}^n$, $n \geq 2$, which, without loss of generality, have edges parallel to axes. The n-dimensional Lebesgue measure of Q is assumed throughout to be equal to one. By $V^{1,p}(Q)$, $1 \leq p \leq \infty$, we denote the *Sobolev space* consisting of all real-valued weakly differentiable functions defined on Q whose first-order derivatives belong to $L^p(Q)$. The classical Morrey theorem tells us that if $p > n$, then every function from $V^{1,p}(Q)$ is almost everywhere equal to a *Hölder-continuous function*. This fact can be stated as a Sobolev-type embedding

$$V^{1,p}(Q) \hookrightarrow C^{0,\alpha}(Q), \quad \alpha = 1 - \tfrac{p}{n},$$

where, for $\alpha \in (0, 1]$, the symbol $C^{0,\alpha}(Q)$ denotes the collection of all Hölder continuous functions, that is, functions for which the seminorm

$$\|f\|_{C^{0,\alpha}(Q)} = \sup_{\substack{x,y\in Q\\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}}$$

is finite.

Given a continuous function $\varphi : (0, \infty) \to (0, \infty)$, the *generalized Hölder space* $C^{0,\varphi}(Q)$ is equipped with the seminorm

$$\|f\|_{C^{0,\varphi}(Q)} = \sup_{\substack{x,y\in Q\\ x\neq y}} \frac{|f(x) - f(y)|}{\varphi(|x-y|)}$$

and is defined as a set of all functions f on Q such that $\|f\|_{C^{0,\varphi}(Q)} < \infty$.

Hölder type spaces constitute an important notion in analysis with a wide range of applications. They are particularly useful for instance in the theory of regularity of solutions of partial differential equations and also in the calculus of variations.

Being determined by a pointwise smoothness of functions, however, the Hölder spaces often, in particular in connection with certain special problems, have to be complemented by the function spaces whose norms measure

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the mean oscillation of functions instead of the pointwise one. This fact leads us to the spaces of *Campanato* and *Morrey* type. It is quite well known that Hölder regularity of solutions to (mostly elliptic) partial differential equations, as well as the corresponding a priori L^p-estimates, are usually obtained through a seemingly different type of regularity, namely that in spaces determined by a control over mean oscillation, combined with embeddings and other relations one has between these spaces and Hölder spaces. An excellent introduction to this topic is given in [2], where also the role of vanishing versions of Campanato spaces (in particular, VMO) in regularity theory is studied, and plenty of references are given therein.

Given a continuous function $\varphi : (0,\infty) \to (0,\infty)$, the *Campanato space* $L^C_\varphi(Q)$ is defined as the space of all real-valued measurable functions f on Q for which the seminorm

$$\|f\|_{L^C_\varphi(Q)} = \sup_{Q'\subset Q} \frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_{Q'} |f(x) - f_{Q'}|\,dx$$

is finite and $f_{Q'} = |Q'|^{-1}\int_{Q'} f(y)\,dy$, the mean value of f over Q'. Here and throughout, $|E|$ denotes the n-dimensional Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. A particularly important instance of a Campanato space is the space BMO of functions with *bounded mean oscillation*, occasionally in literature called also the *John–Nirenberg space*, which corresponds to the choice $\varphi(t) = 1$ for every $t \in (0,1)$ in the definition of $L^C_\varphi(Q)$; for the reference see e.g. [11].

The *Morrey space* $L^M_\varphi(Q)$ is defined as the space of all functions f as above such that the norm

$$\|f\|_{L^M_\varphi(Q)} = \sup_{Q'\subset Q} \frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_{Q'} |f(x)|\,dx$$

is finite.

One of the principal tools for the study of various types of Sobolev embeddings and related problems is the technique based on the so-called *reduction theorem*. By a reduction theorem we usually mean a result that provides a necessary and sufficient condition for a Sobolev embedding in terms of the boundedness of a one-dimensional (usually integral) operator on the so-called *representation spaces* of the rearrangement-invariant spaces in question. This technique has been particularly flourishing in the last fifteen years and has proved to be an indispensable tool in order to obtain sharp results concerning functions in Sobolev-type spaces, their traces, and so on (see, for instance, [5], [6], [9], [12], and the references therein). The techniques used in order to establish the results in this area are naturally restricted to the rearrangement-invariant setting. From this point of view, the involvement of spaces of Hölder, Campanato and Morrey type is very special, since these spaces are not rearrangement-invariant. Hence special, rather fine techniques had to be invented in order to deal with these spaces.

In [4], first-order Sobolev embeddings into spaces of Hölder, Campanato and Morrey type were studied. In particular, optimal target spaces in Sobolev embeddings were established in the framework of all rearrangement-invariant spaces. The special case of the space BMO and some related problems had been treated in the earlier work [3]. In neither of these papers however, a reduction theorem was established. The techniques used in both [3] and [4] avoided the use of a reduction theorem, leaving it as an open question. Although some of the related embedding problems concerning Hölder spaces and their generalizations were resolved in [7], [8] and [10], none of the developed techniques involved reduction theorems to the extend presented in this paper.

