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Integral conditions for Hardy-type operators involving suprema

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Abstract We characterize validity of the weighted inequality

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u(s)\int_s^\infty g(x)\,\mathrm{d}x\Big]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C\left(\int_0^\infty g^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

for all nonnegative functions g on $(0,\infty)$, with exponents in the range $1 \le p < \infty$ and $0 < q < \infty$.

Moreover, we give an integral characterization of the inequality

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u(s)f(s)\Big]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

being satisfied for all nonnegative nonincreasing functions f on $(0,\infty)$ in the case $0 < q < p < \infty$, for which an integral condition was previously unknown.

1 Introduction

In this paper we study the *supremal Hardy-type operators* R_u and S_u defined, for a nonnegative measurable function f on $(0,\infty)$, by

$$R_u f(t) := \sup_{s\in[t,\infty)} u(s)f(s), \qquad t > 0,$$

and

$$S_u f(t) := \sup_{s\in[t,\infty)} u(s)\int_s^\infty f(x)\,\mathrm{d}x, \qquad t > 0,$$

where u is a fixed continuous weight on $(0,\infty)$. The first goal is to characterize boundedness of the operator S_u between weighted Lebesgue spaces $L^p(v)$ and $L^q(w)$ (see Section 2 for the definitions). That is, to provide necessary and sufficient conditions for the inequality

$$\|S_u g\|_{L^q(w)} \le C\|g\|_{L^p(v)} \tag{1}$$

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to hold for all nonnegative measurable functions g on $(0,\infty)$. We do this for the range of exponents $p \in [1,\infty)$ and $q \in (0,\infty)$.

Our second goal is to determine when an analogous inequality holds for the operator R_u restricted to nonincreasing functions. Precisely, we characterize the validity of

$$\|R_u f\|_{L^q(w)} \le C\|f\|_{L^p(v)} \tag{2}$$

for all nonnegative and nonincreasing functions f on $(0,\infty)$, in the range $p,q \in (0,\infty)$.

The second question was studied in [6, Theorem 3.2], and a characterization was found. However, the authors succeeded to find a simple supremal/integral condition only for the case $0 < p \le q < \infty$. (This result is listed here as Theorem 8(i).)

In the case $1 \le p < \infty$, $0 < q < p$, [6] provides only a discrete condition involving a supremum of all "covering sequences" of points partitioning the half-axis $(0,\infty)$. Such condition is unfortunately only hardly verifiable and therefore of little practical use in further applications. In such situations there is always a strong interest in finding a simpler and more explicit condition. We solve this particular problem here in Theorem 8(ii) and provide a condition having a standard integral form.

There is actually more than one way how to solve this problem. In a recent and not yet published paper [5] the authors present a certain reduction method, applying which an integral condition for validity of (2) on nonnegative nonincreasing functions may be derived as well. The resulting characterization is, however, more complicated than the one we derive in here and, in a certain sense, it does not match the condition for $0 < p \le q < \infty$ proved in [6]. More details on this issue are mentioned in Section 4. Reduction methods for weighted inequalities were investigated in more papers, as e.g. [9–11].

Besides the treatment of R_u, the paper [6] offered a complete characterization of the $L^q(w)$–$L^p(v)$ boundedness of another supremal operator

$$T_u f(t) := \sup_{s\in[t,\infty)} u(s)\int_0^s f(x)\,\mathrm{d}x, \quad t>0,$$

where u is a fixed continuous weight and the operator T_u is defined for nonnegative functions f. The interest in studying of this operator stems, among other things, from its relation to the fractional maximal operator. For details, see [6] and the references given therein.

The operator S_u, which we are focusing on in this paper, appears often when *iterated Hardy-type inequalities* and *iterated Hardy-type operators* are studied. It is in fact itself an example of an iterated Hardy-type operator, as it is composed of the dual Hardy operator $H'f(t) := \int_t^\infty f$ and the supremal Hardy-type operator R_u. In a recent work [3], finding a characterization of the $L^q(w)$–$L^p(v)$ boundedness of S_u turns out to be necessary for proving certain embeddings between generalized Lorentz-type spaces with norms based on weighted integral means. This application is the main motivation of this paper.

Another one is, as mentioned before, the goal of finding the missing integral condition for the operator R_u acting on nonincreasing functions in the case $q < p$. It is reached easily once the results regarding S_u are established, since the inequality (2) can be reformulated as a particular case of the inequality (1). It may be worth noting that the process can be also reversed, allowing to characterize (1) for nonnegative functions when knowing the conditions for validity of (2) for nonincreasing functions. In this way, however, some additional assumptions on the weights might be required and they do not seem to be easily removable. Hence, treating S_u first is the preferred choice of action.

The proof technique used here is based on the dyadic discretization of weights, also called the *blocking technique*, which is a common tool for handling weighted inequalities. A comprehensive introduction into this technique is found for example in [13].

To fit the problems investigated in this article, the method needed to be modified and improved in a certain way. Roughly speaking, the key feature is the simultaneous control of both the weights w and u. It seems likely that the same method may be applied to obtain integral conditions in other problems where only discrete conditions or none at all have been known so far.

