BANACH ALGEBRAS OF WEAKLY DIFFERENTIABLE FUNCTIONS

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Abstract. The question is addressed of when a Sobolev type space, built upon a general rearrangement-invariant norm, on an n-dimensional domain, is a Banach algebra under pointwise multiplication of functions. A sharp balance condition among the order of the Sobolev space, the strength of the norm, and the (ir)regularity of the domain is provided for the relevant Sobolev space to be a Banach algebra. The regularity of the domain is described in terms of its isoperimetric function. Related results on the boundedness of the multiplication operator into lower-order Sobolev type spaces are also established. The special cases of Orlicz-Sobolev and Lorentz-Sobolev spaces are discussed in detail. New results for classical Sobolev spaces on possibly irregular domains follow as well.

1. Introduction and main results

The Sobolev space $W^{m,p}(\Omega)$ of those functions in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, whose weak derivatives up to the order m belong to $L^p(\Omega)$, is classically well known to be a Banach space for every $m \in \mathbb{N}$ and $p \in [1, \infty]$. In particular, the sum of any two functions from $W^{m,p}(\Omega)$ always still belongs to $W^{m,p}(\Omega)$. The situation is quite different if the operation of sum is replaced by product. In fact, membership of functions to a Sobolev space need not be preserved under multiplication. Hence, $W^{m,p}(\Omega)$ is not a Banach algebra in general. A standard result in the theory of Sobolev spaces tells us that if Ω is regular, say a bounded domain with the cone property, then $W^{m,p}(\Omega)$ is indeed a Banach algebra if and only if either $p > 1$ and $pm > n$, or $p = 1$ and $m \geq n$. Recall that this amounts to the existence of a constant C such that

$$\|uv\|_{W^m X(\Omega)} \leq C \|u\|_{W^m X(\Omega)} \|v\|_{W^m X(\Omega)} \tag{1.1}$$

for every $u, v \in W^m X(\Omega)$. We refer to Section 6.1 of the monograph [48] for this result, where a comprehensive updated treatment of properties of Sobolev functions under product can be found. See also [1, Theorem 5.23] for a proof of the sufficiency part of the result.

In the present paper abandon this classical setting, and address the question of the validity of an inequality of the form (1.1) in a much more general framework. Assume that Ω is just a domain in $\mathbb{R}^n$, namely an open connected set, with finite Lebesgue measure $|\Omega|$, which, without loss of generality, will be assumed to be equal to 1. Moreover, suppose that $L^p(\Omega)$ is replaced with an arbitrary rearrangement-invariant space $X(\Omega)$, loosely speaking, a Banach space of measurable functions endowed with a norm depending only on the measure of level sets of functions. We refer to the next section for precise definitions concerning function spaces. Let us just recall here that, besides Lebesgue spaces, Lorentz and Orlicz spaces are classical instances of rearrangement-invariant spaces.

Given any $m \in \mathbb{N}$ and any rearrangement-invariant space $X(\Omega)$, consider the m-th order Sobolev type space $\mathcal{V}^m X(\Omega)$ built upon $X(\Omega)$, and defined as the collection of all m times weakly differentiable functions $u : \Omega \to \mathbb{R}$ such that $|\nabla^m u| \in X(\Omega)$. Here, $\nabla^m u$ denotes the vector of all m-th order weak

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derivatives of u, and $|\nabla^m u|$ stands for its length. For notational convenience, we also set $\nabla^0 u = u$ and $\mathcal{V}^0 X(\Omega) = X(\Omega)$. Given any fixed ball $B \subset \Omega$, we define the functional $\|\cdot\|_{\mathcal{V}^m X(\Omega)}$ by

$$\|u\|_{\mathcal{V}^m X(\Omega)} = \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(B)} + \|\nabla^m u\|_{X(\Omega)} \tag{1.2}$$

for $u \in \mathcal{V}^m X(\Omega)$. Observe that in the definition of $\mathcal{V}^m X(\Omega)$ it is only required that the derivatives of the highest order m of u belong to $X(\Omega)$. This assumption does not ensure, for an arbitrary domain Ω, that also u and its derivatives up to the order $m-1$ belong to $X(\Omega)$, or even to $L^1(\Omega)$. However, owing to a standard Poincaré inequality, if $u \in \mathcal{V}^m X(\Omega)$, then $|\nabla^k u| \in L^1(B)$ for $k = 0, \dots, m-1$, for every ball $B \subset \Omega$. It follows that the functional $\|\cdot\|_{\mathcal{V}^m X(\Omega)}$ is a norm on $\mathcal{V}^m X(\Omega)$. Furthermore, a standard argument shows that $\mathcal{V}^m X(\Omega)$ is a Banach space equipped with this norm, which results in equivalent norms under replacements of B with other balls.

We shall exhibit minimal conditions on m, Ω and $\|\cdot\|_{X(\Omega)}$ for $\mathcal{V}^m X(\Omega)$ to be a Banach algebra under pointwise multiplication of functions, namely for an inequality of the form

$$\|uv\|_{\mathcal{V}^m X(\Omega)} \le C \|u\|_{\mathcal{V}^m X(\Omega)} \|v\|_{\mathcal{V}^m X(\Omega)}$$

to hold for some constant C and every $u, v \in \mathcal{V}^m X(\Omega)$. Variants of this inequality, where $\mathcal{V}^m X(\Omega)$ is replaced by a lower-order Sobolev space on the left-hand side, are also dealt with.

In our discussion, we neither a priori assume any regularity on Ω, nor we assume that $X(\Omega)$ is a Lebesgue space (or any other specific space). We shall exhibit a balance condition between the degree of regularity of Ω, the order of differentiation m, and the strength of the norm in $X(\Omega)$ ensuring that $\mathcal{V}^m X(\Omega)$ be a Banach algebra. The dependence on $X(\Omega)$ is only through the representation norm $\|\cdot\|_{X(0,1)}$ of $\|\cdot\|_{X(\Omega)}$. In particular, the associate norm $\|\cdot\|_{X'(0,1)}$ of $\|\cdot\|_{X(0,1)}$, a kind of measure theoretic dual norm of $\|\cdot\|_{X(0,1)}$, will be relevant.
As for our assumptions on the domain Ω, a key role in their formulation will be played by the relative isoperimetric inequality. Let us recall that the discovery of the link between isoperimetric inequalities and Sobolev type inequalities can be traced back to the work of Maz'ya on one hand ([45, 46]), who proved the equivalence of general Sobolev inequalities to either isoperimetric or isocapacitary inequalities, and that of Federer and Fleming on the other hand ([29]) who used the standard isoperimetric inequality by De Giorgi ([26]) to exhibit the best constant in the Sobolev inequality for $W^{1,1}(\mathbb{R}^n)$. The detection of optimal constants in classical Sobolev inequalities continued in the contributions [50], [55], [4], where crucial use of De Giorgi's isoperimetric inequality was again made. An extensive research followed, along diverse directions, on the interplay between isoperimetric and Sobolev inequalities. We just mention the papers [2, 5, 6, 9, 10, 11, 12, 15, 17, 18, 19, 22, 24, 27, 28, 30, 32, 33, 35, 36, 37, 38, 40, 41, 42, 43, 49, 54, 56] and the monographs [13, 14, 16, 31, 34, 47, 53].

Before stating our most general result, let us focus on the situation when m and $X(\Omega)$ are arbitrary, but Ω is still, in a sense, a best possible domain. This is the case when Ω is a John domain. Recall that a bounded open set Ω in $\mathbb{R}^n$ is called a *John domain* if there exist a constant $c \in (0,1)$, an $l \in (0, \infty)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0, l] \to \Omega$, parameterized by arclength, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}(\varpi(r), \partial\Omega) \ge cr \qquad \text{for } r \in [0, l].$$

Lipschitz domains, and domains with the cone property are customary instances of John domains.

When Ω is any John domain, a necessary and sufficient condition for $\mathcal{V}^m X(\Omega)$ to be a Banach algebra is provided by the following result.

Theorem 1.1. *Let $m, n \in \mathbb{N}$, $n \ge 2$. Assume that Ω is a John domain in $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then $\mathcal{V}^m X(\Omega)$ is a Banach algebra if and only if*

$$\|r^{-1+\frac{m}{n}}\|_{X'(0,1)} < \infty. \tag{1.3}$$

As a consequence of Theorem 1.1, and of the characterization of Sobolev embeddings into $L^\infty(\Omega)$, we have the following corollary.

Corollary 1.2. *Let $m, n \in \mathbb{N}$, $n \geq 2$. Assume that Ω is a John domain in $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the Sobolev space $\mathcal{V}^m X(\Omega)$ is a Banach algebra if and only if $\mathcal{V}^m X(\Omega) \to L^\infty(\Omega)$.*

Let us now turn to the general case. Regularity on Ω will be imposed in terms of its isoperimetric function $I_\Omega : [0,1] \to [0,\infty]$, introduced in [45], and given by

$$I_\Omega(s) = \inf\left\{P(E,\Omega) : E \subset \Omega, s \leq |E| \leq \tfrac{1}{2}\right\} \quad \text{if } s \in [0, \tfrac{1}{2}], \tag{1.4}$$

and $I_\Omega(s) = I_\Omega(1-s)$ if $s \in (\frac{1}{2}, 1]$. Here, $P(E,\Omega)$ denotes the perimeter of a measurable set E relative to Ω, which agrees with $\mathcal{H}^{n-1}(\Omega \cap \partial^M E)$, where $\partial^M E$ denotes the essential boundary of E, in the sense of geometric measure theory, and $\mathcal{H}^{n-1}$ stands for $(n-1)$-dimensional Hausdorff measure. The very definition of I_Ω implies the relative isoperimetric inequality in Ω, which tells us that

$$P(E,\Omega) \geq I_\Omega(|E|), \tag{1.5}$$

for every measurable set $E \subset \Omega$. In other words, I_Ω is the largest non-decreasing function in $[0, \frac{1}{2}]$, symmetric about $\frac{1}{2}$, which renders (1.5) true.

The degree of regularity of Ω can be described in terms of the rate of decay of $I_\Omega(s)$ to 0 as $s \to 0$. Heuristically speaking, the faster I_Ω decays to 0, the less regular Ω is. For instance, the isoperimetric function I_Ω of any John domain $\Omega \subset \mathbb{R}^n$ is known to satisfy

$$I_\Omega(s) \approx s^{\frac{1}{n'}} \tag{1.6}$$

near 0, where $n' = \frac{n}{n-1}$. Here, and in what follows, the notation $f \approx g$ mans that the real-valued functions f and g are equivalent, in the sense that there exist positive constants c, C such that $cf(c\cdot) \leq g(\cdot) \leq Cf(C\cdot)$. Notice that (1.6) is the best (i.e. slowest) possible decay of I_Ω, since, if Ω is any domain, then

$$\frac{I_\Omega(s)}{s^{\frac{1}{n'}}} \leq C \quad \text{for } s \in (0,1], \tag{1.7}$$

for some constant C [25, Proposition 4.1].

What enters in our characterization of Sobolev algebras is, in fact, just a lower bound for I_Ω. We shall thus work with classes of domains whose isoperimetric function admits a lower bound in terms of some non-decreasing function $I : (0,1) \to (0,\infty)$. The function I will be continued by continuity at 0 when needed. Given any such function I, we denote by $\mathcal{J}_I$ the collection of all domains $\Omega \subset \mathbb{R}^n$ such that

$$I_\Omega(s) \geq cI(cs) \quad \text{for} \quad s \in (0, \tfrac{1}{2}], \tag{1.8}$$

for some constant $c > 0$. The assumption that $I(t) > 0$ for $t \in (0,1)$ is consistent with the fact that $I_\Omega(t) > 0$ for $t \in (0,1)$, owing to the connectedness of Ω [47, Lemma 5.2.4].
In particular, if $I(s) = s^\alpha$ for $s \in (0,1)$, for some $\alpha \in [\frac{1}{n'}, \infty)$, we denote $\mathcal{J}_I$ simply by $\mathcal{J}_\alpha$, and call it a *Maz'ya class*. Thus, a domain $\Omega \in \mathcal{J}_\alpha$ if there exists a positive constant C such that

$$I_\Omega(s) \geq Cs^\alpha \quad \text{for every } s \in (0, \tfrac{1}{2}].$$

Observe that, thanks to (1.6), any John domain belongs to the class $\mathcal{J}_{\frac{1}{n'}}$.

Our most general result about Banach algebras of Sobolev spaces is stated in the next theorem. Let us emphasize that, if Ω is not a regular domain, it brings new information even in the standard case when $X(\Omega) = L^p(\Omega)$.

Theorem 1.3. *Assume that $m, n \in \mathbb{N}$, $n \geq 2$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that I is a positive non-decreasing function on $(0,1)$. If $\Omega \in \mathcal{J}_I$ and*

$$\left\| \frac{1}{I(t)} \left(\int_0^t \frac{ds}{I(s)} \right)^{m-1} \right\|_{X'(0,1)} < \infty, \tag{1.9}$$

then

$$\mathcal{V}^m X(\Omega) \quad \text{is a Banach algebra,}$$

or, equivalently, there exists a constant C such that

$$\|uv\|_{\mathcal{V}^m X(\Omega)} \leq C \|u\|_{\mathcal{V}^m X(\Omega)} \|v\|_{\mathcal{V}^m X(\Omega)} \tag{1.10}$$

for every $u, v \in \mathcal{V}^m X(\Omega)$.
Conversely, if, in addition,

$$\frac{I(t)}{t^{\frac{1}{n'}}} \quad \text{is equivalent to a non-decreasing function on } (0,1), \tag{1.11}$$

then (1.9) is sharp, in the sense that if $\mathcal{V}^m X(\Omega)$ is a Banach algebra for every $\Omega \in \mathcal{J}_I$, then (1.9) holds.

Remark 1.4. Assumption (1.11) is not restrictive in view of (1.7), and can just be regarded as a qualification of the latter.

Remark 1.5. If condition (1.9) holds for some $X(0,1)$ and m, then necessarily

$$\int_0 \frac{ds}{I(s)} < \infty.$$

This is obvious for $m \geq 2$, whereas it follows from (2.5) below for $m = 1$.