Our main aim in this paper is to establish such a reduction theorem. It will provide us with a completely new characterization of an embedding of the Sobolev space $V^1X(Q)$ into each of the spaces of Hölder, Campanato or Morrey type.

We shall now state the main results. To this end, let us first briefly recall some commonly used notation and notions from the framework of the rearrangement-invariant Banach function spaces (*r.i. spaces* for short). The exact definitions can be found in the following section.

By an r.i. space we understand a Banach function space possessing the property that the norm of a function f depends only on its *nonincreasing rearrangement* f^*. Moreover, for every r.i. space $X(Q)$, there exists a *representation space* $\overline{X}(0,1)$ which is such that the norm of f in $X(Q)$ equals the norm of f^* in $\overline{X}(0,1)$. Let us note that $\overline{X}(0,1)$ is also an r.i. space. We also need the concept of the *associate space* of $X(Q)$ commonly denoted by $X'(Q)$. The associate space of $X(Q)$ coincides with the topological dual of $X(Q)$ in many cases and in the framework of r.i. spaces it is in many cases a suitable substitute for it. We also make use of notation f^{**} for the maximal function of f^*, $n' = \frac{n}{n-1}$ for the conjugate or dual parameter of n and χ_E for the characteristic function of a given set E. Moreover, throughout the paper, the letter C denotes various constants whose value may change from line to line. The constants however remain independent of appropriate quantities.

Our main goal is to characterize embeddings of Sobolev spaces to spaces with controlled oscillation in terms of boundedness of the *reduction operator* H, defined by

$$H : f \longmapsto \int_t^1 r^{-\frac{1}{n'}} f(r)\, dr.$$

The operator H can be considered a mapping from the intersection of the one-dimensional representation space of $X(Q)$ and $\mathcal{M}_+(0,1)$, the set of all a.e. finite nonnegative Lebesgue measurable functions, to an appropriate one-dimensional variant of one of the spaces in question.

Three theorems that follow summarize our main results, but first let us mention a few remarks. Unlike the other two cases, the Hölder-space variant of the reduction theorem requires an additional restriction imposed on φ, in particular that it should be nondecreasing. However, this restriction is quite a natural requirement for this space and has no restrictive effect on the result. For the Morrey-space variant, we may assume, without loss of generality, that

$$\inf_{t>0} \varphi(t) > 0, \tag{1.1}$$

since $L^M_\varphi = \{0\}$ whenever (1.1) does not hold.

Theorem 1.1 *Let $X(Q)$ be an r.i. space and let $\varphi : (0,\infty) \to (0,\infty)$ be a continuous nondecreasing function. Then the following assertions are equivalent:*

(i) *A positive constant C exists such that*

$$\|f\|_{C^{0,\varphi}(Q)} \le C\, \|\nabla f\|_{X(Q)}$$

for every $f \in V^1X(Q)$.

(ii) *A positive constant C exists such that*

$$\|Hf\|_{C^{0,\overline{\varphi}}(0,1)} \le C\, \|f\|_{\overline{X}(0,1)}$$

for every $f \in \mathcal{M}_+(0,1)$, where $\overline{\varphi}(t) = \varphi(t^{\frac{1}{n}})$.

Theorem 1.2 *Let $X(Q)$ be an r.i. space and let $\varphi : (0,\infty) \to (0,\infty)$ be a continuous function. Then the following assertions are equivalent:*

(i) *A positive constant C exists such that*

$$\|f\|_{L^C_\varphi(Q)} \le C\, \|\nabla f\|_{X(Q)}$$

for every $f \in V^1X(Q)$.

(ii) *A positive constant C exists such that*

$$\|Hf\|_{L^C_\varphi(0,1)} \le C\, \|f\|_{\overline{X}(0,1)}$$

for every $f \in \mathcal{M}_+(0,1)$.

Theorem 1.3 *Let $X(Q)$ be an r.i. space and let $\varphi : (0,\infty) \to (0,\infty)$ be a continuous function satisfying* (1.1). *Then the following assertions are equivalent:*

(i) *A positive constant C exists such that*

$$\|f\|_{L^M_\varphi(Q)} \le C\big(\|f\|_{L^1(Q)} + \|\nabla f\|_{X(Q)}\big)$$

for every $f \in V^1X(Q)$.

(ii) *A positive constant C exists such that*

$$\|Hf\|_{L^M_\varphi(0,1)} \le C\, \|f\|_{\overline{X}(0,1)}$$

for every $f \in \mathcal{M}_+(0,1)$.