Let us also briefly describe the structure of the paper. In Section 2 below, we present the definitions and summarize auxiliary results. The main results together with their proofs are included in Section 3. Finally, in the last part, Section 4, we briefly compare the obtained conditions to the alternative characterizations which can be reached by the reduction methods of [5].

2 Definitions and preliminaries

The standard notation $A \lesssim B$ means that there exists a constant C “independent of relevant quantities in A and B” such that $A \le CB$. In this paper, the exact translation of this folklore phrase is that the constant C may depend only on exponents p and q. We write $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$.

The symbol $\mathscr{M}_+$ denotes the cone of all nonnegative Lebesgue-measurable functions on $(0,\infty)$. By $\mathscr{M}_+^\downarrow$ we denote the cone of all nonincreasing functions from $\mathscr{M}_+$.

A *weight* is a function $w \in \mathscr{M}_+$ such that for all $t \in (0,\infty)$ it holds $0 < W(t) < \infty$, where

$$W(t) := \int_0^t w(s)\,\mathrm{d}s.$$

The symbol V has an analogous relation to the weight v.

Let v be a weight and $p \in (0,\infty)$. The *weighted Lebesgue space* $L^p(v) = L^p(v)(0,\infty)$ consists of all real-valued Lebesgue-measurable functions f on $(0,\infty)$ such that

$$\|f\|_{L^p(v)} := \left(\int_0^\infty |f(t)|^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} < \infty.$$

We say that $\mathbb{I} \subseteq \mathbb{Z} \cup \{\pm\infty\}$ is an *index set* if there exist $k_{\min}, k_{\max} \in \mathbb{Z} \cup \{\pm\infty\}$ such that $k_{\min} < k_{\max}$ and

$$\mathbb{I} = \{k \in \mathbb{Z},\ k_{\min} \le k \le k_{\max}\},$$

where the respective inequality is replaced by a sharp one if $k_{\min} = \infty$ or $k_{\max} = \infty$.

Let $\mathbb{I}$ be an index set. A positive sequence $\{b_k\}_{k\in\mathbb{I}}$ is called *strongly increasing*, denoted $b_k \upuparrows$, if

$$\sigma := \inf\left\{\frac{b_{(k+1)}}{b_k},\ k \in \mathbb{I} \setminus \{k_{\max}\}\right\} > 1. \tag{3}$$

Finally, let $n, k \in \mathbb{N}$, $z \in \mathbb{N} \cup \{0\}$, $0 \le k < n$. We write $z \bmod n = k$ if there exists $j \in \mathbb{N} \cup \{0\}$ such that $z = jn + k$. In other words, k is the remainder after division of the number z by the number n.

The proposition below was proved in [12, Proposition 2.1] (although there is a minor error in the estimate of the constant in the original article). It is in fact a key element in the discretization method.

Proposition 1 *Let $\mathbb{I}$ be an index set and let $0 < \alpha < \infty$. Let $\{a_k\}_{k\in\mathbb{I}}$ and $\{b_k\}_{k\in\mathbb{I}}$ be two nonnegative sequences such that $b_k \upuparrows$. Then there exists $C \in (1,\infty)$ such that*

$$\left(\sum_{k=k_{\min}}^{k_{\max}} \left(\sum_{m=k}^{k_{\max}} a_m\right)^{\alpha} b_k^\alpha\right)^{\frac{1}{\alpha}} \le C \left(\sum_{k=k_{\min}}^{k_{\max}} a_k^\alpha b_k^\alpha\right)^{\frac{1}{\alpha}}.$$

The constant C satisfies

$$C \le \begin{cases} 1 + \dfrac{1}{\sigma^\alpha - 1} & \text{if } \alpha \le 1,\\ \left(1 + \dfrac{1}{\sigma^{\frac{1}{\alpha-1}}}\right)^{\alpha-1} \left(1 + \dfrac{1}{\sigma^{\alpha-1} - 1}\right) & \text{if } \alpha > 1, \end{cases} \tag{4}$$

where σ is defined by (3).

Observe that the value of the estimates in (4) decreases with increasing σ. Hence, it suffices to know a lower bound for σ to get a usable constant C. This leads to the following corollary.

Corollary 2 *Let $0 < \alpha < \infty$ and $1 < D < \infty$. Then there exists a constant $C_{\alpha,D} \in (0,\infty)$ such that for any index set $\mathbb{I}$ and any two nonnegative sequences $\{a_k\}_{k\in\mathbb{I}}$ and $\{b_k\}_{k\in\mathbb{I}}$, satisfying $b_{(k+1)} \geq Db_k$ for all $k \in \mathbb{I} \setminus \{k_{\max}\}$, it holds*

$$\left(\sum_{k=k_{\min}}^{k_{\max}} \left(\sum_{m=k}^{k_{\max}} a_m\right)^{\alpha} b_k^{\alpha}\right)^{\frac{1}{\alpha}} \leq C_{\alpha,D} \left(\sum_{k=k_{\min}}^{k_{\max}} a_k^{\alpha} b_k^{\alpha}\right)^{\frac{1}{\alpha}}.$$

Moreover, since $\sup_{k\leq m\leq k_{\max}} a_m \leq \sum_{m=k}^{k_{\max}} a_m$, we obtain another corollary.