An analogue of Corollary 1.2 is provided by the following statement.

Corollary 1.6. *Let $m, n \in \mathbb{N}$, $n \geq 2$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that I is a positive non-decreasing function on $(0,1)$ satisfying (1.11). Then the Sobolev space $\mathcal{V}^m X(\Omega)$ is a Banach algebra for all domains Ω satisfying (1.8) if and only if $\mathcal{V}^m X(\Omega) \to L^\infty(\Omega)$ for every $\Omega \in \mathcal{J}_I$.*

The next corollary of Theorem 1.3 tells us that $\mathcal{V}^m X(\Omega)$ is always a Banach algebra, whatever $X(\Omega)$ is, provided that I_Ω is sufficiently well behaved near 0, depending on m.

Corollary 1.7. *Let $m, n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \in \mathcal{J}_I$ for some positive non-decreasing function I on $(0,1)$. Suppose that*

$$\limsup_{t \to 0_+} \frac{1}{I(t)} \left(\int_0^t \frac{dr}{I(r)} \right)^{m-1} < \infty. \tag{1.12}$$

Then the Sobolev space $\mathcal{V}^m X(\Omega)$ is a Banach algebra for every rearrangement-invariant space $X(\Omega)$.

Remark 1.8. Besides $\mathcal{V}^m X(\Omega)$, one can consider the m-th order Sobolev type space $V^m X(\Omega)$ of those functions u such that the norm

$$\|u\|_{V^m X(\Omega)} = \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(\Omega)} + \|\nabla^m u\|_{X(\Omega)} \tag{1.13}$$

is finite, and the space $W^m X(\Omega)$ of those functions whose norm

$$\|u\|_{W^m X(\Omega)} = \sum_{k=0}^{m} \|\nabla^k u\|_{X(\Omega)}$$

is finite. Both $V^m X(\Omega)$ and $W^m X(\Omega)$ are Banach spaces. Since any rearrangement-invariant space is embedded into $L^1(\Omega)$, one has that

$$W^m X(\Omega) \to V^m X(\Omega) \to \mathcal{V}^m X(\Omega), \tag{1.14}$$

and the inclusions are strict, as noticed above, unless Ω satisfies some additional regularity assumption. In particular, if

$$\inf_{t\in(0,1)} \frac{I_\Omega(t)}{t} > 0, \tag{1.15}$$

then

$$V^m X(\Omega) = \mathcal{V}^m X(\Omega), \tag{1.16}$$

with equivalent norms, as a consequence of [47, Theorem 5.2.3] and of the closed graph theorem. If assumption (1.15) is strengthened to

$$\int_0 \frac{ds}{I_\Omega(s)} < \infty, \tag{1.17}$$

then, in fact,

$$W^m X(\Omega) = V^m X(\Omega) = \mathcal{V}^m X(\Omega), \tag{1.18}$$

up to equivalent norms [25, Proposition 4.5].
If $\Omega \in \mathcal{J}_I$ for some I fulfilling (1.9), then, by Remark 1.5, condition (1.17) is certainly satisfied. Thus, by the first part of Theorem 1.3, assumption (1.9) is sufficient also for $V^m X(\Omega)$ and $W^m X(\Omega)$ to be Banach algebras. Under (1.11), such assumption is also necessary, provided that the function I is a priori known to fulfil either

$$\inf_{t\in(0,1)} \frac{I(t)}{t} > 0, \tag{1.19}$$

or

$$\int_0 \frac{ds}{I(s)} < \infty, \tag{1.20}$$

according to whether $V^m X(\Omega)$ or $W^m X(\Omega)$ is in question. Let us emphasize that these additional assumptions cannot be dispensed with in general. For instance, it is easily seen that $W^m L^\infty(\Omega)$ is always a Banach algebra, whatever m and Ω are.

So far we have analyzed the question of whether $\mathcal{V}^m X(\Omega)$ is a Banach algebra, namely of the validity of inequality (1.10). We now focus on inequalities in the spirit of (1.10), where the space $\mathcal{V}^m X(\Omega)$ is replaced with a lower-order Sobolev space $\mathcal{V}^{m-k} X(\Omega)$ on the left-hand side. The statement of our result in this connection requires the notion of the fundamental function $\varphi_X : [0,1] \to [0,\infty)$ of a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$. Recall that

$$\varphi_X(t) = \|\chi_{(0,t)}\|_{X(0,1)} \quad \text{for } t \in (0,1]. \tag{1.21}$$

Theorem 1.9. *Let m, n, $k \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq m$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that I is a positive non-decreasing function on $(0,1)$. If $\Omega \in \mathcal{J}_I$ and*

$$\sup_{t\in(0,1)} \frac{1}{\varphi_X(t)} \left(\int_0^t \frac{ds}{I(s)} \right)^{m+k} < \infty, \tag{1.22}$$

then

$$\|uv\|_{\mathcal{V}^{m-k} X(\Omega)} \leq C \|u\|_{\mathcal{V}^m X(\Omega)} \|v\|_{\mathcal{V}^m X(\Omega)} \tag{1.23}$$

for every $u, v \in \mathcal{V}^m X(\Omega)$.

Conversely, if, in addition, (1.11) *is fulfilled, then* (1.22) *is sharp, in the sense that if* (1.23) *is satisfied for all domains* $\Omega \in \mathcal{J}_I$*, then* (1.22) *holds.*

Remark 1.10. Considerations on the replacement of the space $\mathcal{V}^m X(\Omega)$ with either the space $V^m X(\Omega)$, or $W^m X(\Omega)$ in Theorem 1.9 can be made, which are analogous to those of Remark 1.8 about Theorem 1.3.

In the borderline case when $k = 0$, condition (1.22) is (essentially) weaker than (1.9) - see Proposition 3.3 and Remark 3.4, Section 3. The next result asserts that it "almost" implies inequality (1.10), in that it yields an inequality of that form, with the borderline terms $u\nabla^m v$ and $v\nabla^m u$ missing in the Leibniz formula for the m-th order derivative of the product uv.

Theorem 1.11. *Let* $m, n \in \mathbb{N}$*,* $m, n \geq 2$*, and let* $\|\cdot\|_{X(0,1)}$ *be a rearrangement-invariant function norm. Assume that* I *is a positive non-decreasing function on* $(0,1)$*. If* $\Omega \in \mathcal{J}_I$ *and*

$$\sup_{t\in(0,1)} \frac{1}{\varphi_X(t)} \left(\int_0^t \frac{ds}{I(s)}\right)^m < \infty \,, \tag{1.24}$$

then there exists a positive constant C *such that*

$$\sum_{k=1}^{m-1} \||\nabla^k u||\nabla^{m-k} v|\|_{X(\Omega)} \leq C\|u\|_{\mathcal{V}^m X(\Omega)}\|v\|_{\mathcal{V}^m X(\Omega)} \tag{1.25}$$

for every $u, v \in \mathcal{V}^m X(\Omega)$*.*
Conversely, if, in addition, (1.11) *is fulfilled, then* (1.24) *is sharp, in the sense that if* (1.25) *is satisfied for all domains* $\Omega \in \mathcal{J}_I$*, then* (1.24) *holds.*

Theorem 1.3 enables us, for instance, to characterize the Lorentz–Zygmund-Sobolev spaces and Orlicz-Sobolev spaces which are Banach algebras for all domains $\Omega \in \mathcal{J}_\alpha$.
Let us first focus on the Lorentz–Zygmund-Sobolev spaces $\mathcal{V}^m L^{p,q;\beta}(\Omega)$. Recall (see e.g. [52, Theorem 9.10.4] or [51, Theorem 7.4]) that a necessary and sufficient condition for $L^{p,q;\beta}(\Omega)$ to be a rearrangement-invariant space is that the parameters p, q, β satisfy either of the following conditions:

$$\begin{cases} 1 < p < \infty, \ 1 \leq q \leq \infty, \ \beta \in \mathbb{R}; \\ p = 1, \ q = 1, \ \beta \geq 0; \\ p = \infty, \ q = \infty, \ \beta \leq 0; \\ p = \infty, \ 1 \leq q < \infty, \ \beta + \frac{1}{q} < 0. \end{cases} \tag{1.26}$$

Proposition 1.12. *Let* $m, n \in \mathbb{N}$*,* $n \geq 2$*, and let* $\alpha \in [\frac{1}{n'}, \infty)$*. Assume that* $1 \leq p, q \leq \infty$*,* $\beta \in \mathbb{R}$ *and one of the conditions in* (1.26) *is in force. Then* $\mathcal{V}^m L^{p,q;\beta}(\Omega)$ *is a Banach algebra for every domain* $\Omega \in \mathcal{J}_\alpha$ *if and only if* $\alpha < 1$ *and either of the following conditions is satisfied:*

$$\begin{cases} m(1-\alpha) > \frac{1}{p}, \\ m(1-\alpha) = \frac{1}{p}, \ q = 1, \ \beta \geq 0, \\ m(1-\alpha) = \frac{1}{p}, \ q > 1, \ \beta > \frac{1}{q'}. \end{cases} \tag{1.27}$$

The Orlicz-Sobolev spaces $\mathcal{V}^m L^A(\Omega)$ are the object of the next result. In particular, it recovers a result from [21], dealing with the case of regular domains.

Proposition 1.13. *Let* $m \in \mathbb{N}$*,* $n \geq 2$*, and* $\alpha \in [\frac{1}{n'}, \infty)$*. Let* A *be a Young function. Then* $\mathcal{V}^m L^A(\Omega)$ *is a Banach algebra for every domain* $\Omega \in \mathcal{J}_\alpha$ *if and only if* $\alpha < 1$ *and either of the following conditions is satisfied:*

$$\begin{cases} m \geq \frac{1}{1-\alpha}, \\ m < \frac{1}{1-\alpha} \quad \text{and} \quad \int^\infty \left(\frac{t}{A(t)}\right)^{\frac{(1-\alpha)m}{1-(1-\alpha)m}} dt < \infty. \end{cases} \tag{1.28}$$

The Lorentz–Zygmund-Sobolev and Orlicz-Sobolev spaces for which the product operator is bounded into a lower-order space for every $\Omega \in \mathcal{J}_\alpha$ can be characterized via Theorem 1.9.

Proposition 1.14. *Let $n, m, k \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq m$, and let $\alpha \in [\frac{1}{n'}, \infty)$. Assume that $1 \leq p, q \leq \infty$, $\beta \in \mathbb{R}$ and one of the conditions in (1.26) is in force. Then, for every domain $\Omega \in \mathcal{J}_\alpha$ there exists a constant C such that*

$$\|uv\|_{\mathcal{V}^{m-k}L^{p,q;\beta}(\Omega)} \leq C\|u\|_{\mathcal{V}^m L^{p,q;\beta}(\Omega)}\|v\|_{\mathcal{V}^m L^{p,q;\beta}(\Omega)}$$

for every $u, v \in \mathcal{V}^m L^{p,q;\beta}(\Omega)$ if and only if $\alpha < 1$, and either of the following conditions is satisfied:

$$\begin{cases} (m+k)(1-\alpha) > \frac{1}{p}, \\ (m+k)(1-\alpha) = \frac{1}{p}, \ \beta \geq 0. \end{cases} \tag{1.29}$$

Proposition 1.15. *Let $n, m, k \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq m$, and let $\alpha \in [\frac{1}{n'}, \infty)$. Let A be a Young function. Then, for every domain $\Omega \in \mathcal{J}_\alpha$ there exists a constant C such that*

$$\|uv\|_{\mathcal{V}^{m-k}L^A(\Omega)} \leq C\|u\|_{\mathcal{V}^m L^A(\Omega)}\|v\|_{\mathcal{V}^m L^A(\Omega)}$$

for every $u, v \in \mathcal{V}^m L^{p,q;\beta}(\Omega)$ if and only if $\alpha < 1$ and either of the following conditions is satisfied:

$$\begin{cases} (m+k) \geq \frac{1}{1-\alpha}, \\ (m+k) < \frac{1}{1-\alpha} \quad \text{and} \quad A(t) \geq Ct^{\frac{1}{(1-\alpha)(m+k)}} \quad \text{for large } \ t\,, \end{cases} \tag{1.30}$$

for some positive constant C.

2. Background

We denote by $\mathcal{M}(\Omega)$ the set of all Lebesgue measurable functions from Ω into $[-\infty, \infty]$. We also define $\mathcal{M}_+(\Omega) = \{u \in \mathcal{M}(\Omega) : u \geq 0\}$, and $\mathcal{M}_0(\Omega) = \{u \in \mathcal{M}(\Omega) : u \text{ is finite a.e. in } \Omega\}$.

The *decreasing rearrangement* $u^* : (0,1) \to [0, \infty]$ of a function $u \in \mathcal{M}(\Omega)$ is defined as

$$u^*(s) = \sup\{t \in \mathbb{R} : \ |\{x \in \Omega : \ |u(x)| > t\}| > s\} \quad \text{for } s \in (0,1).$$

We also define $u^{**} : (0,1) \to [0, \infty]$ as

$$u^{**}(s) = \frac{1}{s}\int_0^s u^*(r)\, dr.$$

We say that a functional $\|\cdot\|_{X(0,1)} : \mathcal{M}_+(0,1) \to [0, \infty]$ is a *function norm*, if, for all f, g and $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{M}_+(0,1)$, and every $\lambda \geq 0$, the following properties hold:

(P1) $\|f\|_{X(0,1)} = 0$ if and only if $f = 0$ a.e.; $\|\lambda f\|_{X(0,1)} = \lambda\|f\|_{X(0,1)}$;
$\|f + g\|_{X(0,1)} \leq \|f\|_{X(0,1)} + \|g\|_{X(0,1)}$;
(P2) $f \leq g$ a.e. implies $\|f\|_{X(0,1)} \leq \|g\|_{X(0,1)}$;
(P3) $f_j \nearrow f$ a.e. implies $\|f_j\|_{X(0,1)} \nearrow \|f\|_{X(0,1)}$;
(P4) $\|1\|_{X(0,1)} < \infty$;
(P5) $\int_0^1 f(x)\, dx \leq C\|f\|_{X(0,1)}$ for some constant C independent of f.