2 Preliminaries

In this section we give detailed definitions and recall some basic properties of the notions involved. We shall also state and prove a theorem which will be essential for the proofs of the main results in the next section. We give only a brief insight into the framework of rearrangement. The extensive and detailed knowledge about rearrangements, Banach function spaces and r.i. spaces can be found in [1].

Definition 2.1 Let f be a real-valued measurable function on Q. Then f^* denotes the *nonincresing rearrangement* of f given by

$$f^*(t) = \sup\{s \geq 0 : |\{x \in Q : |f(x)| > s\}| > t\}, \quad 0 < t < 1,$$

and f^{**} denotes the *maximal function* of f^* given by

$$f^{**}(t) = t^{-1} \int_0^t f^*(r)\,dr, \quad 0 < t \leq 1.$$

A *rearrangement invariant Banach function space* $X(Q)$ is a space endowed with the norm $\|.\|_{X(Q)}$ satisfying the following conditions.

(RI$_1$) If $f, g \in X(Q)$ and $0 \leq g(x) \leq f(x)$ for a.e. $x \in Q$, then $\|g\|_{X(Q)} \leq \|f\|_{X(Q)}$.
(RI$_2$) If $f, f_n \in X(Q)$ and $0 \leq f_n(x) \nearrow f(x)$ for a.e. $x \in Q$, then $\|f_n\|_{X(Q)} \nearrow \|f\|_{X(Q)}$.
(RI$_3$) $\|1\|_{X(Q)} < \infty$.
(RI$_4$) There is a positive constant $C > 0$ such that $\int_Q f(x)\,dx \leq C\,\|f\|_{X(Q)}$ for every $f \in X(Q)$.
(RI$_5$) If $f, g \in X(Q)$ and $f^*(x) = g^*(x)$ for every $x \in Q$, then $\|g\|_{X(Q)} = \|f\|_{X(Q)}$.

One of the fundamental inequalities in the theory of r.i. spaces is the *Hardy–Littlewood inequality* which can be formulated in our framework as

$$\int_{Q'} |f(x)g(x)|\,dx \leq \int_0^{|Q'|} f^*(r)g^*(r)\,dr \tag{2.1}$$

for every $Q' \subset \mathbb{R}^n$ and f, g measurable functions on Q'.

Throughout the paper we work with the boundedness of a one-dimensional linear reduction operator H acting between spaces of positive measurable functions on $(0, 1)$. More precisely, let $H : \mathcal{M}_+(0, 1) \to \mathcal{M}_+(0, 1)$ be a operator given by

$$H : f \longmapsto \int_t^1 r^{-\frac{1}{n'}} f(r)\,dr.$$

By the boundedness of H from an r.i. space $X(0, 1)$ into another r.i. space $Y(0, 1)$, we mean that the quantity

$$\|T\| = \sup\left\{ \left\|T(f)\right\|_{Y(0,1)} : f \in X(0, 1) \cap \mathcal{M}_+(0, 1), \|f\|_{X(0,1)} \leq 1 \right\}$$

is finite, i.e. there exists a positive constant $C = \|T\|$, such that

$$\left\|T(f)\right\|_{Y(0,1)} \leq C\,\|f\|_{X(0,1)}$$

for every $f \in X(0, 1) \cap \mathcal{M}_+(0, 1)$.

Let $X(Q)$ be an r.i. Banach function space. Then

- $X^{'}(Q)$ denotes the *associate space* of $X(Q)$ given by

$$X^{'}(Q) = \left\{ f \text{ measurable function on } Q : \int_Q |f(x)g(x)|\,dx < \infty \text{ for every } g \in X(Q) \right\}$$

  and endowed with the norm

$$\|f\|_{X'(Q)} = \sup_{\|g\|_{X(Q)} \leq 1} \int_Q |f(x)g(x)|\,dx.$$

Let us note that $X'(Q)$ is also an r.i. space, $X''(Q) = X(Q)$ and the Hölder type inequality

$$\int_Q |f(x)g(x)|\,dx \leq \|f\|_{X(Q)}\,\|g\|_{X'(Q)}$$

holds for any measurable functions f, g on Q.

- $\overline{X}(0,1)$ denotes a unique r.i. space on $(0,1)$ called the *representation space* of $X(Q)$, endowed with the norm $\|.\|_{\overline{X}(0,1)}$ that satisfies $\|f\|_{X(Q)} = \|f^*\|_{\overline{X}(0,1)}$ and is given by

$$\|f\|_{\overline{X}(0,1)} = \sup_{\|g\|_{X'(Q)}\leq 1} \int_0^1 f^*(r)g^*(r)\,dr.$$

- $V^1X(Q)$ denotes a *first-order Sobolev space* built upon $X(Q)$ given by

$$V^1X(Q) = \left\{f : f \text{ is a real-valued weakly differentiable function on } Q \text{ and } |\nabla f| \in X(Q)\right\}.$$

Definition 2.2 Let f be a function on a measurable set $E \subset \mathbb{R}^n$, where $0 < |E| < \infty$. Then the value c is a *median* of f in E if

$$E \cap \{x \in \mathbb{R}^n : f(x) > c\}| \leq \frac{1}{2}|E| \quad \text{and} \quad |E \cap \{x \in \mathbb{R}^n : f(x) < c\}| \leq \frac{1}{2}|E|.$$

The following theorem is a variant of the well-known result from [3, Lemma 4.1] and it is essential for the proofs in the next section. In our variant we assume that f is nonnegative function with the median equal to zero and that the domain of the Sobolev functions has finite measure.