Corollary 3 *Let $0 < \alpha < \infty$ and $1 < D < \infty$. Then there exists a constant $C_{\alpha,D} \in (0,\infty)$ such that for any index set $\mathbb{I}$ and any two nonnegative sequences $\{a_k\}_{k\in\mathbb{I}}$ and $\{b_k\}_{k\in\mathbb{I}}$, satisfying $b_{(k+1)} \geq Db_k$ for all $k \in \mathbb{I} \setminus \{k_{\max}\}$, it holds*

$$\left(\sum_{k=k_{\min}}^{k_{\max}} \left(\sup_{k\leq m\leq k_{\max}} a_m\right)^{\alpha} b_k^{\alpha}\right)^{\frac{1}{\alpha}} \leq C_{\alpha,D} \left(\sum_{k=k_{\min}}^{k_{\max}} a_k^{\alpha} b_k^{\alpha}\right)^{\frac{1}{\alpha}}.$$

Now we recall a useful property of $L^p(v)$-spaces. If v is a weight, $p \in (1,\infty)$ and $0 \leq x < y \leq \infty$, Hölder inequality yields

$$\int_x^y h(s)\,\mathrm{d}s \leq \left(\int_x^y h^p(s)v(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \left(\int_x^y v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}}$$

for any nonnegative measurable function h on (x,y). Moreover, the well-known description of the dual space to an L^p-space gives the following saturation property

$$\left(\int_x^y v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} = \sup_{\substack{h\in L^p(v)\\ \|h\|_{L^p(v)}\neq 0}} \frac{\int_x^y |h(s)|\,\mathrm{d}s}{\left(\int_x^y |h(s)|^p v(s)\,\mathrm{d}s\right)^{\frac{1}{p}}}.$$

In particular, if $\int_x^y v^{1-p'}(s)\,\mathrm{d}s < \infty$, there exists a nonnegative function $g \in L^p(v) \cap L^1$ such that

$$2\frac{\int_x^y g(s)\,\mathrm{d}s}{\left(\int_x^y g^p(s)v(s)\,\mathrm{d}s\right)^{\frac{1}{p}}} \geq \sup_{\substack{h\in L^p(v)\\ \|h\|_{L^p(v)}\neq 0}} \frac{\int_x^y |h(s)|\,\mathrm{d}s}{\left(\int_x^y |h(s)|^p v(s)\,\mathrm{d}s\right)^{\frac{1}{p}}} = \left(\int_x^y v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}}.$$

Moreover, the function g may be taken such that $\|g\|_{L^p(v)} = 1$, in which case we get

$$\left(\int_x^y v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} \leq 2\int_x^y g(s)\,\mathrm{d}s < \infty.$$

This property is used throughout the text and referred to as the *duality of L^p-spaces.* Similar results, of course, exist for l^p-spaces consisting of sequences. We summarize them in the next two propositions.

Proposition 4 *Let $\mathbb{I}$ be an index set and let $\{a_k\}_{k\in\mathbb{I}}$ and $\{b_k\}_{k\in\mathbb{I}}$ be two nonnegative sequences.*

(i) *Let $0 < p \leq q < \infty$. Then*

$$\left(\sum_{k\in\mathbb{I}} a_k^q b_k\right)^{\frac{1}{q}} \leq \left(\sum_{k\in\mathbb{I}} a_k^p\right)^{\frac{1}{p}} \sup_{j\in\mathbb{I}} b_j^{\frac{1}{q}}.$$

(ii) *Let $0 < q < p < \infty$. Then*

$$\left(\sum_{k\in\mathbb{I}} a_k^q b_k\right)^{\frac{1}{q}} \leq \left(\sum_{k\in\mathbb{I}} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k\in\mathbb{I}} b_k^{\frac{p}{p-q}}\right)^{\frac{p-q}{pq}}.$$

Proof Case (i) is proved using convexity of the $\frac{q}{p}$-th power (with $p \leq q$) and the Jensen inequality. Case (ii) follows from the Hölder inequality with the pair of exponents $\frac{p}{q}$ and $\frac{p}{p-q}$.

Proposition 5 *Let $\mathbb{I}$ be an index set and $\{b_k\}_{k\in\mathbb{I}}$ be a nonnegative sequence. Let $0<q<p<\infty$. Then*

$$\left(\sum_{k\in\mathbb{I}} b_k^{\frac{p}{p-q}}\right)^{\frac{p-q}{pq}} = \sup_{\{a_k\}_{k\in\mathbb{I}}} \frac{\left(\sum_{k\in\mathbb{I}} a_k^q b_k\right)^{\frac{1}{q}}}{\left(\sum_{k\in\mathbb{I}} a_k^p\right)^{\frac{1}{p}}},$$

where the supremum is taken over all positive sequences $\{a_k\}_{k\in\mathbb{I}}$. In particular, if $\sum_{k\in\mathbb{I}} b_k^{\frac{p}{p-q}} < \infty$, then there exists a nonnegative sequence $\{a_k\}_{k\in\mathbb{I}}$ such that $\sum_{k\in\mathbb{I}} a_k^p = 1$ and

$$\left(\sum_{k\in\mathbb{I}} b_k^{\frac{p}{p-q}}\right)^{\frac{p-q}{pq}} \le 2\left(\sum_{k\in\mathbb{I}} a_k^q b_k\right)^{\frac{1}{q}} < \infty.$$

Obviously, in accordance with the other terminology of ours, Proposition 5 could be called "duality of l^p-spaces".