If, in addition,

(P6) $\|f\|_{X(0,1)} = \|g\|_{X(0,1)}$ whenever $f^* = g^*$,

we say that $\|\cdot\|_{X(0,1)}$ is a *rearrangement-invariant function norm.*

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, the space $X(\Omega)$ is defined as the collection of all functions $u \in \mathcal{M}(\Omega)$ such that the expression

$$\|u\|_{X(\Omega)} = \|u^*\|_{X(0,1)}$$

is finite. Such expression defines a norm on $X(\Omega)$, and the latter is a Banach space endowed with this norm, called a *rearrangement-invariant space.* Moreover, $X(\Omega) \subset \mathcal{M}_0(\Omega)$ for any rearrangement-invariant space $X(\Omega)$.

With any rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, it is associated another functional on $\mathcal{M}_+(0,1)$, denoted by $\|\cdot\|_{X'(0,1)}$, and defined, for $g \in \mathcal{M}_+(0,1)$, by

$$\|g\|_{X'(0,1)} = \sup_{\substack{f\in\mathcal{M}_+(0,1)\\ \|f\|_{X(0,1)}\leq 1}} \int_0^1 f(s)g(s)\,ds. \tag{2.1}$$

It turns out that $\|\cdot\|_{X'(0,1)}$ is also an rearrangement invariant function norm, which is called the *associate function norm* of $\|\cdot\|_{X(0,1)}$. The rearrangement invariant space $X'(\Omega)$ built upon the function norm $\|\cdot\|_{X'(0,1)}$ is called the *associate space* of $X(\Omega)$. Given an rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, the *Hölder inequality*

$$\int_\Omega |u(x)v(x)|\,dx \leq \|u\|_{X(\Omega)}\|v\|_{X'(\Omega)} \tag{2.2}$$

holds for every $u \in X(\Omega)$ and $v \in X'(\Omega)$. For every rearrangement-invariant space $X(\Omega)$ the identity $X''(\Omega) = X(\Omega)$ holds and, moreover, for every $f \in \mathcal{M}(\Omega)$, we have that

$$\|u\|_{X(\Omega)} = \sup_{v\in\mathcal{M}(\Omega);\ \|v\|_{X'(\Omega)}\leq 1} \int_0^1 |u(x)v(x)|\,dx. \tag{2.3}$$

The fundamental functions of a rearrangement-invariant space $X(\Omega)$ and its associate space $X'(\Omega)$ satisfy

$$\varphi_X(t)\varphi_{X'}(t) = t \quad \text{for every } t \in (0,1). \tag{2.4}$$

Since we are assuming that Ω has finite measure,

$$L^\infty(\Omega) \to X(\Omega) \to L^1(\Omega) \tag{2.5}$$

for every rearrangement-invariant space $X(\Omega)$.
A basic property of rearrangements is the *Hardy-Littlewood inequality* which tells us that, if $u, v \in \mathcal{M}(\Omega)$, then

$$\int_\Omega |u(x)v(x)|dx \leq \int_0^1 u^*(t)v^*(t)dt. \tag{2.6}$$

A key fact concerning rearrangement-invariant function norms is the *Hardy–Littlewood–Pólya principle* which states that if, for some $u, v \in \mathcal{M}(\Omega)$,

$$\int_0^t u^*(s)\,ds \leq \int_0^t v^*(s)\,ds \text{ for } t \in (0,1), \tag{2.7}$$

then

$$\|u\|_{X(\Omega)} \leq \|v\|_{X(\Omega)}$$

for every rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$. Moreover,

$$\|uv\|_{X(\Omega)} \leq \|u^*v^*\|_{X(0,1)} \tag{2.8}$$

for every rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, and all functions $u, v \in \mathcal{M}(\Omega)$. Inequality (2.8) follows from the inequality

$$\int_0^t (uv)^*(s)\,ds \leq \int_0^t u^*(s)v^*(s)\,ds,$$

and the Hardy–Littlewood–Pólya principle.

We refer the reader to [7] for proofs of the results recalled above, and for a comprehensive treatment of rearrangement-invariant spaces.

Let $1 \le p, q \le \infty$ and $\beta \in \mathbb{R}$. We define the functional $\|\cdot\|_{L^{p,q;\beta}(0,1)}$ as

$$\|f\|_{L^{p,q;\beta}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} \log^\beta\left(\tfrac{2}{s}\right) f^*(s) \right\|_{L^q(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$. If one of the conditions in (1.26) is satisfied, then $\|\cdot\|_{L^{p,q;\beta}(0,1)}$ is equivalent to a rearrangement-invariant function norm, called a *Lorentz–Zygmund norm* (for details see e.g [8], [51] or [52]). The corresponding rearrangement-invariant space $L^{p,q;\beta}(\Omega)$ is called the *Lorentz–Zygmund space*. When $\beta = 0$, the space $L^{p,q;0}(\Omega)$ is denoted by $L^{p,q}(\Omega)$ and called *Lorentz space*. It is known (e.g. [51, Theorem 6.11] or [52, Theorem 9.6.13]) that

$$(L^{p,q;\beta})'(\Omega) = \begin{cases} L^{p',q';-\beta}(\Omega) & \text{if } p < \infty; \\ L^{(1,q';-\beta-1)}(\Omega) & \text{if } p = \infty,\ 1 \le q < \infty,\ \beta + \frac{1}{q} < 0; \\ L^1(\Omega) & \text{if } p = \infty,\ q = \infty,\ \beta = 0, \end{cases} \tag{2.9}$$

where $L^{(p,q;\beta)}(\Omega)$ denotes the function space defined analogously to $L^{p,q;\beta}(\Omega)$ but with the functional $\|\cdot\|_{L^{(p,q;\beta)}(0,1)}$ given by

$$\|f\|_{L^{(p,q;\beta)}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} \log^\beta\left(\tfrac{2}{s}\right) f^{**}(s) \right\|_{L^q(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$. Note that, if $\beta = 0$, then

$$L^{p,q;0}(\Omega) = L^{p,q}(\Omega),$$

a standard Lorentz space. In particular, if $p = q$, then

$$L^{p,p}(\Omega) = L^p(\Omega),$$

a Lebesgue space.

A generalization of the Lebesgue spaces in a different direction is provided by the Orlicz spaces. Let $A : [0,\infty) \to [0,\infty]$ be a Young function, namely a convex (non-trivial), left-continuous function vanishing at 0. The Orlicz space $L^A(\Omega)$ is the rearrangement-invariant space associated with the *Luxemburg function norm* defined as

$$\|f\|_{L^A(0,1)} = \inf\left\{\lambda > 0 : \int_0^1 A\left(\frac{f(s)}{\lambda}\right) ds \le 1\right\}$$

for $f \in \mathcal{M}_+(0,1)$. In particular, $L^A(\Omega) = L^p(\Omega)$ if $A(t) = t^p$ for some $p \in [1,\infty)$, and $L^A(\Omega) = L^\infty(\Omega)$ if $A(t) = \infty\chi_{(1,\infty)}(t)$.

The associate function norm of $\|\cdot\|_{L^A(0,1)}$ is equivalent to the function norm $\|\cdot\|_{L^{\widetilde{A}}(0,1)}$, where $\widetilde{A}$ is the Young conjugate of A defined as

$$\widetilde{A}(t) = \sup\{ts - A(s) : s \ge 0\} \qquad \text{for } t \ge 0.$$

3. Key one-dimensional inequalities

In this section we shall state and prove two key assertions concerning one-dimensional inequalities involving non-increasing functions and rearrangement-invariant spaces defined on an interval. Both these results are of independent interest and they constitute a new approach to inequalities involving products of functions.

Assume that I is a positive non-decreasing function on $(0,1)$. We denote by H_I the operator defined at every nonnegative measurable function g on $(0,1)$ by

$$H_I g(t) = \int_t^1 \frac{g(r)}{I(r)}\, dr \quad \text{for } t \in (0,1).$$

Moreover, given $m \in \mathbb{N}$, we set

$$H_I^m = \underbrace{H_I \circ H_I \circ \cdots \circ H_I}_{m-\text{times}}.$$

We also denote by H_I^0 the identity operator. It is easily verified that

$$H_I^m g(t) = \frac{1}{(m-1)!} \int_t^1 \frac{g(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \quad \text{for } m \in \mathbb{N} \text{ and } t \in (0,1),$$

see [25, Remark 8.2 (iii)].

Let $X(0,1)$ be a rearrangement-invariant space and let $m \in \mathbb{N}$. If the function I satisfies (1.19), then the optimal (smallest) rearrangement-invariant space $X_m(0,1)$ such that

$$H_I^m : X(0,1) \to X_m(0,1) \tag{3.1}$$

is endowed with the function norm $\|\cdot\|_{X_m(0,1)}$, whose associate function norm is given by

$$\|g\|_{X_m'(0,1)} = \left\| \frac{1}{I(t)} \int_0^t g^*(s) \left(\int_s^t \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{X'(0,1)} \tag{3.2}$$

for $g \in \mathcal{M}(0,1)$ [25, Proposition 8.3].

The following lemma provides us with a pointwise inequality involving the operator H_I^k for $k = 1, \dots, m-1$ for I satisfying (1.20). For every such I we denote by ψ_I the function defined by

$$\psi_I(t) = \left(\int_0^t \frac{dr}{I(r)} \right)^m \quad \text{for } t \in (0,1). \tag{3.3}$$

Lemma 3.1. *Assume that $m, k \in \mathbb{N}$, $m \geq 2$, $1 \leq k \leq m-1$. Let I be positive a non-decreasing function on $(0,1)$ satisfying (1.20) and let ψ_I be the function defined by (3.3). There exists a constant $C = C(m)$ such that, if $g \in \mathcal{M}_+(0,1)$ and*

$$g^*(t) \leq \frac{1}{\psi_I(t)} \quad \textit{for } \ t \in (0,1), \tag{3.4}$$

then

$$H_I^k g^*(t) \leq C g^*(t)^{1-\frac{k}{m}} \quad \textit{for } \ t \in (0,1) \textit{ and } k \in \{1, \dots, m\}. \tag{3.5}$$

Proof. Fix $g \in \mathcal{M}_+(0,1)$ such that (3.4) holds, and any $t \in (0,1)$. Define $a \in (0,1]$ by the identity

$$g^*(t) = \frac{1}{\psi_I(a)} \quad \text{if } g^*(t) > \frac{1}{\psi_I(1)}$$

and $a = 1$ otherwise. Note that the definition is correct since ψ_I is continuous and strictly increasing on $(0,1)$, and $\lim_{t \to 0_+} \psi_I(t) = 0$.
Assume first that $g^*(t) > \frac{1}{\psi_I(1)}$. Then (3.4) and the monotonicity of ψ_I imply that $t \leq a \leq 1$. We thus get

$$g^*(s) \leq \begin{cases} g^*(t) & \text{if } s \in [t,a), \\ \frac{1}{\psi_I(s)} & \text{if } s \in [a,1). \end{cases}$$

Fix $k \in \{1, \dots, m-1\}$. Then, consequently,

$$\begin{aligned} (k-1)! H_I^k g^*(t) &= \int_t^a \frac{g^*(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds + \int_a^1 \frac{g^*(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds \\ &\leq g^*(t) \int_t^a \frac{1}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds + \int_a^1 \frac{1}{\psi_I(s) I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds. \end{aligned}$$

By the definition of a,

$$g^*(t) \int_t^a \frac{1}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds = \frac{g^*(t)}{k} \left(\int_t^a \frac{dr}{I(r)} \right)^k \leq \frac{g^*(t)}{k} \left(\int_0^a \frac{dr}{I(r)} \right)^k = \frac{g^*(t)}{k} \psi_I(a)^{\frac{k}{m}} = \frac{1}{k} g^*(t)^{1-\frac{k}{m}}$$

and

$$\int_a^1 \frac{1}{\psi_I(s)I(s)}\left(\int_t^s \frac{dr}{I(r)}\right)^{k-1} ds = \int_a^1 \frac{1}{I(s)}\frac{\left(\int_t^s \frac{dr}{I(r)}\right)^{k-1}}{\left(\int_0^s \frac{dr}{I(r)}\right)^{m}} ds \le \int_a^1 \frac{1}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{k-m-1} ds$$
$$= \tfrac{1}{m-k}\left[\left(\int_0^a \frac{dr}{I(r)}\right)^{k-m} - \left(\int_0^1 \frac{dr}{I(r)}\right)^{k-m}\right]$$
$$\le \tfrac{1}{m-k}\left(\int_0^a \frac{dr}{I(r)}\right)^{k-m} = \tfrac{1}{m-k}\psi_I(a)^{\frac{k}{m}-1} = \tfrac{1}{m-k}g^*(t)^{1-\frac{k}{m}}.$$

Assume next that $g^*(t) \le \frac{1}{\psi_I(1)}$. Then $a = 1$. Similarly as above, we have that

$$(k-1)!H_I^k g^*(t) = \int_t^1 \frac{g^*(s)}{I(s)}\left(\int_t^s \frac{dr}{I(r)}\right)^{k-1} ds \le g^*(t)\int_t^1 \frac{1}{I(s)}\left(\int_t^s \frac{dr}{I(r)}\right)^{k-1} ds$$
$$= \frac{g^*(t)}{k}\left(\int_t^1 \frac{dr}{I(r)}\right)^{k} \le \frac{g^*(t)}{k}\psi_I(1)^{\frac{k}{m}} \le \frac{1}{k}g^*(t)^{1-\frac{k}{m}}.$$

Altogether, inequality (3.5) follows. □

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$ and $p \in (1,\infty)$, we define the functional $\|\cdot\|_{X^p(0,1)}$ by

$$\|g\|_{X^p(0,1)} = \|g^p\|_{X(0,1)}^{\frac{1}{p}} \quad \text{for } g \in \mathcal{M}_+(0,1).$$

The functional $\|\cdot\|_{X^p(0,1)}$ is also an rearrangement invariant function norm. Moreover, the inequality

$$\|fg\|_{X(0,1)} \le \|f\|_{X^p(0,1)}\|g\|_{X^{p'}(0,1)} \tag{3.6}$$

holds for every $f, g \in \mathcal{M}(0,1)$ (see e.g. [44, Lemma 1]).