Theorem 2.3 *Let $X(Q')$ be an r.i. space, $|Q'| < \infty$, and let $f \in V^1X(Q')$ be a nonnegative function with the median in Q' equal to zero. Then f^* is locally absolutely continuous and there exists a constant C, depending only on n, such that*

$$\left\| r^{\frac{1}{n'}} \left(-\frac{df^*}{dr}(r)\right)\right\|_{\overline{X}(0,|Q'|)} \leq C\,\|\nabla f\|_{X(Q')}\,. \tag{2.3}$$

P r o o f. The proof simply follows from [3, Lemma 4.1/(ii)], since with our assumption we have $f^\circ = f^*$ and $f^*(r) = 0$ for $r \geq \frac{|Q'|}{2}$ and thus also

$$\left\| r^{\frac{1}{n'}} \left(-\frac{df^*}{dr}(r)\right)\right\|_{\overline{X}(0,|Q'|)} \leq C\left\| h_Q(r)\left(-\frac{df^\circ}{dr}(r)\right)\right\|_{\overline{X}(0,|Q'|)},$$

where f° and h_Q are objects defined in [3]. □

3 Proofs of the main results

P r o o f o f T h e o r e m 1 . 1 First we assume that (i) holds. Then, by [4, Theorem 1.3], one also has

$$\sup_{0<t<1} \frac{1}{\varphi(t^{\frac{1}{n}})} \left\| r^{-\frac{1}{n'}}\chi_{(0,t)}(r)\right\|_{\overline{X}'(0,1)} < \infty. \tag{3.1}$$

Thus, for every $f \in \mathcal{M}_+(0,1)$, we have

$$\begin{aligned}
\|Hf\|_{C^{0,\overline{\varphi}}(0,1)} &= \sup_{\substack{a,b\in(0,1)\\ a\neq b}} \frac{|Hf(a) - Hf(b)|}{\overline{\varphi}(|a-b|)} \\
&= \sup_{\substack{a,b\in(0,1)\\ a\neq b}} \frac{|\int_a^1 r^{-\frac{1}{n'}} f(r)\,dr - \int_b^1 r^{-\frac{1}{n'}} f(r)\,dr|}{\overline{\varphi}(|a-b|)} \\
&= \sup_{0<a<b<1} \frac{|\int_a^b r^{-\frac{1}{n'}} f(r)\,dr|}{\overline{\varphi}(|a-b|)}
\end{aligned}$$

$$\stackrel{(2.1)}{\le} \sup_{0<a<b<1} \frac{\int_0^{|a-b|} r^{-\frac{1}{n'}} f^*(r)\,dr}{\overline{\varphi}(|a-b|)}$$
$$= \sup_{0<t<1} \frac{\int_0^{t} r^{-\frac{1}{n'}} f^*(r)\,dr}{\overline{\varphi}(t)}$$
$$\stackrel{(2.2)}{\le} \sup_{0<t<1} \frac{1}{\varphi(t^{\frac{1}{n}})} \left\| r^{-\frac{1}{n'}} \chi_{(0,t)}(r) \right\|_{\overline{X}'(0,1)} \left\| f^* \right\|_{\overline{X}(0,1)}$$
$$\stackrel{(3.1)}{\le} C \left\| f^* \right\|_{\overline{X}(0,1)},$$

and (ii) is satisfied.

Conversely, we assume that (ii) holds. Moreover, for every couple $x, y \in Q$, there exists a cube $Q_{x,y} \subset Q$, such that $x, y \in Q_{x,y}$ and

$$|Q_{x,y}|^{\frac{1}{n}} = \sup_{i\in\{1,\dots,n\}} |x_i - y_i| \le |x-y|. \tag{3.2}$$