3 Main results

Theorem 6 *Let v, w be weights and let u be a continuous weight. Consider the inequality*

$$\left(\int_0^\infty \Big[\sup_{x\in[t,\infty)} u(x)\int_x^\infty g(s)\,\mathrm{d}s\Big]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_{(5)}\left(\int_0^\infty g^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}. \tag{5}$$

(i) *Let $1<p\le q<\infty$. Then the inequality (5) holds for all $g\in\mathscr{M}_+$ if and only if*

$$A_{(6)} := \sup_{t\in(0,\infty)} \left(\int_0^t w(x)\sup_{z\in[x,t]} u^q(z)\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_t^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} < \infty. \tag{6}$$

Moreover, the least constant $C_{(5)}$ such that (5) holds for all $g\in\mathscr{M}_+$ satisfies $C_{(5)}\approx A_{(6)}$.

(ii) *Let $1<p<\infty$ and $0<q<p<\infty$. Set $r:=\frac{pq}{p-q}$. Then the inequality (5) holds for all $g\in\mathscr{M}_+$ if and only if*

$$A_{(7)} = \left(\int_0^\infty W^{\frac{r}{p}}(t)w(t)\sup_{z\in[t,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\mathrm{d}t\right)^{\frac{1}{r}} < \infty \tag{7}$$

and

$$A_{(8)} = \left(\int_0^\infty \left(\int_0^t w(x)\sup_{y\in[x,t]} u^q(y)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\mathrm{d}t\right)^{\frac{1}{r}} < \infty. \tag{8}$$

Moreover, the least constant $C_{(5)}$ such that (5) holds for all $g\in\mathscr{M}_+$ satisfies $C_{(5)}\approx A_{(7)}+A_{(8)}$.

Proof For the start, let us assume that there exists a finite $K\in\mathbb{Z}$ such that $\int_0^\infty w = 2^K$. It is possible to find a sequence of points $\{t_k\}_{k=-\infty}^K$ such that for every $k\in\mathbb{Z}$, $k<K$ it holds $t_k\in(0,\infty)$, $t_k>t_{k-1}$ and $\int_0^{t_k} w = 2^k$. We also define $t_K:=\infty$. For every $k\in\mathbb{Z}$ such that $k\le K-1$ define the *k-th segment*

$$\Delta_k := [t_k, t_{k+1}).$$

Then it holds

$$2^k = \int_0^{t_k} w(s)\,\mathrm{d}s = \int_{\Delta_k} w(s)\,\mathrm{d}s = 2\int_{\Delta_{(k-1)}} w(s)\,\mathrm{d}s. \tag{9}$$

Throughout the proof, we use the notation

$$U(x,y) := \sup_{z\in[x,y)} u(z)$$

for any $0 \le x < y \le \infty$. If the interval $[x,y)$ is the k-th segment, we write shortly

$$U(\Delta_k) := U(t_k, t_{(k+1)}).$$

Observe that it holds

$$U(x,z) \le U(x,y) + U(y,z) \quad \text{whenever} \quad 0 \le x \le y \le z \le \infty. \tag{10}$$

Choose a fixed $\mu \in \mathbb{Z}$ such that $\mu \le K-2$. Define the finite set $\mathbb{Z}_\mu := \{k \in \mathbb{Z},\ \mu \le k \le K-1\}$. Now we construct a subset of indices in the following way: At first, set $k_0 := \mu$ and $k_1 := \mu + 1$. We continue inductively.

(S) Let $k_0, \dots, k_n$ be already defined. Then:

(a) If $k_n = K$, define $N := n-1$ and stop the procedure.

(b) If $k_n < K$, proceed as follows. If there exists any index $j \in \mathbb{Z}$ such that $k_n < j \le K$ and

$$\sum_{k=k_n}^{j-1} 2^k U^q(\Delta_k) \ge 2 \sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k),$$

then define $k_{(n+1)}$ as the smallest such index j and proceed again with step (S). If no such j exists, set $N := n$, define $k_{(N+1)} := K$ and so finish the construction.

In this way, we obtain a set of indices $\{k_0, \dots, k_N\} \subseteq \mathbb{Z}_\mu$ and $k_{(N+1)} = K$.