The following lemma, of possible independent interest, is a major tool in the proofs of our main results.

Lemma 3.2. *Assume that $m \in \mathbb{N}$, $m \ge 2$. Let I be a positive non-decreasing function on $(0,1)$, and let ψ_I be the function defined by (3.3). Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the following statements are equivalent:*

(i) *Condition (1.20) holds, and there exists a positive constant C such that*

$$\sup_{t\in(0,1)} g^{**}(t)\psi_I(t) \le C\|g\|_{X(0,1)} \tag{3.7}$$

for every $g \in \mathcal{M}_+(0,1)$.

(ii) *Condition (1.20) holds, and there exists a positive constant C such that*

$$\sup_{t\in(0,1)} g^{*}(t)\psi_I(t) \le C\|g\|_{X(0,1)} \tag{3.8}$$

for every $g \in \mathcal{M}_+(0,1)$.

(iii) *For every $k \in \mathbb{N}$, $1 \le k \le m-1$, there is a positive constant C such that*

$$\|g\|_{X^{\frac{m}{m-k}}(0,1)} \le C\|g\|_{X_k(0,1)} \tag{3.9}$$

for every $g \in \mathcal{M}_+(0,1)$.

(iv) *For every $k \in \mathbb{N}$, $1 \le k \le m-1$, there exists a positive constant C such that*

$$\|fg\|_{X(0,1)} \le C\|f\|_{X_k(0,1)}\|g\|_{X_{m-k}(0,1)} \tag{3.10}$$

for every $f, g \in \mathcal{M}_+(0,1)$.

(v) *There exists $k \in \mathbb{N}$, $1 \le k \le m-1$, and a positive constant C such that (3.10) holds.*

(vi) *There exists $k \in \mathbb{N}$, $1 \le k \le m-1$, and a positive constant C such that*

$$\|H_I^k f \, H_I^{m-k} g\|_{X(0,1)} \le C\|f\|_{X(0,1)}\|g\|_{X(0,1)} \tag{3.11}$$

for every $f, g \in \mathcal{M}_+(0,1)$.

(vii) *There exists a positive constant C such that*

$$\left(\int_0^t \frac{ds}{I(s)}\right)^m \le C\varphi_X(t) \quad \textit{for } t \in (0,1). \tag{3.12}$$

Proof. (i)⇒(ii) This implication is trivial thanks to the universal pointwise estimate $g^*(t) \le g^{**}(t)$ which holds for every $g \in \mathcal{M}(0,1)$ and every $t \in (0,1)$.

(ii)⇒(iii) Fix $k \in \mathbb{N}$, $1 \le k \le m-1$. In view of the optimality of the space $X_k(0,1)$ in $H_I^k : X(0,1) \to X_k(0,1)$, mentioned above Lemma 3.1, the assertion will follow once we show that

$$H_I^k : X(0,1) \to X^{\frac{m}{m-k}}(0,1). \tag{3.13}$$

Let $g \in X(0,1)$, $g \not\equiv 0$, and let C be the constant from (3.8). Define $h = \frac{g}{C\|g\|_{X(0,1)}}$. Inequality (3.8) implies that $h^*(t) \le \frac{1}{\psi_I(t)}$ for $t \in (0,1)$, whence, by Lemma 3.1,

$$H_I^k h^*(t) \le C' h^*(t)^{1-\frac{k}{m}} \quad \text{for } t \in (0,1),$$

for some constant $C' = C'(m,k)$. Thus,

$$\begin{aligned}\|H_I^k g^*\|_{X^{\frac{m}{m-k}}(0,1)} &= C\|g\|_{X(0,1)}\|H_I^k h^*\|_{X^{\frac{m}{m-k}}(0,1)} \le C'C\|g\|_{X(0,1)}\|{h^*}^{1-\frac{k}{m}}\|_{X^{\frac{m}{m-k}}(0,1)} \\ &= C'C\|g\|_{X(0,1)}\|h^*\|_{X(0,1)}^{1-\frac{k}{m}} = C'C^{\frac{k}{m}}\|g\|_{X(0,1)}.\end{aligned}$$

By [25, Corollary 9.8], this is equivalent to the existence of a positive constant $C(m,k,X)$ such that

$$\|H_I^k g\|_{X^{\frac{m}{m-k}}(0,1)} \le C(m,k,X)\|g\|_{X(0,1)}$$

for every $g \in \mathcal{M}_+(0,1)$. Hence, (3.13) follows.

(iii)⇒(iv) Fix $k \in \{1,\dots,m-1\}$ and let $f, g \in \mathcal{M}_+(0,1)$. On applying first (3.9) to f in place of g, and then (3.9) again, this time with k replaced by $m-k$, we obtain

$$\|f\|_{X^{\frac{m}{m-k}}(0,1)} \le C\|f\|_{X_k(0,1)} \quad \text{and} \quad \|g\|_{X^{\frac{m}{k}}(0,1)} \le C\|g\|_{X_{m-k}(0,1)}.$$

Combining these estimates with (3.6), with $p = \frac{m}{m-k}$, yields

$$\|fg\|_{X(0,1)} \le C^2\|f\|_{X_k(0,1)}\|g\|_{X_{m-k}(0,1)},$$

and (iv) follows.

(iv)⇒(v) This implication is trivial.

(v)⇒(vi) Let $k \in \mathbb{N}$, $1 \le k \le m-1$, be such that (3.10) holds, and let $f, g \in \mathcal{M}_+(0,1)$. On making use of (3.10) with $H_I^k f$ and $H_I^{m-k} g$ in the place of f and g, respectively, we obtain

$$\|H_I^k f \, H_I^{m-k} g\|_{X(0,1)} \le C\|H_I^k f\|_{X_k(0,1)}\|H_I^{m-k} g\|_{X_{m-k}(0,1)}.$$

It follows from (3.1) that

$$H_I^k : X(0,1) \to X_k(0,1) \quad \text{and} \quad H_I^{m-k} : X(0,1) \to X_{m-k}(0,1).$$

Coupling these facts with the preceding inequality implies that

$$\|H_I^k f \, H_I^{m-k} g\|_{X(0,1)} \le C'\|f\|_{X(0,1)}\|g\|_{X(0,1)}$$

for some positive constant $C' = C'(m,k,I,X)$ but independent of $f, g \in \mathcal{M}_+(0,1)$. Thus, the property (vi) follows.

(vi)⇒(vii) Let $k \in \mathbb{N}$, $1 \le k \le m-1$, be such that (3.11) holds. Assume, for the time being, that $m < 2k$. On replacing, if necessary, $\|\cdot\|_{X(0,1)}$ with the equivalent norm $C\|\cdot\|_{X(0,1)}$, we may suppose, without loss of generality, that $C = 1$ in (3.11). Thus,

$$\|(H_I^k f)(H_I^{m-k} g)\|_{X(0,1)} \le \|f\|_{X(0,1)}\|g\|_{X(0,1)} \tag{3.14}$$

for every $f, g \in \mathcal{M}_+(0,1)$.
Let $b > -1$. Fix $a \in (0,1]$ and $\varepsilon \in (0,a)$. Set

$$f(t) = \chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^b, \quad g(t) = \chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{k}{m-k}+k-m} \tag{3.15}$$

for $t \in (0,1)$. One can verify that

$$H_I^k f(t) = \frac{1}{(b+1)\dots(b+k)}\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)} \quad \text{for } t \in (\varepsilon,a). \tag{3.16}$$

Since we are assuming that $m < 2k$ and $b > -1$,

$$(b+k)\tfrac{k}{m-k} + k - m > (-1+k)\tfrac{k}{2k-k} + k - 2k = -1. \tag{3.17}$$

Hence, analogously to (3.16),

$$H_I^{m-k} g(t) = \frac{1}{[(b+k)\frac{k}{m-k}+k-m+1]\dots[(b+k)\frac{k}{m-k}]}\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{k}{m-k}} \quad \text{for } t \in (\varepsilon,a). \tag{3.18}$$

Note that

$$(b+k)\tfrac{k}{m-k} + k - m = b\tfrac{m-k}{k} + (b+k)\tfrac{m}{m-k}\tfrac{2k-m}{k}. \tag{3.19}$$

Set $p = \frac{k}{m-k}$. The assumption $m < 2k$ guarantees that $p > 1$. Moreover, $p' = \frac{k}{2k-m}$. Thus, inequality (3.6), applied to this choice of p, and equation (3.19) tell us that

$$\begin{aligned} \|g\|_{X(0,1)} &= \left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{k}{m-k}+k-m}\right\|_{X(0,1)} \\ &\le \left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{b}\right\|_{X(0,1)}^{\frac{m-k}{k}} \left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)}^{\frac{2k-m}{k}} \\ &= \|f\|_{X(0,1)}^{\frac{m}{k}-1}\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)}^{\frac{2k-m}{k}}. \end{aligned} \tag{3.20}$$

Coupling (3.14) with (3.20) yields

$$\|H_I^k f\, H_I^{m-k} g\|_{X(0,1)} \le \|f\|_{X(0,1)}^{\frac{m}{k}}\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)}^{\frac{2k-m}{k}}. \tag{3.21}$$

By (3.16) and (3.18),

$$H_I^k f(t)\, H_I^{m-k} g(t) = \frac{1}{(b+1)\dots(b+k)[(b+k)\frac{k}{m-k}+k-m+1]\dots[(b+k)\frac{k}{m-k}]}\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}$$

for $t \in (\varepsilon,a)$. On setting

$$B(b) = (b+1)\dots(b+k)\big[(b+k)\tfrac{k}{m-k}+k-m+1]\dots[(b+k)\tfrac{k}{m-k}\big] \tag{3.22}$$

and making use of (3.21), we obtain

$$\begin{aligned}
\frac{1}{B(b)}\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)} &= \left\|\chi_{(\varepsilon,a)}\, H_I^k f\, H_I^{m-k} g\right\|_{X(0,1)} \\
&\le \left\|H_I^k f\, H_I^{m-k} g\right\|_{X(0,1)} \\
&\le \|f\|_{X(0,1)}^{\frac{m}{k}} \left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)}^{\frac{2k-m}{k}}.
\end{aligned} \tag{3.23}$$

The function

$$(0,1) \ni t \mapsto \chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}$$

is bounded and hence, by (2.5), belongs to $X(0,1)$. Thus,

$$\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)} < \infty.$$

Moreover, $1-\frac{2k-m}{k} = \frac{m}{k}-1$. Therefore, (3.23) yields

$$\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)}^{\frac{m}{k}-1} \le B(b)\, \|f\|_{X(0,1)}^{\frac{m}{k}}.$$

On raising this inequality to the power $\frac{k}{m}$, and recalling the definition of f, we get

$$\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)}^{1-\frac{k}{m}} \le B(b)^{\frac{k}{m}} \left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{b}\right\|_{X(0,1)}. \tag{3.24}$$

Next, assume that $m=2k$. Note that (3.17) holds also in this case. Let f and g be defined by (3.15) again. Then

$$(b+k)\tfrac{k}{m-k} + k - m = b,$$

whence $f=g$. Moreover, since $k=m-k$, we also have that $H_I^k f = H_I^{m-k} g$. Therefore, (3.14), (3.16) and (3.18) imply that

$$\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{2(b+k)}\right\|_{X(0,1)}^{\frac{1}{2}} \le (b+1)\dots(b+k)\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a \frac{dr}{I(r)}\right)^{b}\right\|_{X(0,1)}.$$

The assumption $m=2k$ entails that $B(b) = [(b+1)\dots(b+k)]^2$, and $\frac{k}{m}=\frac{1}{2}$. Hence, (3.24) follows. We have thus established (3.24) whenever $m \le 2k$. From now on, we keep this assumption in force. Define $b_0 = 0$ and, for $j \in \mathbb{N}$,

$$b_j = (b_{j-1}+k)\tfrac{m}{m-k},$$

namely

$$b_j = m\big[(\tfrac{m}{m-k})^j - 1\big]. \tag{3.25}$$

We next set, for $j \in \mathbb{N}$,

$$K_j = \prod_{i=0}^{j-1} B(b_i)^{\frac{k}{m}(\frac{m-k}{m})^i}.$$

Let us note that the assumption $2k \ge m$ implies that $B(b_j) \ge 1$ for $j \in \mathbb{N}\cup\{0\}$, and hence $K_j \ge 1$ as well.

We claim that, for every $j \in \mathbb{N}$,

$$\left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_j} \right\|_{X(0,1)}^{(\frac{m-k}{m})^j} \leq K_j \|\chi_{(\varepsilon,a)}\|_{X(0,1)}. \tag{3.26}$$

Indeed, choosing $b = 0$ in (3.24), yields (3.26) for $j = 1$. Assume now that (3.26) holds for some fixed $j \in \mathbb{N}$. Then, by (3.24) with $b = b_j$,

$$\left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_{j+1}} \right\|_{X(0,1)}^{(\frac{m-k}{m})^{j+1}} \leq B(b_j)^{\frac{k}{m}(\frac{m-k}{m})^j} \left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_j} \right\|_{X(0,1)}^{(\frac{m-k}{m})^j}.$$

Thus, by the induction assumption,

$$\left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_{j+1}} \right\|_{X(0,1)}^{(\frac{m-k}{m})^{j+1}} \leq B(b_j)^{\frac{k}{m}(\frac{m-k}{m})^j} K_j \left\| \chi_{(\varepsilon,a)} \right\|_{X(0,1)} = K_{j+1} \|\chi_{(\varepsilon,a)}\|_{X(0,1)},$$

and (3.26) follows.