Now we arbitrarily fix $f \in V^1X(Q)$, $x, y \in Q$, and denote by g the restriction of f to $Q_{x,y}$. Without loss of generality we can assume that $g \in V^1X(Q_{x,y})$ is a nonnegative function and its median in $Q_{x,y}$ equals zero, since for arbitrary $g \in V^1X(Q_{x,y})$ we have

$$\begin{aligned} |g(x)-g(y)| &= |(g-c)^+(x) - (g-c)^+(y) + (g-c)^-(y) - (g-c)^-(x)| \\ &\le |(g-c)^+(x) - (g-c)^+(y)| + |(g-c)^-(x) - (g-c)^-(y)|, \end{aligned}$$

where c denotes the median of g in $Q_{x,y}$,

$$\left\| \nabla (g-c)^+ \right\|_{X(Q_{x,y})} \le \|\nabla f\|_{X(Q)} \quad \text{and} \quad \left\| \nabla (g-c)^- \right\|_{X(Q_{x,y})} \le \|\nabla f\|_{X(Q)} \tag{3.3}$$

and $(g-c)^+(x)$, $(g-c)^-(x)$ are nonnegative $V^1X(Q_{x,y})$ functions with the median in $Q_{x,y}$ equal to zero. Then for a.e. earlier fixed $x, y \in Q$ we have

$$|f(x)-f(y)| = |g(x)-g(y)| \le \operatorname*{ess\,sup}_{z\in Q_{x,y}} g(z) - \operatorname*{ess\,inf}_{z\in Q_{x,y}} g(z) = \int_0^{|Q_{x,y}|} \left(-\frac{dg^*}{dr}(r)\right) dr. \tag{3.4}$$

Furthermore, using the nondecreasing property of φ for the second inequality and identifying λ with $|Q_{x,y}|$, we can compute

$$\begin{aligned} \frac{|f(x)-f(y)|}{\varphi(|x-y|)} &\stackrel{(3.4)}{\le} \frac{\int_0^{|Q_{x,y}|} \left(-\frac{dg^*}{dr}(r)\right) dr}{\varphi(|x-y|)} \stackrel{(3.2)}{\le} \frac{\int_0^{|Q_{x,y}|} \left(-\frac{dg^*}{dr}(r)\right) dr}{\varphi(|Q_{x,y}|^{\frac{1}{n}})} \\ &\le \sup_{0<b<\lambda} \frac{\int_0^{b} \left(-\frac{dg^*}{dr}(r)\right) dr}{\varphi(b^{\frac{1}{n}})} \le \sup_{0<a<b<\lambda} \frac{\int_a^{b} \left(-\frac{dg^*}{dr}(r)\right) dr}{\varphi(|a-b|^{\frac{1}{n}})} \\ &= \sup_{\substack{a,b\in(0,\lambda)\\ a\neq b}} \frac{|g^*(a)-g^*(b)|}{\varphi(|a-b|^{\frac{1}{n}})} = \left\| g^*(r) \right\|_{C^{0,\overline{\varphi}}(0,\lambda)} \\ &= \left\| g^*(r)\chi_{(0,\lambda)}(r) \right\|_{C^{0,\overline{\varphi}}(0,1)} = \left\| H\left(r^{\frac{1}{n'}} \left(-\frac{dg^*}{dr}(r)\right) \chi_{(0,\lambda)}(r) \right) \right\|_{C^{0,\overline{\varphi}}(0,1)} \\ &\stackrel{\text{(ii)}}{\le} C \left\| r^{\frac{1}{n'}} \left(-\frac{dg^*}{dr}(r)\right) \chi_{(0,\lambda)}(r) \right\|_{\overline{X}(0,1)} = C \left\| r^{\frac{1}{n'}} \left(-\frac{dg^*}{dr}(r)\right) \right\|_{\overline{X}(0,\lambda)} \\ &\stackrel{(2.3)}{\le} C \left\| \nabla g \right\|_{X(Q_{x,y})} \le C \left\| \nabla f \right\|_{X(Q)}. \end{aligned}$$

Since x, y were chosen arbitrarily, we can take the supremum of the above inequality over $x, y \in Q$, $x \neq y$, to obtain

$$\|f\|_{C^{0,\varphi}(Q)} = \sup_{\substack{x,y\in Q\\ x\neq y}} \frac{|f(x)-f(y)|}{\varphi(|x-y|)} \le C\,\|\nabla f\|_{X(Q)}$$

and thus (i) in the theorem assertion holds. The proof is complete. □

P r o o f o f T h e o r e m 1 . 2 First we assume that (i) holds. By [4, Theorem 1.1], we conclude that

$$\sup_{0<t<1} \frac{1}{t\varphi(t^{\frac{1}{n}})} \left\| r^{\frac{1}{n}} \chi_{(0,t)}(r) \right\|_{\overline{X}'(0,1)} < \infty. \tag{3.5}$$