To continue, we may call the interval $[t_{k_n}, t_{k_n+1})$ the *n-th block*. For every $n \in \mathbb{N}$ such that $n \le N$, it holds either

$$k_{(n+1)} = k_n + 1,$$

which means that the n-th block consists only of one segment (the k_n-th one), or

$$k_{(n+1)} > k_n + 1,$$

which means that the n-th block consists of more than one segment. If the latter is the case, we will say that $n \in \mathbb{A}$. Precisely, we put

$$\mathbb{A} := \{n \in \mathbb{N},\ n \le N,\ k_{(n+1)} > k_n + 1\}.$$

Notice that this set may be empty but it is always satisfied

$$\mathbb{Z}_\mu = \{k_{(n+1)} - 1\}_{n=0}^N \cup \left\{k \in \mathbb{Z},\ k_n \le k \le k_{(n+1)} - 2\right\}_{n \in \mathbb{A}}. \tag{11}$$

In plain words, each segment is either the last segment (i.e. the one with the highest index k) in a block, or it lies in a block which contains multiple segments but this particular segment is not the last one of them.

From the way it was constructed it follows that the system has the following properties. At first, for every $n \in \mathbb{N}$ such that $n < N$ it holds

$$\sum_{k=k_n}^{k_{(n+1)}-1} 2^k U^q(\Delta_k) \ge 2 \sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k). \tag{12}$$

This is not necessarily true for the last, N-th bloc, but it will not be an issue. Next, for all $n \in \mathbb{A}$ we have

$$\sum_{k=k_n}^{k_{(n+1)}-2} 2^k U^q(\Delta_k) < 2 \sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k). \tag{13}$$

Furthermore, by iterating (12) it is shown that, for every $n \in \mathbb{N}$, $n \le N$,

$$\sum_{k=\mu}^{k_n-1} 2^k U^q(\Delta_k) = \sum_{i=0}^{n-1} \sum_{k=k_i}^{k_{(i+1)}-1} 2^k U^q(\Delta_k) \le \sum_{i=0}^{n-1} 2^{i-n+1} \sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k) \le 2 \sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k),$$

hence

$$\sum_{k=\mu}^{k_n-1} 2^k U^q(\Delta_k) \le 2 \sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k). \tag{14}$$

Now suppose that $n \in \mathbb{N}$, $n \le N$, $k \in \mathbb{Z}$ is such that $k < k_{(n+1)}$ and $t \in \Delta_k$. Then we have

$$\begin{aligned}
\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x &= \int_{t_\mu}^{t_k} w(x)U^q(x,t)\,\mathrm{d}x + \int_{t_k}^{t} w(x)U^q(x,t)\,\mathrm{d}x \\
&\lesssim \int_{t_\mu}^{t_k} w(x)U^q(x,t_k)\,\mathrm{d}x + \int_{t_\mu}^{t_k} w(x)\,\mathrm{d}x\; U^q(t_k,t) + \int_{t_k}^{t} w(x)U^q(x,t)\,\mathrm{d}x \qquad (15)\\
&\le \sum_{j=\mu}^{k-1} \int_{\Delta_j} w(x)\,\mathrm{d}x\; U^q(t_j,t_k) + \int_{t_\mu}^{t_{(k+1)}} w(x)\,\mathrm{d}x\; U^q(t_k,t) \\
&\lesssim \sum_{j=\mu}^{k-1} 2^j U^q(t_j,t_k) + 2^k U^q(t_k,t) \qquad (16)\\
&= \sum_{j=\mu}^{k-1} 2^j \left(\sum_{i=j}^{k-1} U(\Delta_i) \right)^q + 2^k U^q(t_k,t) \\
&\lesssim \sum_{j=\mu}^{k-1} 2^j U^q(\Delta_j) + 2^k U^q(t_k,t). \qquad (17)
\end{aligned}$$

Step (15) follows by (10), step (16) is due to (9) and (17) holds by Corollary 2. Next, if $k \le k_n$, then

$$\sum_{j=\mu}^{k-1} 2^j U^q(\Delta_j) \le \sum_{j=\mu}^{k_n-1} 2^j U^q(\Delta_j) \lesssim \sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j),$$

where the second inequality follows by (14). If $k > k_n$, then $n \in \mathbb{A}$, $k_n + 1 \le k \le k_{(n+1)} - 1$ and we get

$$\sum_{j=\mu}^{k-1} 2^j U^q(\Delta_j) \le \sum_{j=\mu}^{k_{(n+1)}-2} 2^j U^q(\Delta_j) = \sum_{j=\mu}^{k_n-1} 2^j U^q(\Delta_j) + \sum_{j=k_n}^{k_{(n+1)}-2} 2^j U^q(\Delta_j) \lesssim \sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j).$$

The last inequality is granted by (13) and (14). We have proved that

$$\sum_{j=\mu}^{k-1} 2^j U^q(\Delta_j) \lesssim \sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j).$$

Inserting this into the inequality obtained at (17), we finally receive

$$\int_{t_\mu}^{t} w(x)U^q(x,t)\,\mathrm{d}x \lesssim \sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j) + 2^k U^q(t_k,t) \tag{18}$$

for any $n \in \mathbb{N}$, $n \le N$, $k \in \mathbb{Z}$, $k < k_{(n+1)}$ and $t \in \Delta_k$.