Letting $\varepsilon \to 0^+$ in (3.26) and making use of property (P3) of rearrangement-invariant function norms yields

$$\left\| \chi_{(0,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_j} \right\|_{X(0,1)}^{(\frac{m-k}{m})^j} \leq K_j \|\chi_{(0,a)}\|_{X(0,1)}. \tag{3.27}$$

Fix $j \in \mathbb{N}$, and set

$$g_j(t) = \chi_{(0,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_j} \quad \text{for } t \in (0,1).$$

Then, owing to (3.27)

$$\|g_j\|_{X(0,1)}^{(\frac{m-k}{m})^j} \leq K_j \|\chi_{(0,a)}\|_{X(0,1)}. \tag{3.28}$$

By the Hölder inequality and (2.4),

$$\int_0^a g_j(t)\, dt \leq \|g_j\|_{X(0,1)} \|\chi_{(0,a)}\|_{X'(0,1)} = \frac{a\|g_j\|_{X(0,1)}}{\|\chi_{(0,a)}\|_{X(0,1)}}.$$

By (3.28),

$$\frac{1}{a} \int_0^a g_j(t)\, dt \leq K_j^{\frac{1}{(\frac{m-k}{m})^j}} \|\chi_{(0,a)}\|_{X(0,1)}^{\frac{1-(\frac{m-k}{m})^j}{(\frac{m-k}{m})^j}}.$$

We have thus shown that

$$\left(\frac{1}{a} \int_0^a g_j(t)\, dt \right)^{\frac{(\frac{m-k}{m})^j}{1-(\frac{m-k}{m})^j}} \leq K_j^{\frac{1}{1-(\frac{m-k}{m})^j}} \|\chi_{(0,a)}\|_{X(0,1)} \tag{3.29}$$

for $j \in \mathbb{N}$.

Observe that

$$\lim_{j\to\infty}\left(\frac{1}{a}\int_0^a g_j(t)\,dt\right)^{\frac{(\frac{m-k}{m})^j}{1-(\frac{m-k}{m})^j}} = \lim_{j\to\infty}\left(\frac{1}{a}\int_0^a\left(\int_t^a\frac{dr}{I(r)}\right)^{b_j}dt\right)^{\frac{1}{(\frac{m}{m-k})^j-1}} \tag{3.30}$$
$$= \lim_{j\to\infty}\left(\frac{1}{a}\int_0^a\left(\int_t^a\frac{dr}{I(r)}\right)^{m((\frac{m}{m-k})^j-1)}dt\right)^{\frac{1}{(\frac{m}{m-k})^j-1}}$$
$$= \left\|\chi_{(0,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^m\right\|_{L^\infty(0,1)} = \left(\int_0^a\frac{dr}{I(r)}\right)^m.$$

On the other hand, since $\frac{m-k}{m}<1$,

$$1-\left(\tfrac{m-k}{m}\right)^j \geq 1-\tfrac{m-k}{m} = \tfrac{k}{m}$$

for $j\in\mathbb{N}$. As observed above, $K_j\geq 1$ for every $j\in\mathbb{N}$. By (3.22), if $b>-1$, then

$$B(b)\leq (b+k)^k((b+k)\tfrac{k}{m-k})^{m-k}\leq (b+k)^m(\tfrac{m}{m-k})^m.$$

With the choice $b=b_j$, the last chain and (3.25) yield

$$B(b_j)\leq (m((\tfrac{m}{m-k})^j-1)+k)^m(\tfrac{m}{m-k})^m\leq (m(\tfrac{m}{m-k})^{j+1})^m$$

for $j\in\mathbb{N}\cup\{0\}$. Altogether, we deduce that

$$K_j^{\frac{1}{1-(\frac{m-k}{m})^j}}\leq K_j^{\frac{m}{k}}=\prod_{i=0}^{j-1}B(b_i)^{(\frac{m-k}{m})^i}\leq\prod_{i=0}^{\infty}\left(\left(m\left(\tfrac{m}{m-k}\right)^{i+1}\right)^m\right)^{(\frac{m-k}{m})^i}<\infty.$$

On setting

$$K=\prod_{i=0}^{\infty}\left(\left(m\left(\tfrac{m}{m-k}\right)^{i+1}\right)^m\right)^{(\frac{m-k}{m})^i},$$

we have that

$$\limsup_{j\to\infty}K_j^{\frac{1}{1-(\frac{m-k}{m})^j}}\leq K. \tag{3.31}$$

Therefore, combining (3.29), (3.30) and (3.31) tells us that

$$\left(\int_0^a\frac{dr}{I(r)}\right)^m\leq K\|\chi_{(0,a)}\|_{X(0,1)}.$$

Hence, inequality (3.12) follows. Thus, property (vii) is proved when $m\leq 2k$. However, if this is not the case, then $m\leq 2(m-k)$. The same argument as above, applied with $m-k$ in the place of k, leads to the conclusion.

(vii)⇒(i) Let $g\in\mathcal{M}(0,1)$. Then, by (3.12),

$$\sup_{t\in(0,1)}g^{**}(t)\psi_I(t)\leq C\sup_{t\in(0,1)}g^{**}(t)\varphi_X(t).$$

One has that

$$\sup_{t\in(0,1)}g^{**}(t)\varphi_X(t)\leq\|g\|_{X(0,1)}$$

for every $g\in\mathcal{M}(0,1)$ (see e.g. It is a classical fact (see e.g. [7, Chapter 2, Proposition 5.9]). Combining the last two estimates yields

$$\sup_{t\in(0,1)}g^{**}(t)\psi_I(t)\leq C\|g\|_{X(0,1)}$$

for $g\in\mathcal{M}(0,1)$, namely (3.7). □

We conclude this section by showing that assumption (1.9) is actually essentially stronger than (1.24).

Proposition 3.3. *Let $m, n \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that I is a positive non-decreasing function on $(0,1)$. If (1.9) holds, then (1.24) holds as well.*

Proof. By the Hölder inequality in rearrangement-invariant spaces,

$$\left(\int_0^t \frac{ds}{I(s)}\right)^m = m\int_0^t \frac{1}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{m-1} ds \leq m\left\|\frac{1}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{m-1}\right\|_{X'(0,1)} \|\chi_{(0,t)}\|_{X(0,1)}$$
$$= m\left\|\frac{1}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{m-1}\right\|_{X'(0,1)} \varphi_X(t) \qquad \text{for } t \in (0,1).$$

Hence, the assertion follows. □

Remark 3.4. It is easily seen that (1.9) is in fact essentially stronger than (1.24), in general. Indeed, let $I(t) = t^\alpha$ for some $\alpha \in \mathbb{R}$ such that $\alpha \geq \frac{1}{n'}$ and $\alpha > \frac{1}{m'}$, and let $\|\cdot\|_{X(0,1)} = \|\cdot\|_{L^q(0,1)}$ with $q = \frac{1}{m(1-\alpha)}$. Then (1.24) holds but (1.9) does not. In other words, by Theorems 1.3 and 1.9, $\mathcal{V}^m L^{\frac{1}{m(1-\alpha)}}(\Omega)$ is not a Banach algebra for every $\Omega \in \mathcal{J}_\alpha$, yet it satisfies (1.25) for every $\Omega \in \mathcal{J}_\alpha$.

4. Proofs of the main results

Here, we accomplish the proofs of the results stated in Section 1.
A result to be exploited in our proofs is an embedding theorem for the space $\mathcal{V}^m X(\Omega)$, which tells us that, under assumption (1.19),

$$V^m X(\Omega) \to X_m(\Omega) \tag{4.1}$$

where $X_m(\Omega)$ is the rearrangement-invariant space built upon the function norm $\|\cdot\|_{X_m(0,1)}$ given by (3.1), and that $X_m(\Omega)$ is the optimal (smallest) such rearrangement-invariant space [25, Theorem 5.4] (see also [27], [19] and [39] for earlier proofs in special cases).

The next three lemmas are devoted to certain "worst possible" domains whose isoperimetric function has a prescribed decay. Such domains will be of use in the proof of the necessity of our conditions in the main results.

Lemma 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and let I be a positive , non-decreasing function satisfying (1.11). Then there exists a positive non-decreasing function $\widehat{I}$ in $(0,1)$ such that $\widehat{I} \in C^1(0,1)$, $\widehat{I}^{n'}$ is convex on $(0,1)$, and*

$$\widehat{I}(s) \approx I(s) \quad \textit{for } s \in (0,1). \tag{4.2}$$

Proof. By (1.11), there exists a non-decreasing function $\varsigma : (0,1) \to (0,\infty)$ such that

$$\frac{I(s)^{n'}}{s} \approx \varsigma(s) \qquad \text{for } s \in (0,1). \tag{4.3}$$

Set

$$I_1(s) = \left(\int_0^s \varsigma(r)\, dr\right)^{1/n'} \qquad \text{for } s \in (0,1).$$

Then $I_1 \in C^0(0,1)$, and $I_1^{n'}$ is convex in $(0,1)$. Moreover, we claim that

$$I_1(s) \approx I(s) \qquad \text{for } s \in (0,1). \tag{4.4}$$

Indeed, by the monotonicity of ς,

$$\tfrac{1}{2}\varsigma(s/2) \le \frac{1}{s}\int_{s/2}^{s} \varsigma(r)dr \le \frac{I_1(s)^{n'}}{s} = \frac{1}{s}\int_0^s \varsigma(r)dr \le \varsigma(s) \qquad \text{for } s \in (0,1). \tag{4.5}$$

Equation (4.4) follows from (4.3) and (4.5). Similarly, on setting

$$\widehat{I}(s) = \left(\int_0^s \frac{I_1(r)^{n'}}{r} dr\right)^{1/n'} \qquad \text{for } s \in (0,1),$$

one has that $\widehat{I} \in C^1(0,1)$, $\widehat{I}^{n'}$ is convex in $(0,1)$, and

$$\widehat{I}(s) \approx I_1(s) \qquad \text{for } s \in (0,1). \tag{4.6}$$

Coupling (4.4) with (4.6) yields (4.2). Thus, the function $\widehat{I}$ has the required properties. □

Lemma 4.2. *Let $n \in \mathbb{N}$, $n \ge 2$, and let I be a positive, non-decreasing function on $(0,1)$ satisfying (1.11). Then there exist $L \in (0,\infty]$ and a convex function $\eta : (0,L) \to (0,\infty)$ such that the domain $\Omega_I \subset \mathbb{R}^n$, defined by*

$$\Omega_I = \{x \in \mathbb{R}^n \colon x = (x', x_n),\ x_n \in (0,L),\ x' \in \mathbb{R}^{n-1},\ |x'| < \eta(x_n)\}, \tag{4.7}$$

satisfies $|\Omega_I| = 1$ and

$$I_{\Omega_I}(s) \approx I(s) \quad \text{for } s \in (0, \tfrac{1}{2}]. \tag{4.8}$$

Proof. By Lemma 4.1, we can assume with no loss of generality that $I \in C^1(0,1)$ and $I^{n'}$ is convex in $(0,1)$. Let $L \in (0,\infty]$ be defined by

$$L = \int_0^1 \frac{dr}{I(r)}, \tag{4.9}$$

and let $M : [0,L) \to (0,1]$ be the function implicitly defined as

$$\int_{M(t)}^1 \frac{dr}{I(r)} = t \qquad \text{for } t \in [0,L). \tag{4.10}$$

The function M strictly decreases from 1 to 0. In particular, M is continuously differentiable in $(0,L)$, and

$$I(M(r)) = -M'(r) \qquad \text{for } r \in (0,L)\,. \tag{4.11}$$

On defining $\eta : (0,L) \to (0,\infty)$ as

$$\eta(r) = \left(\frac{I(M(r))}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \qquad \text{for } r \in (0,L), \tag{4.12}$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional unit ball, we have that $\eta(r) > 0$ for $r \in (0,L)$, and, by (4.11),

$$|\{x \in \Omega_I : t < x_n < L\}| = \omega_{n-1}\int_t^L \eta(r)^{n-1}\,dr = \int_t^L I(M(r))\,dr = \int_t^L -M'(r)\,dr = M(t) \quad \text{for } t \in (0,L). \tag{4.13}$$

Moreover, η is convex. To see this, notice that, by (4.11),

$$\begin{aligned} \omega_{n-1}^{\frac{1}{n-1}}\eta'(r) = \big(I(M(r))^{\frac{1}{n-1}}\big)' &= \frac{1}{n-1} I'(M(r))I(M(r))^{\frac{1}{n-1}-1}M'(r) \\ &= -\frac{1}{n-1} I'(M(r))I(M(r))^{\frac{1}{n-1}} \quad \text{for } r \in (0,L). \end{aligned} \tag{4.14}$$

Thus, since $M(r)$ is decreasing, $\eta'(r)$ is increasing if and only if $I'(s)I(s)^{\frac{1}{n-1}}$ is increasing, and this is in turn equivalent to the convexity of $I(s)^{n'}$. Equation (4.14) also tells us that

$$\eta'(0^+) = -\frac{1}{n-1}\omega_{n-1}^{-\frac{1}{n-1}} I'(1^-)I(1^-)^{\frac{1}{n-1}} > -\infty.$$

By (4.13), with $t=0$, the set Ω_I as in (4.7), with η given by (4.12), satisfies $|\Omega_I| = 1$. Furthermore, owing to the properties of η, from either [47, Example 5.3.3.1] or [47, Example 5.3.3.2], according to whether $L < \infty$ or $L = \infty$, we infer that there exist positive constants C_1 and C_2 such that

$$C_1\, \eta(M^{-1}(s))^{n-1} \le I_{\Omega_I}(s) \le C_2\, \eta(M^{-1}(s))^{n-1} \qquad \text{for } s \in (0, \tfrac{1}{2}]. \tag{4.15}$$

Hence, by (4.12),

$$\frac{C_1}{\omega_{n-1}}\, I(s) \le I_{\Omega_I}(s) \le \frac{C_2}{\omega_{n-1}}\, I(s) \qquad \text{for } s \in (0, \tfrac{1}{2}], \tag{4.16}$$

and (4.8) follows. □

Lemma 4.3. *Let $n \in \mathbb{N}$, $n \ge 2$. Let I be a positive non-decreasing function on $(0,1)$ such that $I \in C^1(0,1)$ and $I^{n'}$ is convex in $(0,1)$. Let L, M, η and Ω_I be as in Lemma 4.2. Given $h \in \mathcal{M}_+(0,1)$, let $F : \Omega_I \to [0,\infty)$ be the function defined by*

$$F(x) = h(M(x_n)) \quad \text{for } x \in \Omega_I.$$

Then

$$F^*(t) = h^*(t) \quad \text{for } t \in (0,1). \tag{4.17}$$

Proof. On making use of (4.12), of a change of variables and of (4.11), we obtain that

$$\begin{aligned} |\{x \in \Omega_I \colon F(x) > \lambda\}| &= |\{x \in \Omega_I \colon h(M(x_n)) > \lambda\}| = \omega_{n-1} \int_{\{r\in(0,L)\colon h(M(r))>\lambda\}} \eta(r)^{n-1}\, dr \\ &= \int_{\{r\in(0,L)\colon h(M(r))>\lambda\}} I(M(r))\, dr = \int_{\{t\in(0,1)\colon h(t)>\lambda\}} -\frac{I(t)}{M'(M^{-1}(t))}\, dt \\ &= |\{t \in (0,1) \colon h(t) > \lambda\}| \qquad \text{for } \lambda > 0. \end{aligned}$$

Equation (4.17) hence follows, via the definition of the decreasing rearrangement. □

Proposition 4.4. *Assume that $m, n \in \mathbb{N}$, $n \ge 2$, and let I be a positive non-decreasing function satisfying (1.11). Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that for every domain $\Omega \in \mathcal{J}_I$, the Sobolev space $\mathcal{V}^m X(\Omega)$ is a Banach algebra. Then condition (1.9) holds. Moreover,*

$$\mathcal{V}^m X(\Omega) \to L^\infty(\Omega) \tag{4.18}$$

for every such domain Ω.