Then for arbitrary $(a, b) \subset (0, 1)$ and $f \in \mathcal{M}_+(0, 1)$ we have

$$\begin{aligned}
\int_a^b |Hf(s) - (Hf)_{(a,b)}|\,ds &= \int_a^b \left| Hf(s) - \frac{1}{(b-a)} \int_a^b Hf(t)\,dt \right| ds \\
&\le \frac{1}{(b-a)} \int_a^b \int_a^b |Hf(s) - Hf(t)|\,dt\,ds \\
&= \frac{1}{(b-a)} \int_a^b \int_a^b \int_{\min\{s,t\}}^{\max\{s,t\}} r^{-\frac{1}{n'}} f(r)\,dr\,dt\,ds \\
&= \frac{1}{(b-a)} \left(\int_a^b \int_a^s \int_t^s r^{-\frac{1}{n'}} f(r)\,dr\,dt\,ds + \int_a^b \int_s^b \int_s^t r^{-\frac{1}{n'}} f(r)\,dr\,dt\,ds \right) \\
&= \frac{1}{(b-a)} \left(\int_a^b \int_a^s \int_a^r r^{-\frac{1}{n'}} f(r)\,dt\,dr\,ds + \int_a^b \int_s^b \int_r^b r^{-\frac{1}{n'}} f(r)\,dt\,dr\,ds \right) \\
&= \frac{1}{(b-a)} \left(\int_a^b \int_r^b \int_a^r r^{-\frac{1}{n'}} f(r)\,dt\,ds\,dr + \int_a^b \int_a^r \int_r^b r^{-\frac{1}{n'}} f(r)\,dt\,ds\,dr \right) \\
&= \frac{2}{(b-a)} \int_a^b \frac{(r-a)(b-r)}{r^{\frac{1}{n'}}} f(r)\,dr \le \frac{2}{(b-a)} \int_a^b \frac{(r-a)(b-a)}{(r-a)^{\frac{1}{n'}}} f(r)\,dr \\
&\overset{(2.1)}{\le} 2 \int_0^{b-a} (b-a-r)^{\frac{1}{n}} f^*(r)\,dr \overset{(2.2)}{\le} 2\,\|f^*\|_{\overline{X}(0,1)} \left\| (b-a-r)^{\frac{1}{n}} \chi_{(0,b-a)}(r) \right\|_{\overline{X}'(0,1)} \\
&= 2\,\|f^*\|_{\overline{X}(0,1)} \left\| r^{\frac{1}{n}} \chi_{(0,b-a)}(r) \right\|_{\overline{X}'(0,1)},
\end{aligned}$$

where

$$\left\| (b-a-r)^{\frac{1}{n}} \chi_{(0,b-a)}(r) \right\|_{\overline{X}'(0,1)} = \left\| r^{\frac{1}{n}} \chi_{(0,b-a)}(r) \right\|_{\overline{X}'(0,1)}$$

holds, since $((.)^{\frac{1}{n}} \chi_{(0,b-a)}(.))^*(r) = (b-a-r)^{\frac{1}{n}} \chi_{(0,b-a)}(r)$ and $\overline{X}'(0,1)$ is r.i. Now, multiplying both sides of the inequality by $((b-a)\varphi((b-a)^{\frac{1}{n}}))^{-1}$ and taking the supremum over $(a, b) \subset (0, 1)$, we obtain

$$\begin{aligned}
\|Hf\|_{L^C_\varphi(0,1)} &= \sup_{(a,b)\subset(0,1)} \frac{1}{(b-a)\varphi((b-a)^{\frac{1}{n}})} \int_a^b |Hf(s) - (Hf)_{(a,b)}|\,ds \\
&\le 2\,\|f^*\|_{\overline{X}(0,1)} \sup_{(a,b)\subset(0,1)} \frac{1}{(b-a)\varphi((b-a)^{\frac{1}{n}})} \left\| r^{\frac{1}{n}} \chi_{(0,b-a)}(r) \right\|_{\overline{X}'(0,1)} \\
&= 2\,\|f^*\|_{\overline{X}(0,1)} \sup_{0<t<1} \frac{1}{t\varphi(t^{\frac{1}{n}})} \left\| r^{\frac{1}{n}} \chi_{(0,t)}(r) \right\|_{\overline{X}'(0,1)} \\
&\overset{(3.5)}{\le} 2C\,\|f^*\|_{\overline{X}(0,1)}
\end{aligned}$$

and (ii) is satisfied.

Conversely, we assume that (ii) holds and that $f \in V^1X(Q)$. Thus $|\nabla f| \in X(Q)$ and by the condition (RL_4) we have

$$\int_Q |\nabla f(x)|\,dx \le C\,\|\nabla f\|_{X(Q)}\,,$$

which immediately gives that $|\nabla f| \in L^1(Q)$ and, since Q has a Lipschitz boundary, $f \in L^1(Q)$ by the Sobolev embedding theorem. Now, for any $Q' \subset Q$, we denote by g the restriction of f to Q' and identify λ with $|Q'|$. Without loss of generality we can assume that $g \in V^1X(Q')$ is a nonnegative function and its median in Q' equals zero, since for arbitrary $g \in V^1X(Q')$ we have

$$\begin{aligned}|g(x) - g_{Q'}| &= |(g-c)^+(x) - ((g-c)^+)_{Q'} + ((g-c)^-)_{Q'} - (g-c)^-(x)|\\ &\le |(g-c)^+(x) - ((g-c)^+)_{Q'}| + |(g-c)^-(x) - ((g-c)^-)_{Q'}|,\end{aligned}$$