Yet another useful inequality reads

$$\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j) \lesssim \int_{t_{k_{(n-1)}-1}}^{t_{k_n}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t \tag{19}$$

for any $n \in \mathbb{N}$ such that $n \le N$. Indeed, this follows from the following observation:

$$\begin{aligned}
\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j) &\lesssim \sum_{j=k_{(n-1)}}^{k_n-1} \int_{\Delta_{j-1}} w(t)\,\mathrm{d}t\; U^q(\Delta_j) \le \sum_{j=k_{(n-1)}}^{k_n-1} \int_{\Delta_{j-1}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t \\
&= \int_{t_{k_{(n-1)}-1}}^{t_{(k_n-1)}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t \le \int_{t_{k_{(n-1)}-1}}^{t_{k_n}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t,
\end{aligned}$$

in which we also used (9) to establish the first inequality.

We have prepared the core of the proof method now and may begin with the main part, which is split into proving sufficiency of the respective A-conditions for validity of (5), and their necessity.

Sufficiency. Choose a function $g \in L^p(v)$. We start by estimating

$$\begin{aligned}
&\left[\int_{t_\mu}^{\infty}\left(\sup_{x\in[t,\infty)} u(x)\int_x^\infty g\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}}\\
&=\left[\sum_{k\in\mathbb{Z}_\mu}\int_{\Delta_k} w(t)\left(\sup_{x\in[t,\infty)} u(x)\int_x^\infty g\right)^q \mathrm{d}t\right]^{\frac{1}{q}}\\
&\le\left[\sum_{k\in\mathbb{Z}_\mu}\int_{\Delta_k} w(t)\,\mathrm{d}t\left(\sup_{x\in[t_k,\infty)} u(x)\int_x^\infty g\right)^q\right]^{\frac{1}{q}}\\
&=\left[\sum_{k\in\mathbb{Z}_\mu}2^k\left(\sup_{x\in[t_k,\infty)} u(x)\int_x^\infty g\right)^q\right]^{\frac{1}{q}} && (20)\\
&\approx\left[\sum_{k\in\mathbb{Z}_\mu}2^k\left(\sup_{x\in\Delta_k} u(x)\int_x^\infty g\right)^q\right]^{\frac{1}{q}} && (21)\\
&\approx\left[\sum_{k\in\mathbb{Z}_\mu}2^k\left(\sup_{x\in\Delta_k} u(x)\int_x^{t_{(k+1)}} g\right)^q\right]^{\frac{1}{q}}+\left[\sum_{k\in\mathbb{Z}_\mu}2^k U^q(\Delta_k)\left(\int_{t_{(k+1)}}^\infty g\right)^q\right]^{\frac{1}{q}}\\
&=: B_1+B_2.
\end{aligned}$$

Step (20) follows from (9), and step (21) from Corollary 3. Moreover, B_2 can be further estimated as follows.

$$\begin{aligned}
B_2 &\approx\left[\sum_{n=1}^{N}2^{k_n-1}U^q(\Delta_{k_n-1})\left(\int_{t_{k_n}}^\infty g\right)^q\right]^{\frac{1}{q}}+\left[\sum_{n\in\mathbb{A}}\sum_{k=k_n}^{k_{(n+1)}-2}2^k U^q(\Delta_k)\left(\int_{t_{(k+1)}}^\infty g\right)^q\right]^{\frac{1}{q}} && (22)\\
&\le\left[\sum_{n=1}^{N}2^{k_n-1}U^q(\Delta_{k_n-1})\left(\int_{t_{k_n}}^\infty g\right)^q\right]^{\frac{1}{q}}+\left[\sum_{n\in\mathbb{A}}\sum_{k=k_n}^{k_{(n+1)}-2}2^k U^q(\Delta_k)\left(\int_{t_{(k_n+1)}}^\infty g\right)^q\right]^{\frac{1}{q}}\\
&\lesssim\left[\sum_{n=1}^{N}2^{k_n-1}U^q(\Delta_{k_n-1})\left(\int_{t_{k_n}}^\infty g\right)^q\right]^{\frac{1}{q}}+\left[\sum_{n\in\mathbb{A}}\sum_{k=k_{(n-1)}}^{k_n-1}2^k U^q(\Delta_k)\left(\int_{t_{(k_n+1)}}^\infty g\right)^q\right]^{\frac{1}{q}} && (23)\\
&\le\left[\sum_{n=1}^{N}2^{k_n-1}U^q(\Delta_{k_n-1})\left(\int_{t_{k_n}}^\infty g\right)^q\right]^{\frac{1}{q}}+\left[\sum_{n\in\mathbb{A}}\sum_{k=k_{(n-1)}}^{k_n-1}2^k U^q(\Delta_k)\left(\int_{t_{k_n}}^\infty g\right)^q\right]^{\frac{1}{q}}\\
&\lesssim\left[\sum_{n=1}^{N}\sum_{k=k_{(n-1)}}^{k_n-1}2^k U^q(\Delta_k)\left(\int_{t_{k_n}}^\infty g\right)^q\right]^{\frac{1}{q}}\\
&\lesssim\left[\sum_{n=1}^{N}\sum_{k=k_{(n-1)}}^{k_n-1}2^k U^q(\Delta_k)\left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} g\right)^q\right]^{\frac{1}{q}} && (24)\\
&\le\left[\sum_{n=1}^{N}\sum_{k=k_{(n-1)}}^{k_n-1}2^k U^q(\Delta_k)\left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}\right)^{\frac{q}{p'}}\left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} g^p v\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} && (25)\\
&=: B_3.
\end{aligned}$$

In here, step (22) follows from (11), and step (23) from (13). In (24) we used Corollary 2, considering also (12). Step (25) follows by Hölder inequality.