Proof. By Lemma 4.1, we can assume, without loss of generality, that $I \in C^1(0,1)$ and $I^{n'}$ is convex in $(0,1)$. Let L, M, η and Ω_I be as in Lemma 4.2. Let $f, g \in \mathcal{M}_+(0,1)$. We define the functions $u, v : \Omega_I \to [0,\infty]$ by

$$u(x) = \int_{M(x_n)}^1 \frac{1}{I(r_1)} \int_{r_1}^1 \frac{1}{I(r_2)} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{I(r_m)} dr_m\, dr_{m-1} \ldots dr_1 \quad \text{for } x \in \Omega_I, \tag{4.19}$$

and

$$v(x) = \int_{M(x_n)}^1 \frac{1}{I(r_1)} \int_{r_1}^1 \frac{1}{I(r_2)} \cdots \int_{r_{m-1}}^1 \frac{g(r_m)}{I(r_m)} dr_m\, dr_{m-1} \ldots dr_1 \quad \text{for } x \in \Omega_I. \tag{4.20}$$

Then the functions u and v are m times weakly differentiable in Ω_I. Since u is a non-decreasing function of the variable x_n,

$$\begin{aligned}|\nabla u(x)| = \frac{\partial u}{\partial x_n}(x) &= -\frac{M'(x_n)}{I(M(x_n))}\int_{M(x_n)}^1 \frac{1}{I(r_2)}\int_{r_2}^1 \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{I(r_m)} dr_m\, dr_{m-1} \ldots dr_2 \\ &= \int_{M(x_n)}^1 \frac{1}{I(r_2)}\int_{r_2}^1 \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{I(r_m)} dr_m\, dr_{m-1} \ldots dr_2 \quad \text{for a.e. } x \in \Omega_I,\end{aligned}$$

where the last equality holds by (4.11). Similarly, if $1 \le k \le m-1$,
(4.21)

$$|\nabla^k u(x)| = \frac{\partial^k u}{\partial x_n^k}(x) = \int_{M(x_n)}^1 \frac{1}{I(r_{k+1})}\int_{r_{k+1}}^1 \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{I(r_m)} dr_m\, dr_{m-1} \ldots dr_{k+1} \quad \text{for a.e. } x \in \Omega_I,$$

and

$$|\nabla^m u(x)| = \frac{\partial^m u}{\partial x_n^m}(x) = f(M(x_n)) \quad \text{for a.e. } x \in \Omega_I. \tag{4.22}$$

Thus, if $0 \le k \le m$, then

$$|\nabla^k u(x)| = \frac{\partial^k u}{\partial x_n^k}(x) = H_I^{m-k} f(M(x_n)) \quad \text{for a.e. } x \in \Omega_I, \tag{4.23}$$

where, as agreed, $H_I^0 f = f$. Analogously, if $0 \le k \le m$,

$$|\nabla^k v(x)| = \frac{\partial^k v}{\partial x_n^k}(x) = H_I^{m-k} g(M(x_n)) \quad \text{for a.e. } x \in \Omega_I. \tag{4.24}$$

By the Leibniz Rule,

$$\begin{aligned}|\nabla^m (uv)(x)| = \frac{\partial^m (uv)}{\partial x_n^m}(x) &= \sum_{k=0}^m \binom{m}{k} \frac{\partial^k u}{\partial x_n^k}(x) \frac{\partial^{m-k} v}{\partial x_n^{m-k}}(x) \\ &= \sum_{k=0}^m \binom{m}{k} H_I^{m-k} f(M(x_n)) H_I^k g(M(x_n)) \quad \text{for a.e. } x \in \Omega_I.\end{aligned}$$

Since $\Omega_I \in \mathcal{J}_I$, the space $\mathcal{V}^m X(\Omega_I)$ is a Banach algebra by our assumption. Therefore, in particular, there exists a positive constant C such that

$$\|\nabla^m (uv)\|_{X(\Omega_I)} \le C\|u\|_{\mathcal{V}^m X(\Omega_I)} \|v\|_{\mathcal{V}^m X(\Omega_I)},$$

namely,

$$\left\| \sum_{k=0}^m \binom{m}{k} H_I^{m-k} f(M(x_n)) H_I^k g(M(x_n)) \right\|_{X(\Omega_I)} \le C\|u\|_{\mathcal{V}^m X(\Omega_I)} \|v\|_{\mathcal{V}^m X(\Omega_I)}. \tag{4.25}$$

Now, if $0 \le k \le m$, then

$$H_I^{m-k} f(M(x_n)) H_I^k g(M(x_n)) \ge 0 \quad \text{for a.e. } x \in \Omega_I.$$

Therefore, we can disregard the terms with $k \ne 0$ in (4.25), and obtain

$$\|H_I^m f(M(x_n))\, g(M(x_n))\|_{X(\Omega_I)} \le C\|u\|_{\mathcal{V}^m X(\Omega_I)} \|v\|_{\mathcal{V}^m X(\Omega_I)}. \tag{4.26}$$

By Lemma 4.3, the function $F_{m,I} : \Omega_I \to [0,\infty)$, defined as

$$F_{m,I}(x) = H_I^m f(M(x_n)) g(M(x_n)) \qquad \text{for } x \in \Omega_I,$$

is such that

$$F_{m,I}^* = (H_I^m fg)^*\,.$$

Hence,

$$\|H_I^m f(M(x_n))g(M(x_n))\|_{X(\Omega_I)} = \|H_I^m fg\|_{X(0,1)}. \tag{4.27}$$

As for the terms on the right-hand side, note that, by (1.2), (4.22) and Lemma 4.3,

$$\|u\|_{\mathcal{V}^m X(\Omega)} = \|f\|_{X(0,1)} + \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(B)}, \tag{4.28}$$

where B is any ball in Ω_I. It is readily verified from (4.13) that there exists a constant $c > 0$ such that $M(x_n) \ge c$ for every $x \in B$. Thus,

$$\frac{1}{I(r)} \le \frac{1}{I(c)} \quad \text{if } x \in B \text{ and } r \in [M(x_n), 1].$$

Hence, by (4.19) and (4.21), if $0 \le k \le m-1$,

$$\begin{aligned} |\nabla^k u(x)| &\le \frac{1}{I(c)^{m-k}} \int_{M(x_n)}^1 \int_{r_{k+1}}^1 \cdots \int_{r_{m-1}}^1 f(r_m) dr_m \, dr_{m-1} \, \dots dr_{k+1} \le \frac{C_1}{I(c)^{m-k}} \int_{M(x_n)}^1 f(r_m) dr_m \\ &\le C_2 \int_0^1 f(r)\, dr \qquad \text{for a.e. } x \in B, \end{aligned}$$

for suitable positive constants C_1 and C_2. Consequently, by (2.5), if $0 \le k \le m-1$

$$\|\nabla^k u\|_{L^1(B)} \le C_2 |B| \|f\|_{L^1(0,1)} \le C_3 \|f\|_{X(0,1)}$$

for a suitable constant C_3. Hence, via (4.28),

$$\|u\|_{\mathcal{V}^m X(\Omega)} \le C \|f\|_{X(0,1)} \tag{4.29}$$

for some constant C. Analogously,

$$\|v\|_{\mathcal{V}^m X(\Omega)} \le C \|g\|_{X(0,1)}. \tag{4.30}$$

From (4.26), (4.27), (4.29) and (4.30) we infer that

$$\|g H_I^m f\|_{X(0,1)} \le C \|f\|_{X(0,1)} \|g\|_{X(0,1)} \tag{4.31}$$

for some positive constant C.
We now claim that

$$\|H_I^m f\|_{L^\infty(0,1)} \le \sup_{g \ge 0, \|g\|_{X(0,1)} \le 1} \|g H_I^m f\|_{X(0,1)}. \tag{4.32}$$

Indeed, if $\lambda < \|H_I^m f\|_{L^\infty(0,1)}$, then there exists a set $E \subset (0,1)$ of positive measure such that $H_I^m f \ge \lambda \chi_E$. Thus,

$$\sup_{g \ge 0, \|g\|_{X(0,1)} \le 1} \|g H_I^m f\|_{X(0,1)} \ge \lambda \sup_{g \ge 0, \|g\|_{X(0,1)} \le 1} \|g \chi_E\|_{X(0,1)} = \lambda\,,$$

whence (4.32) follows. Coupling inequality (4.32) with (4.31) tells us that

$$\|H_I^m f\|_{L^\infty(0,1)} \le C \|f\|_{X(0,1)}$$

for every $f \in X(0,1)$. The last inequality can be rewritten in the form

$$\int_0^1 \frac{f(r)}{I(r)} \left(\int_0^r \frac{dt}{I(t)} \right)^{m-1} dr \le C \|f\|_{X(0,1)},$$

and hence, by (2.1),

$$\left\| \frac{1}{I(t)} \left(\int_0^t \frac{ds}{I(s)} \right)^{m-1} \right\|_{X'(0,1)} \le C,$$

namely (1.9).

In particular, we have established (1.20), and hence also (1.17). Therefore, by Remark 1.8, the three spaces $V^m X(\Omega)$, $\mathcal{V}^m X(\Omega)$ and $W^m X(\Omega)$ coincide. Moreover, thanks to (1.9), we may apply [25, Corollary 5.5] and obtain thereby that $V^m X(\Omega) \to L^\infty(\Omega)$ for every $\Omega \in \mathcal{J}_I$. This establishes (4.18). Since $V^m X(\Omega) = \mathcal{V}^m X(\Omega)$, the proof is complete. □

The following theorem shows that the embedding into the space of essentially bounded functions is necessary for a Banach space to be a Banach algebra, in a quite general framework.

Theorem 4.5. *Assume that $Z(\Omega)$ is a Banach algebra such that $Z(\Omega) \subset \mathcal{M}(\Omega)$ and*

$$Z(\Omega) \to L^{1,\infty}(\Omega). \tag{4.33}$$

Then

$$Z(\Omega) \to L^\infty(\Omega). \tag{4.34}$$

Proof. Since $Z(\Omega)$ is a Banach algebra, there exists a constant $C \geq 1$ such that

$$\|uv\|_{Z(\Omega)} \leq C\|u\|_{Z(\Omega)}\|v\|_{Z(\Omega)} \tag{4.35}$$

for every $u, v \in Z(\Omega)$. Suppose, by contradiction, that (4.34) fails. Then there exists a function $w \in Z(\Omega)$ such that

$$\|w\|_{L^\infty(\Omega)} > 2C\|w\|_{Z(\Omega)},$$

where C is the constant from (4.35). In other words, the set

$$E = \{x \in \Omega;\ |w(x)| > 2C\|w\|_{Z(\Omega)}\}$$

has positive Lebesgue measure. Fix $j \in \mathbb{N}$. Applying (4.35) $(j-1)$-times, we obtain

$$\|w^j\|_{Z(\Omega)} \leq C^{j-1}\|w\|^j_{Z(\Omega)}.$$

Combining this inequality with (4.33) yields

$$\lambda|\{x \in \Omega : |w(x)|^j > \lambda\}| \leq C' C^{j-1}\|w\|^j_{Z(\Omega)}.$$

for some constant C', and for every $\lambda > 0$. In particular, the choice $\lambda = (2C\|w\|_{Z(\Omega)})^j$ yields

$$(2C\|w\|_{Z(\Omega)})^j|E| \leq C' C^{j-1}\|w\|^j_{Z(\Omega)},$$

namely

$$2^j|E| \leq CC^{-1}.$$

However, this is impossible, since $|E| > 0$ and j is arbitrary. □

Given a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$, with $\gamma_i \in \mathbb{N} \cup \{0\}$ for $i = 1, \dots, n$, set $|\gamma| = \gamma_1 + \cdots + \gamma_n$, and $D^\gamma u = \frac{\partial^{|\gamma|} u}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}$ for $u : \Omega \to \mathbb{R}$. Moreover, given two multi-indices γ and δ, we write $\gamma \leq \delta$ to denote that $\gamma_i \leq \delta_i$ for $i = 1, \dots, n$. Accordingly, by $\gamma < \delta$ we mean that $\gamma \leq \delta$ and $\gamma_i < \delta_i$ for at least one $i \in \{1, \dots, n\}$.