where c denotes the median of g in Q' and we use inequalities (3.3) with the properties immediately after them with Q' in place of $Q_{x,y}$. Then

$$\begin{aligned}\frac{1}{|Q'|\varphi(|Q'|^{\frac1n})}\int_{Q'}|f(x)-f_{Q'}|\,dx &= \frac{1}{|Q'|\varphi(|Q'|^{\frac1n})}\int_{Q'}|g(x)-g_{Q'}|\,dx\\ &= \frac{1}{\lambda\varphi(\lambda^{\frac1n})}\int_0^\lambda |g^*(t)-g^*_{(0,\lambda)}|\,dt\\ &\le \sup_{0<a<b<\lambda}\frac{1}{(b-a)\varphi((b-a)^{\frac1n})}\int_a^b |g^*(t)-g^*_{(a,b)}|\,dt\\ &= \left\|g^*(r)\right\|_{L^C_\varphi(0,\lambda)} = \left\|g^*(r)\chi_{(0,\lambda)}(r)\right\|_{L^C_\varphi(0,1)}\\ &= \left\|H\left(r^{\frac{1}{n'}}\left(-\frac{dg^*}{dr}(r)\right)\chi_{(0,\lambda)}(r)\right)\right\|_{L^C_\varphi(0,1)}\\ &\overset{\text{(ii)}}{\le} C\left\|r^{\frac{1}{n'}}\left(-\frac{dg^*}{dr}(r)\right)\chi_{(0,\lambda)}(r)\right\|_{\overline{X}(0,1)}\\ &= C\left\|r^{\frac{1}{n'}}\left(-\frac{dg^*}{dr}(r)\right)\right\|_{\overline{X}(0,\lambda)}\\ &\overset{(2.3)}{\le} C\,\|\nabla g\|_{X(Q')} \le C\,\|\nabla f\|_{X(Q)}\,.\end{aligned}$$

Since Q' was chosen arbitrarily, we can take the supremum of the above inequality over $Q' \subset Q$ to obtain

$$\|f\|_{L^C_\varphi(Q)} = \sup_{Q'\subset Q}\frac{1}{|Q'|\varphi(|Q'|^{\frac1n})}\int_{Q'}|f(x)-f_{Q'}|\,dx \le C\,\|\nabla f\|_{X(Q)}$$

and thus (i) in the theorem assertion holds. The proof is complete. □

Proof of Theorem 1.3 First we assume that (i) holds. Then, by [4, Theorem 1.2] we obtain

$$\sup_{0<t<1}\frac{1}{\varphi(t^{\frac1n})}\left\|r^{-\frac{1}{n'}}\chi_{(t,1)}(r)\right\|_{\overline{X}'(0,1)} < \infty, \tag{3.6}$$

whence, using [4, Lemma 3.2], we can also infer that the estimate

$$\sup_{0<t<1}\frac{1}{t\varphi(t^{\frac1n})}\left\|r^{\frac1n}\chi_{(0,t)}(r)\right\|_{\overline{X}'(0,1)} < \infty \tag{3.7}$$

is true. Then for arbitrary $(a,b) \subset (0,1)$ and $f \in \mathcal{M}_+(0,1)$ we have

$$\int_a^b Hf(s)\,ds = \int_a^b\int_s^1 r^{-\frac{1}{n'}}f(r)\,dr\,ds$$

$$
\begin{aligned}
&= \int_a^1 \int_a^{\min\{r,b\}} r^{-\frac{1}{n'}} f(r)\, ds\, dr \\
&= \int_a^b \int_a^r r^{-\frac{1}{n'}} f(r)\, ds\, dr + \int_b^1 \int_a^b r^{-\frac{1}{n'}} f(r)\, ds\, dr \\
&\le \int_a^b \frac{(r-a) f(r)}{(r-a)^{\frac{1}{n'}}}\, dr + (b-a) \int_{b-a}^1 r^{-\frac{1}{n'}} f(r)\, dr \\
&= \int_a^b (r-a)^{\frac{1}{n}} f(r)\, dr + (b-a) \int_{b-a}^1 r^{-\frac{1}{n'}} f(r)\, dr \\
&\overset{(2.1)}{\le} \int_0^{b-a} r^{\frac{1}{n}} f^*(r)\, dr + (b-a) \int_{b-a}^1 r^{-\frac{1}{n'}} f(r)\, dr \\
&\overset{(2.2)}{\le} \left\| f^* \right\|_{\overline{X}(0,1)} \left\| r^{\frac{1}{n}} \chi_{(0,b-a)}(r) \right\|_{\overline{X}'(0,1)} + \left\| f^* \right\|_{\overline{X}(0,1)} (b-a) \left\| r^{-\frac{1}{n'}} \chi_{(b-a,1)}(r) \right\|_{\overline{X}'(0,1)}.
\end{aligned}
$$