The above estimates resulting in B_1 and B_3 are valid independently of the relation between p and q. The rest will be split into the cases (i) and (ii).

(i) Let $1 < p \le q < \infty$. Suppose that $A_{(6)} < \infty$. The goal is to show that $C_{(5)} \lesssim A_{(6)}$. First, we get

$$
\begin{aligned}
B_1 &\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^k \sup_{x\in\Delta_k} u^q(x) \left(\int_x^{t_{(k+1)}} v^{1-p'} \right)^{\frac{q}{p'}} \left(\int_x^{t_{(k+1)}} g^p v \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} && (26)\\
&\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^k \sup_{x\in\Delta_k} u^q(x) \left(\int_x^{t_{(k+1)}} v^{1-p'} \right)^{\frac{q}{p'}} \left(\int_{\Delta_k} g^p v \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&\le \sup_{k\in\mathbb{Z}_\mu} 2^{\frac{k}{q}} \sup_{x\in\Delta_k} u(x) \left(\int_x^{t_{(k+1)}} v^{1-p'} \right)^{\frac{1}{p'}} \left(\sum_{k\in\mathbb{Z}_\mu} \int_{\Delta_k} g^p v \right)^{\frac{1}{p}} && (27)\\
&\le \sup_{k\in\mathbb{Z}_\mu} 2^{\frac{k}{q}} \sup_{x\in\Delta_k} u(x) \left(\int_x^{t_{(k+1)}} v^{1-p'} \right)^{\frac{1}{p'}} \|g\|_{L^p(v)} \\
&= \sup_{k\in\mathbb{Z}_\mu} \left(\int_0^{t_k} w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \sup_{x\in\Delta_k} u(x) \left(\int_x^{t_{(k+1)}} v^{1-p'} \right)^{\frac{1}{p'}} \|g\|_{L^p(v)} && (28)\\
&\le \sup_{k\in\mathbb{Z}_\mu} \sup_{x\in\Delta_k} \left(\int_0^{x} w(t) U^q(t,x)\,\mathrm{d}t \right)^{\frac{1}{q}} \left(\int_x^{\infty} v^{1-p'} \right)^{\frac{1}{p'}} \|g\|_{L^p(v)} \\
&= A_{(6)} \|g\|_{L^p(v)}.
\end{aligned}
$$

Step (26) follows from Hölder inequality, step (27) from Proposition 4(i). In (28) we used (9).

We proceed with B_3.

$$
\begin{aligned}
B_3 &\le \sup_{\substack{n\in\mathbb{N}\\ n\le N}} \left(\sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k) \right)^{\frac{1}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'} \right)^{\frac{1}{p'}} \left(\sum_{n=1}^{N} \int_{t_{k_n}}^{t_{k_{(n+1)}}} g^p v \right)^{\frac{1}{p}} && (29)\\
&\le \sup_{\substack{n\in\mathbb{N}\\ n\le N}} \left(\sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k) \right)^{\frac{1}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'} \right)^{\frac{1}{p'}} \|g\|_{L^p(v)} \\
&\lesssim \sup_{\substack{n\in\mathbb{N}\\ n\le N}} \left(\int_0^{t_{k_n}} w(t) U^q(t,t_{k_n})\,\mathrm{d}t \right)^{\frac{1}{q}} \left(\int_{t_{k_n}}^{\infty} v^{1-p'} \right)^{\frac{1}{p'}} \|g\|_{L^p(v)} && (30)\\
&\le A_{(6)} \|g\|_{L^p(v)}.
\end{aligned}
$$

Step (29) follows by Proposition 4(i), and (30) is due to (19).

At this point we have proved that for an arbitrary $\mu \in \mathbb{Z}$ such that $\mu \le K-2$ and an arbitrarily chosen $g \in L^p(v)$ it holds

$$
\left[\int_{t_\mu}^\infty \left(\sup_{x\in[t,\infty)} u(x) \int_x^\infty g \right)^q w(t)\,\mathrm{d}t \right]^{\frac{1}{q}} \lesssim A_{(6)} \|g\|_{L^p(v)},
$$

where the constant contained the symbol "$\lesssim$" is independent of g, u, v, w and μ. If needed, the reader may verify the independence of μ by re-checking all the estimates above. Now let $\mu \to -\infty$, then $t_\mu \downarrow 0$ and the monotone convergence theorem yields

$$
\left[\int_{0}^\infty \left(\sup_{x\in[t,\infty)} u(x) \int_x^\infty g \right)^q w(t)\,\mathrm{d}t \right]^{\frac{1}{q}} \lesssim A_{(6)} \|g\|_{L^p(v)}.
$$