Proof of Theorem 1.9. Let $1 \leq k \leq m$. Fix any multi-index γ satisfying $|\gamma| \leq m - k$. Assumption (1.22) implies that

$$\int_0^1 \frac{ds}{I(s)} < \infty. \tag{4.36}$$

From (1.22) and (4.36), we obtain that there exists a positive constant C such that

$$\left(\int_0^t \frac{ds}{I(s)}\right)^{2m-|\gamma|} \leq \left(\int_0^1 \frac{ds}{I(s)}\right)^{m-k-|\gamma|} \left(\int_0^t \frac{ds}{I(s)}\right)^{m+k} \leq C\varphi_X(t). \tag{4.37}$$

Suppose that δ is any multi-index fulfilling $\delta \leq \gamma$. Since (1.22) holds, Lemma 3.2, implication (vii) $\Rightarrow$ (iii), combined with (4.1), tells us that

$$V^{m-|\delta|}X(\Omega) \to X^{\frac{2m-|\gamma|}{m-|\gamma|+|\delta|}}(\Omega) \quad \text{and} \quad V^{m-|\gamma|+|\delta|}(\Omega) \to X^{\frac{2m-|\gamma|}{m-|\delta|}}(\Omega). \tag{4.38}$$

On applying (3.6) with $p = \frac{2m-|\gamma|}{m-|\gamma|+|\delta|}$ and the two embeddings in (4.38), we deduce that

$$\begin{aligned} \|D^\delta u D^{\gamma-\delta} v\|_{X(\Omega)} &\leq \|D^\delta u\|_{X^{\frac{2m-|\gamma|}{m-|\gamma|+|\delta|}}(\Omega)} \|D^{\gamma-\delta} v\|_{X^{\frac{2m-|\gamma|}{m-|\delta|}}(\Omega)} \\ &\leq C\|D^\delta u\|_{V^{m-|\delta|}X(\Omega)} \|D^{\gamma-\delta} v\|_{V^{m-|\gamma|+|\delta|}X(\Omega)} \leq C'\|u\|_{V^m X(\Omega)}\|v\|_{V^m X(\Omega)}, \end{aligned} \tag{4.39}$$

for some constants C and C', and for every $u, v \in V^m X(\Omega)$. In particular, inequality (4.39) implies that $\sum_{\delta \leq \gamma} |D^\delta u D^{\gamma-\delta} v| \in L^1(\Omega)$. Hence, via [3, Ex. 3.17], we deduce that the function uv is $(m-k)$-times weakly differentiable and

$$D^\gamma(uv) = \sum_{\delta \leq \gamma} \frac{\gamma!}{\delta!(\gamma-\delta)!} D^\delta u D^{\gamma-\delta} v.$$

It follows from (4.39) that

$$\|D^\gamma(uv)\|_{X(\Omega)} \leq \sum_{\delta \leq \gamma} \frac{\gamma!}{\delta!(\gamma-\delta)!} \|D^\delta u D^{\gamma-\delta} v\|_{X(\Omega)} \leq C\|u\|_{V^m X(\Omega)}\|v\|_{V^m X(\Omega)},$$

for some constant C. Thus,

$$\|uv\|_{W^{m-k}X(\Omega)} = \sum_{|\gamma| \leq m-k} \|D^\gamma(uv)\|_{X(\Omega)} \leq C\|u\|_{V^m X(\Omega)}\|v\|_{V^m X(\Omega)} \tag{4.40}$$

for some constant C, and for every $u, v \in V^m X(\Omega)$. Since (4.36) is in force, $W^{m-k}X(\Omega) = \mathcal{V}^{m-k}X(\Omega)$ and $V^m X(\Omega) = \mathcal{V}^m X(\Omega)$, up to equivalent norms. As a consequence of (4.40), inequality (1.23) follows.

In order to prove the converse assertion, observe that, by Lemma 4.1, we can assume with no loss of generality that $I \in C^1(0,1)$ and $I^{n'}$ is convex. Let L, M, η and Ω_I be as in Lemma 4.2. Since, by (4.8), $\Omega_I \in \mathcal{J}_I$, condition (1.23) is fulfilled. Thus, there exists a positive constant C such that

$$\|\nabla^{m-k}(uv)\|_{X(\Omega_I)} \leq C\|u\|_{\mathcal{V}^m X(\Omega_I)}\|v\|_{\mathcal{V}^m X(\Omega_I)} \tag{4.41}$$

for every u, $v \in \mathcal{V}^m X(\Omega_I)$. Given $f, g \in \mathcal{M}_+(0,1)$, define $u, v : \Omega_I \to [0,\infty]$ as in (4.19) and (4.20), respectively. The functions u and v are m-times weakly differentiable in Ω_I. Furthermore, by (4.23) and (4.24),

$$\begin{aligned} |\nabla^{m-k}(uv)(x)| &= \frac{\partial^{m-k}(uv)}{\partial x_n^{m-k}}(x) = \sum_{\ell=0}^{m-k} \binom{m-k}{\ell} \frac{\partial^\ell u}{\partial x_n^\ell}(x) \frac{\partial^{m-k-\ell} v}{\partial x_n^{m-k-\ell}}(x) \\ &= \sum_{\ell=0}^{m-k} \binom{m-k}{\ell} H_I^{m-\ell} f(M(x_n)) H_I^{k+\ell} g(M(x_n)) \\ &\geq H_I^m f(M(x_n)) H_I^k g(M(x_n)) \quad \text{for a.e. } x \in \Omega_I. \end{aligned} \tag{4.42}$$

From (4.41), (4.42), (4.29) and (4.30) we infer that

$$\|H_I^m f\, H_I^k g\|_{X(0,1)} = \|H_I^m f(M(x_n)) H_I^k g(M(x_n))\|_{X(\Omega)} \leq C\|f\|_{X(0,1)}\|g\|_{X(0,1)}$$

for some constant C, and for every $f, g \in \mathcal{M}_+(0,1)$. Since $1 \leq k \leq m \leq m+k-1$, it follows from Lemma 3.2, implication (vi) $\Rightarrow$ (vii), that (1.22) holds. $\square$

Proof of Theorem 1.11. Let $\Omega \in \mathcal{J}_I$, $\|\cdot\|_{X(0,1)}$ and $1 \le k \le m$ be such that (1.22) is fulfilled. By Theorem 1.9, there exists a positive constant C such that

$$\sum_{l=0}^{m-k} \||\nabla^l u||\nabla^{m-k-l} v|\|_{X(\Omega)} \le C\|u\|_{\mathcal{V}^m X(\Omega)}\|v\|_{\mathcal{V}^m X(\Omega)} \tag{4.43}$$

for every $u, v \in \mathcal{V}^m X(\Omega)$.
In order to prove (1.25), it suffices to show that there exists $C > 0$ such that

$$\|D^\gamma u\, D^\delta v\|_{X(\Omega)} \le C\|u\|_{\mathcal{V}^m X(\Omega)}\|v\|_{\mathcal{V}^m X(\Omega)} \tag{4.44}$$

for every $u, v \in \mathcal{V}^m X(\Omega)$ and every multi-indices γ and δ satisfying $|\gamma| + |\delta| = m$, $|\gamma| \ge 1$ and $|\delta| \ge 1$. Fix such u, v, γ and δ. Without loss of generality we may assume that $|\gamma| \le |\delta|$. Let σ be an arbitrary multi-index such that $\sigma \le \delta$ and $|\sigma| = |\gamma|$. Assumption (1.24) ensures that condition (1.22) is fulfilled, with m and k replaced with $m - |\gamma|$ and $|\gamma|$. Thus, owing to (4.43), applied with u, v, m and k replaced by $D^\gamma u$, $D^{\delta_0} v$, $m - |\gamma|$ and $|\gamma|$, respectively, (and disregarding the terms with $\ell > 0$ on the left-hand side) we obtain that

$$\begin{aligned}\|D^\gamma u\, D^\delta v\|_{X(\Omega)} &\le \|D^\gamma u|\nabla^{m-2|\gamma|} D^\sigma v|\|_{X(\Omega)} \\ &\le C\|D^\gamma u\|_{\mathcal{V}^{m-|\gamma|} X(\Omega)}\|D^\sigma v\|_{\mathcal{V}^{m-|\gamma|} X(\Omega)} \\ &\le C\|u\|_{\mathcal{V}^m X(\Omega)}\|v\|_{\mathcal{V}^m X(\Omega)},\end{aligned}$$

whence (4.44) follows.
A proof of the necessity of condition (1.24), under (1.25) and (1.11) follows along the same lines as in the proof of (1.22) in Theorem 1.9, and will be omitted for brevity. □

We shall now prove a general sufficient condition for the space $W^m X(\Omega)$ to be a Banach algebra.

Proposition 4.6. *Let $m, n \in \mathbb{N}$, $n \ge 2$, and let I be a positive non-decreasing function in* $(0, 1)$*. Assume that $\Omega \in \mathcal{J}_I$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement invariant function norm. If* (1.9) *holds, then the Sobolev space $W^m X(\Omega)$ is a Banach algebra.*

Proof. It suffices to show that, for each pair of multi-indices γ and δ such that $|\gamma| \le m$ and $\delta \le \gamma$, there exists a positive constant C such that inequality

$$\|D^\delta u D^{\gamma-\delta} v\|_{X(\Omega)} \le C\|u\|_{W^m X(\Omega)}\|v\|_{W^m X(\Omega)}. \tag{4.45}$$

holds for all u, $v \in W^m X(\Omega)$. Indeed, such inequality implies, in particular, that $\sum_{\delta\le\gamma} |D^\delta u D^{\gamma-\delta} v| \in L^1(\Omega)$. Hence, once again, one can use [3, Ex. 3.17] to deduce that the function uv is m-times weakly differentiable in Ω, and that, for each γ with $1 \le |\gamma| \le m$,

$$D^\gamma(uv) = \sum_{\delta\le\gamma} \frac{\gamma!}{\delta!(\gamma-\delta)!} D^\delta u D^{\gamma-\delta} v. \tag{4.46}$$

In order to prove (4.45), let us begin by noting that, by (1.9) and (2.3),

$$\int_0^1 \frac{g(t)}{I(t)} \left(\int_0^t \frac{dr}{I(r)}\right)^{m-1} dt \le C\|g\|_{X(0,1)}, \tag{4.47}$$

for some constant C and every function $g \in \mathcal{M}_+(0,1)$. Since (4.47) can be rewritten in the form

$$\left\|\int_t^1 \frac{g(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)}\right)^{m-1} ds\right\|_{L^\infty(0,1)} \le C\|g\|_{X(0,1)},$$

it follows from [25, Theorem 5.1] that $V^m X(\Omega) \to L^\infty(\Omega)$. Thus, by (1.14), $W^m X(\Omega) \to L^\infty(\Omega)$ as well.
Assume now that γ is an arbitrary multi-index such that $0 \le |\gamma| \le m$. Then, for every $u, v \in W^m X(\Omega)$,

$$\|u D^\gamma v\|_{X(\Omega)} \le \|u\|_{L^\infty(\Omega)} \|D^\gamma v\|_{X(\Omega)} \le C\|u\|_{W^m X(\Omega)} \|v\|_{W^m X(\Omega)},$$

and, analogously,

$$\|(D^\gamma u)v\|_{X(\Omega)} \le C\|u\|_{W^m X(\Omega)} \|v\|_{W^m X(\Omega)}.$$

This establishes (4.45) whenever $|\gamma| \le m$ and either $\delta = 0$ or $\delta = \gamma$.

Assume now that $|\delta| \ge 1$ and $\delta < \gamma$. It follows from Proposition 3.3 that (1.9) implies (1.24), and hence also (1.22) for every $k \in \mathbb{N}$. Clearly,

$$\|D^\delta u\, D^{\gamma-\delta} v\|_{X(\Omega)} \le \|\, |\nabla^{|\delta|} u|\, |\nabla^{|\gamma-\delta|} v|\, \|_{X(\Omega)}. \tag{4.48}$$

On the other hand, we claim that

$$\|\, |\nabla^{|\delta|} u|\, |\nabla^{|\gamma-\delta|} v|\, \|_{X(\Omega)} \le C\|u\|_{\mathcal{V}^m X(\Omega)} \|v\|_{\mathcal{V}^m X(\Omega)} \tag{4.49}$$

for some positive C and for every $u, v \in \mathcal{V}^m X(\Omega)$. Indeed, if $|\gamma| = m$, then the claim is a straightforward consequence of Theorem 1.11. If $|\gamma| < m$, then the claim follows from inequality (4.43), applied with the choice $k = m - |\gamma|$. Combining inequalities (4.48), (4.49) and the first embedding in (1.14) completes the proof. □

Proof of Theorem 1.3. Assume that (1.9) is satisfied. Then, by Proposition 4.6, the space $W^m X(\Omega)$ is a Banach algebra for all $\Omega \in \mathcal{J}_\alpha$. Moreover, as we have already observed, condition (1.9) implies (1.17). Therefore, by Remark 1.8, the spaces $W^m X(\Omega)$ and $\mathcal{V}^m X(\Omega)$ coincide. Consequently, $\mathcal{V}^m X(\Omega)$ is a Banach algebra.
The second part of the theorem is a straightforward consequence of Proposition 4.4. □

Proof of Corollary 1.6. Assume that the embedding $\mathcal{V}^m X(\Omega) \to L^\infty(\Omega)$ holds for every $\Omega \in \mathcal{J}_I$. Let Ω_I be the domain defined by (4.7). Given $f \in \mathcal{M}_+(0,1)$, define u by (4.19). It follows from (4.23) with $k = 0$ that $u(x) = H_I^m f(M(x_n))$ for a.e. $x \in \Omega_I$ and M defined by (4.10). Therefore, $\|u\|_{L^\infty(\Omega)} = \|H_I^m f\|_{L^\infty(0,1)}$. Furthermore, by (4.29), we get $\|u\|_{\mathcal{V}^m X(\Omega)} \le C\|f\|_{X(0,1)}$ for some constant $C > 0$. Hence, our assumptions imply that there exists a positive constant C such that

$$\|H_I^m f\|_{L^\infty(0,1)} \le C\|f\|_{X(0,1)} \tag{4.50}$$

for every nonnegative function $f \in X(0,1)$. Inequality (4.50) implies (1.9), as was again observed in the course of proof of Proposition 4.4. By Theorem 1.3, this tells us that $\mathcal{V}^m X(\Omega)$ is a Banach algebra for every $\Omega \in \mathcal{J}_I$.
The converse implication follows at once from Proposition 4.4. □

Proof of Corollary 1.7. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then, by the assumption and (2.5), condition (1.9) is satisfied. Hence, owing to Theorem 1.3, the space $\mathcal{V}^m X(\Omega)$ is a Banach algebra. □

Proof of Theorem 1.1. Assume first that $\mathcal{V}^m X(\Omega)$ is a Banach algebra. Due to (1.6), condition (1.17) is satisfied. Hence, by Remark 1.8, the spaces $\mathcal{V}^m X(\Omega)$ and $W^m X(\Omega)$ coincide. In particular, $W^m X(\Omega)$ is a Banach algebra. Furthermore, by (2.5) and trivial inclusions, we clearly have

$$W^m X(\Omega) \to X(\Omega) \to L^1(\Omega) \to L^{1,\infty}(\Omega),$$

hence the assumption (4.33) of Theorem 4.5 is satisfied. Thus, as a special case of Theorem 4.5 we obtain that $W^m X(\Omega) \to L^\infty(\Omega)$. Therefore, $W^m X(\Omega) \to L^\infty(\Omega)$. On the other hand, by [25, Theorem 6.1], this embedding is equivalent to the inequality

$$\int_0^1 g(s) s^{\frac{m}{n}-1}\, ds \le C\|g\|_{X(0,1)}$$

for some constant C, and for every nonnegative function $g \in X(0,1)$. Hence, via the very definition of associate function norm, we obtain (1.3).