Now, multiplying both sides of the inequality by $\left((b-a)\varphi\left((b-a)^{\frac{1}{n}}\right)\right)^{-1}$ and taking the supremum over $(a,b) \subset (0,1)$, we obtain

$$
\begin{aligned}
\| Hf \|_{L^M_\varphi(0,1)} &= \sup_{(a,b)\subset(0,1)} \frac{1}{(b-a)\varphi((b-a)^{\frac{1}{n}})} \int_a^b Hf(s)\, ds \\
&\le \left\| f^* \right\|_{\overline{X}(0,1)} \sup_{(a,b)\subset(0,1)} \frac{1}{(b-a)\varphi((b-a)^{\frac{1}{n}})} \left\| r^{\frac{1}{n}} \chi_{(0,b-a)}(r) \right\|_{\overline{X}'(0,1)} \\
&\quad + \left\| f^* \right\|_{\overline{X}(0,1)} \sup_{(a,b)\subset(0,1)} \frac{1}{\varphi((b-a)^{\frac{1}{n}})} \left\| r^{-\frac{1}{n'}} \chi_{(b-a,1)}(r) \right\|_{\overline{X}'(0,1)} \\
&\le \left\| f^* \right\|_{\overline{X}(0,1)} \sup_{0<t<1} \frac{1}{t\varphi(t^{\frac{1}{n}})} \left\| r^{\frac{1}{n}} \chi_{(0,t)}(r) \right\|_{\overline{X}'(0,1)} \\
&\quad + \left\| f^* \right\|_{\overline{X}(0,1)} \sup_{0<t<1} \frac{1}{\varphi(t^{\frac{1}{n}})} \left\| r^{-\frac{1}{n'}} \chi_{(t,1)}(r) \right\|_{\overline{X}'(0,1)} \\
&\overset{(3.6),(3.7)}{\le} C \left\| f^* \right\|_{\overline{X}(0,1)}
\end{aligned}
$$

and (ii) follows.

Conversely, we assume that (ii) holds and that $f \in V^1X(Q)$. Thus $|\nabla f| \in X(Q)$ and by the condition (RI$_4$) we have

$$
\int_Q |\nabla f(x)|\, dx \le C \, \|\nabla f\|_{X(Q)},
$$

which immediately gives $|\nabla f| \in L^1(Q)$ and, since Q has a Lipschitz boundary, $f \in L^1(Q)$ by the Sobolev embedding theorem. Moreover, let f be a nonnegative function and its median in Q equals zero. Then

$$
\begin{aligned}
\frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_{Q'} |f(x)|\, dx &\le \frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_0^{|Q'|} f^*(t)\, dt \le \sup_{0<b<1} \frac{1}{b\varphi(b^{\frac{1}{n}})} \int_0^b f^*(t)\, dt \\
&\le \sup_{0<a<b<1} \frac{1}{(b-a)\varphi((b-a)^{\frac{1}{n}})} \int_a^b f^*(t)\, dt = \left\| f^*(r) \right\|_{L^M_\varphi(0,1)} \\
&\le \left\| H\left(r^{\frac{1}{n'}} \left(-\frac{df^*}{dr}(r) \right) \right) \right\|_{L^M_\varphi(0,1)} \overset{(ii)}{\le} C \left\| r^{\frac{1}{n'}} \left(-\frac{df^*}{dr}(r) \right) \right\|_{\overline{X}(0,1)} \\
&\overset{(2.3)}{\le} C \, \|\nabla f\|_{X(Q)}.
\end{aligned}
$$

For arbitrary $f \in V^1X(Q)$, denoting by c the median of f in Q, we have

$$|f(x)| \le |f(x) - c| + |c| = (f(x) - c)^+ + (f(x) - c)^- + |c|,$$

the inequalities (3.3) with the properties immediately after them for f instead of g and Q' in place of $Q_{x,y}$ and

$$\frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_{Q'} |c|\,dx = \frac{|c|}{\varphi(|Q'|^{\frac{1}{n}})} \le C \int_Q |f(x)|\,dx = C\,\|f\|_{L^1(Q)}$$

obtained by (1.1) and the properties of median. Thus, combining the previous computations, we obtain

$$\frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_{Q'} |f(x)|\,dx \le C\left(\|\nabla f\|_{X(Q)} + \|f\|_{L^1(Q)}\right),$$

for arbitrary $f \in V^1X(Q)$. Since Q' was chosen arbitrarily, we can take the supremum of the above inequality over $Q' \subset Q$ to have

$$\|f\|_{L^M_\varphi(Q)} = \sup_{Q' \subset Q} \frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_{Q'} |f(x)|\,dx \le C\left(\|\nabla f\|_{X(Q)} + \|f\|_{L^1(Q)}\right)$$

and thus (i) in the theorem assertion holds. The proof is complete. □

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