Conversely, assume that (1.3) holds. Since Ω is a John domain, inequality (1.8) is satisfied with $I(t) = t^{\frac{1}{n'}}$, $t \in (0,1)$. Hence, it follows from Theorem 1.3 that the space $\mathcal{V}^m X(\Omega)$ is a Banach algebra. □

Proof of Proposition 1.12. If $I(t) = t^\alpha$, with $\alpha \in [\frac{1}{n'}, \infty)$, then condition (1.11) is clearly satisfied. Hence, by Theorem 1.3, the space $\mathcal{V}^m L^{p,q;\beta}(\Omega)$ is a Banach algebra for every $\Omega \in \mathcal{J}_\alpha$ if and only if (1.9) holds. Owing to Remark 1.5, condition (1.9) entails that $\alpha < 1$. Thus,

$$\frac{1}{I(t)}\left(\int_0^t \frac{ds}{I(s)}\right)^{m-1} = (\tfrac{1}{1-\alpha})^{m-1} t^{m(1-\alpha)-1} \quad \text{for } t \in (0,1).$$

Therefore, it only remains to analyze under which conditions the power function $t^{m(1-\alpha)-1}$ belongs to $(L^{p,q;\beta})'(0,1)$. It is easily verified, via (2.9), that this is the case if and only if one of the conditions in (1.27) is satisfied. □

Proof of Proposition 1.13. By the same argument as in the proof of Proposition 1.12 we deduce that for $\mathcal{V}^m L^A(\Omega)$ to be a Banach algebra it is necessary that $\alpha < 1$. If $m \geq \frac{1}{1-\alpha}$, then (1.12) holds with $I(t) = t^\alpha$, $t \in (0,1)$, hence, by Corollary 1.7, $\mathcal{V}^m L^A(\Omega)$ is a Banach algebra whatever A is. If $m < \frac{1}{1-\alpha}$, then assumption (1.9) of Theorem 1.3 is equivalent to

$$\|r^{m(1-\alpha)-1}\|_{L^{\widetilde{A}}(0,1)} < \infty. \tag{4.51}$$

The latter condition is, in turn, equivalent to

$$\int_0 \widetilde{A}\big(r^{m(1-\alpha)-1}\big)\, dr < \infty\,, \tag{4.52}$$

namely to

$$\int^\infty \frac{\widetilde{A}(t)}{t^{1+\frac{1}{1-m(1-\alpha)}}}\, dt < \infty. \tag{4.53}$$

By [20, Lemma 2.3], equation (4.53) is equivalent to

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{(1-\alpha)m}{1-(1-\alpha)m}} dt < \infty.$$

□

Proof of Proposition 1.14. Condition (1.11) is satisfied if $I(t) = t^\alpha$, with $\alpha \in [\frac{1}{n'}, \infty)$. By Theorem 1.9, condition (1.23) for $X(\Omega) = L^{p,q;\beta}(\Omega)$ holds for every $\Omega \in \mathcal{J}_\alpha$ if and only if (1.22) is in force. In turn, inequality (1.22) entails (1.20), whence $\alpha < 1$. Now,

$$\varphi_{L^{p,q;\beta}}(t) \approx \begin{cases} t^{\frac{1}{p}}(\log\frac{2}{t})^\beta & \text{if } 1 < p < \infty,\ 1 \leq q \leq \infty,\ \beta \in \mathbb{R}; \\ t(\log\frac{2}{t})^\beta & \text{if } p = 1,\ q = 1,\ \beta \geq 0; \\ 1 & \text{if } p = \infty,\ q = \infty,\ \beta = 0; \\ (\log\frac{2}{t})^{\beta+\frac{1}{q}} & \text{if } p = \infty,\ 1 \leq q < \infty,\ \beta + \frac{1}{q} < 0, \end{cases} \tag{4.54}$$

for $t \in (0,1)$ (see e.g. [52, Lemma 9.4.1, page 318]). On the other hand,

$$\left(\int_0^t s^{-\alpha}\, ds\right)^{m+k} = (1-\alpha)^{-m-k} t^{(1-\alpha)(m+k)} \quad \text{for } t \in (0,1). \tag{4.55}$$

Thus, inequality (1.22) holds, with $X(0,1) = L^{p,q;\beta}(0,1)$, if and only if one of the conditions in (1.29) is satisfied. □

Proof of Proposition 1.15. As observed in the above proof, we may assume that $\alpha < 1$, and, by Theorem 1.9, reduce (1.23), with $I(t) = t^\alpha$, $t \in (0,1)$, and $X(\Omega) = L^A(\Omega)$, to the validity of (1.22). It is easily seen that

$$\varphi_{L^A}(t) = \frac{1}{A^{-1}(1/t)} \quad \text{for } t \in (0,1).$$

Thus, the conclusion follows via (4.55). □

References

[1] R.A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.
[2] A. Alvino, V. Ferone and G. Trombetti, *Moser-type inequalities in Lorentz spaces*, Potential Anal. **5** (1996), 273–299.
[3] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Clarendon Press, Oxford, 2000.
[4] T. Aubin, *Problèmes isopérimetriques et espaces de Sobolev*, J. Diff. Geom. **11** (1976), 573–598.
[5] F. Barthe, P. Cattiaux and C. Roberto, *Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry*, Rev. Mat. Iberoam. **22** (2006), 993–1067.
[6] T. Bartsch, T. Weth and M. Willem, *A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator*, Calc. Var. Partial Differential Equations **18** (2003), 253–268.
[7] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics Vol. 129, Academic Press, Boston 1988.
[8] C. Bennett and K. Rudnick, *On Lorentz–Zygmund spaces*, Dissertationes Math. **175** (1980), 1–67.
[9] S.G. Bobkov and C. Houdré, *Some connections between isoperimetric and Sobolev-type inequalities*, Mem. Am. Math. Soc. **129** (1997), viii+111.
[10] S.G. Bobkov and M. Ledoux, *From Brunn-Minkowski to sharp Sobolev inequalities*, Ann. Mat. Pura Appl. **187** (2008), 389–384.
[11] S. Buckley and P. Koskela, *Sobolev-Poincaré implies John*, Math. Res. Lett. **2** (1995), 577–593.
[12] S. Buckley and P. Koskela, *Criteria for embeddings of Sobolev-Poincaré type*, Int. Math. Res. Not. **18** (1996), 881–902.
[13] Yu.D. Burago and V.A. Zalgaller, *Geometric inequalities*, Springer, Berlin, 1988.
[14] L. Capogna, D. Danielli, S.D. Pauls and J.T. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Birkhauser, Basel, 2007.
[15] E.A. Carlen and C. Kerce, *On the cases of equality in Bobkov's inequality and Gaussian rearrangement*, Calc. Var. Partial Differential Equations **13** (2001), 1–18.
[16] I. Chavel, *Isoperimetric inequalities: differential geometric aspects and analytic perspectives*, Cambridge University Press, Cambridge, 2001.
[17] J. Cheeger, *A lower bound for the smallest eigevalue of the Laplacian, in Problems in analysis, 195–199*, Princeton Univ. Press, Princeton, 1970.
[18] A. Cianchi, *A sharp embedding theorem for Orlicz–Sobolev spaces*, Indiana Univ. Math. J. **45** (1996), 39–65.
[19] A. Cianchi, *Symmetrization and second-order Sobolev inequalities*, Ann. Mat. Pura Appl. **183** (2004), 45–77.
[20] A. Cianchi, *Optimal Orlicz-Sobolev embeddings*, Rev. Mat. Iberoamericana **20** (2004), 427–474.
[21] A. Cianchi, *Orlicz–Sobolev algebras*, Potential Anal. **28** (2008), 379–388.
[22] A. Cianchi, N. Fusco, F. Maggi and A. Pratelli, *The sharp Sobolev inequality in quantitative form*, J. Eur. Math. Soc. **11** (2009), 1105–1139.
[23] A. Cianchi and L. Pick, *Sobolev embeddings into BMO, VMO and L_∞*, Ark. Mat. **36** (1998), 317–340.
[24] A. Cianchi and L. Pick, *Optimal Gaussian Sobolev embeddings*, J. Funct. Anal. **256** (2009), 3588–3642.
[25] A. Cianchi, L. Pick and L. Slavíková, *Higher-order Sobolev embeddings and isoperimetric inequalities*, Adv. Math, to appear.
[26] E. De Giorgi, *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita*, Atti. Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. **5** (1958), 33–44.
[27] D. E. Edmunds, R. Kerman and L. Pick, *Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms*, J. Funct. Anal. **170** (2000), 307–355.
[28] L. Esposito, V. Ferone, B. Kawohl, C. Nitsch, and C. Trombetti, *The longest shortest fence and sharp Poincaré-Sobolev inequalities*, Arch. Ration. Mech. Anal. **206** (2012), 821–851.
[29] H. Federer and W. Fleming, *Normal and integral currents*, Annals of Math. **72** (1960), 458–520.
[30] F. Gazzola and H.-C. Grunau, *Critical dimensions and higher order Sobolev inequalities with remainder terms*, NoDEA Nonlinear Differential Equations Appl. **8** (2001), 35–44.

[31] F. Gazzola, H.-C. Grunau and G. Sweers, *Polyharmonic boundary value problems. Positivity preserving and non-linear higher order elliptic equations in bounded domains*, Lecture Notes in Mathematics, 1991. Springer-Verlag, Berlin, 2010.

[32] A. Grygor'yan, *Isoperimetric inequalities and capacities on Riemannian manifolds*, The Maz'ya Anniversary Collection, vol. 1, Operator Theory, Advances and Applications **110**, Birkhäuser, Basel, 1999, 139–153.

[33] P. Hajłasz and P. Koskela, *Isoperimetric inequalites and imbedding theorems in irregular domains*, J. London Math. Soc. **58** (1998), 425–450.

[34] E. Hebey, *Analysis on manifolds: Sobolev spaces and inequalities*, Courant Lecture Notes in Mathematics **5**, AMS, Providence, 1999.

[35] D. Hoffman and J. Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Comm. Pure Appl. Math. **27** (1974), 715–727.

[36] D. Hoffmann and J. Spruck, *A correction to: "Sobolev and isoperimetric inequalities for Riemannian submanifolds" (Comm. Pure Appl. Math. 27 (1974), 715–725)*, Comm. Pure Appl. Math. **28** (1975), 765–766.

[37] T. Kilpeläinen and J. Malý, *Sobolev inequalities on sets with irregular boundaries*, Z. Anal. Anwend. **19** (2000), 369–380.

[38] V.S. Klimov, *Imbedding theorems and geometric inequalities*, Izv. Akad. Nauk SSSR **40** (1976), 645–671 (Russian); English translation: Math. USSR Izv. **10** (1976), 615–638.

[39] R. Kerman and L. Pick, *Optimal Sobolev imbeddings*, Forum Math. **18, 4** (2006), 535–570.

[40] V.I. Kolyada, *Estimates on rearrangements and embedding theorems*, Mt. Sb. **136** (1988), 3–23 (Russian); English translation: Math. USSR Sb. **64** (1989), 1–21.

[41] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. II.*, Rev. Mat. Iberoamericana **1** (1985), 45–121.

[42] P.-L. Lions F. Pacella and M. Tricarico, *Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions*, Indiana Univ. Math. J. **37** (1988), 301–324.

[43] E. Lutwak D. Yang and G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Diff. Geom. **62** (2002), 17–38.

[44] L. Maligranda and L.E. Persson, *Generalized duality of some Banach function spaces*, Nederl. Akad. Wetensch. Indag. Math. **51** (1989), 323–338.

[45] V.G. Maz'ya, *Classes of regions and imbedding theorems for function spaces*, Dokl. Akad. Nauk. SSSR **133** (1960), 527–530 (Russian); English translation: Soviet Math. Dokl. **1** (1960), 882–885.

[46] V.G. Maz'ya, *On p-conductivity and theorems on embedding certain functional spaces into a C-space*, Dokl. Akad. Nauk. SSSR **140** (1961), 299–302 (Russian); English translation: Soviet Math. Dokl. **3** (1962).

[47] V.G. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations*, Springer, Berlin, 2011.

[48] V.G. Maz'ya and T.O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Pitman, Boston, 1985.

[49] E. Milman, *On the role of convexity in functional and isoperimetric inequalities*, Proc. London Math. Soc. **99** (2009), 32–66.

[50] J. Moser, *A sharp form of an inequality by Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[51] B. Opic and L. Pick, *On generalized Lorentz-Zygmund spaces*, Math. Inequal. Appl. **2** (1999), 391–467.

[52] L. Pick, A. Kufner, O. John and S. Fučík, *Function Spaces, Volume 1*, 2nd Revised and Extended Edition, De Gruyter Series in Nonlinear Analysis and Applications 14, Walter De Gruyter GmbH, Berlin/Boston 2013, ISBN 978-3-11-025041-1, xv+479 pp.

[53] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, Cambridge University Press, Cambridge, 2002.

[54] C.A. Swanson, *The best Sobolev constant*, Apl.Anal. **47** (1992), 227–239.

[55] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.

[56] G. Zhang, *The affine Sobolev inequality*, J. Diff. Geom. **53** (1999), 183–202.

[57] W.P. Ziemer, *Weakly differentiable functions*, Graduate texts in Math. **120**, Springer, Berlin, 1989.

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