Recall that until now we have assumed that $\int_0^\infty w = 2^K$ with $K \in \mathbb{Z}$, and therefore for all weights w such that $\int_0^\infty w < \infty$ (one may multiply w by a constant and use homogeneity). To prove the statement for a general weight w, suppose that $\int_0^\infty w = \infty$ and $A_{(6)} < \infty$. Find, e.g. by truncation, a sequence of weights $\{w_K\}_{K=1}^\infty$ such that $\int_0^\infty w_K = 2^K$ and $w_K \uparrow w$ pointwise as $K \to \infty$. By the previous part of the proof, for all $K \in \mathbb{N}$ we have

$$
\begin{aligned}
\left[\int_0^\infty \left(\sup_{x\in[t,\infty)} u(x)\int_x^\infty g\right)^q w_K(t)\,\mathrm{d}t\right]^{\frac1q} &\lesssim \sup_{x>0}\left(\int_0^x \sup_{x\in[t,x]} u^q(y) w_K(t)\,\mathrm{d}t\right)^{\frac1q}\left(\int_x^\infty v^{1-p'}\right)^{\frac{1}{p'}} \|g\|_{L^p(v)}\\
&\le \sup_{x>0}\left(\int_0^x \sup_{x\in[t,x]} u^q(y) w(t)\,\mathrm{d}t\right)^{\frac1q}\left(\int_x^\infty v^{1-p'}\right)^{\frac{1}{p'}} \|g\|_{L^p(v)}\\
&= A_{(6)}\|g\|_{L^p(v)}.
\end{aligned}
$$

Letting $K \to \infty$, by the monotone convergence theorem it follows

$$
\left[\int_0^\infty \left(\sup_{x\in[t,\infty)} u(x)\int_x^\infty g\right)^q w(t)\,\mathrm{d}t\right]^{\frac1q} \lesssim A_{(6)}\|g\|_{L^p(v)}.
$$

The function $g \in L^p(v)$ is arbitrary and the constant in "$\lesssim$" does not depend on g, hence (5) holds and the optimal $C_{(5)}$ must satisfy $C_{(5)} \lesssim A_{(6)}$ in the case $1 < p \le q < \infty$.

(ii) Let $1 < p < \infty$ and $0 < q < p$. Assume $A_{(7)} + A_{(8)} < \infty$. Then for B_1 we have

$$
\begin{aligned}
B_1 &\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^k \sup_{x\in\Delta_k} u^q(x)\left(\int_x^{t_{(k+1)}} v^{1-p'}\right)^{\frac{q}{p'}}\left(\int_x^{t_{(k+1)}} g^p v\right)^{\frac qp}\right]^{\frac1q} \qquad (31)\\
&\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^k \sup_{x\in\Delta_k} u^q(x)\left(\int_x^{t_{(k+1)}} v^{1-p'}\right)^{\frac{q}{p'}}\left(\int_{\Delta_k} g^p v\right)^{\frac qp}\right]^{\frac1q}\\
&\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^{\frac{kr}{q}} \sup_{x\in\Delta_k} u^r(x)\left(\int_x^{t_{(k+1)}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac1r}\left(\sum_{k\in\mathbb{Z}_\mu}\int_{\Delta_k} g^p v\right)^{\frac1p} \qquad (32)\\
&\le \left[\sum_{k\in\mathbb{Z}_\mu} 2^{\frac{kr}{q}} \sup_{x\in\Delta_k} u^r(x)\left(\int_x^{t_{(k+1)}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac1r}\|g\|_{L^p(v)}\\
&\lesssim \left[\sum_{k\in\mathbb{Z}_\mu} \int_{\Delta_k} W^{\frac rp}(t)w(t)\,\mathrm{d}t \sup_{x\in\Delta_k} u^r(x)\left(\int_x^{t_{(k+1)}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac1r}\|g\|_{L^p(v)} \qquad (33)\\
&\le \left[\sum_{k\in\mathbb{Z}_\mu} \int_{\Delta_k} W^{\frac rp}(t)w(t) \sup_{x\in[t,\infty)} u^r(x)\left(\int_x^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac1r}\|g\|_{L^p(v)}\\
&= A_{(7)}\|g\|_{L^p(v)}.
\end{aligned}
$$

Here, (31) follows from the Hölder inequality, step (32) makes use of Proposition 4(ii) and in (33) one applies the property (9).

Before we continue with B_3, let us notice that for any $t \in (0,\infty)$ it holds

$$
\sup_{y\in[t,\infty)}\sup_{z\in[t,y]} u(z)\left(\int_y^\infty v^{1-p'}\right)^{\frac{1}{p'}} = \sup_{z\in[t,\infty]} u(z)\sup_{y\in[z,\infty)}\left(\int_y^\infty v^{1-p'}\right)^{\frac{1}{p'}} = \sup_{z\in[t,\infty]} u(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{1}{p'}}. \qquad (34)
$$

Define $k_{-1} := k_0 - 1 = \mu - 1$. Now it is possible to write

$$
\begin{aligned}
B_3 &\le \left[\sum_{n=1}^{N} \left(\sum_{k=k_{(n-1)}}^{k_n - 1} 2^k U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \left(\sum_{n=1}^{N} \int_{t_{k_n}}^{t_{k_{(n+1)}}} g^p v \right)^{\frac{1}{p}} \qquad (35)\\
&\le \left[\sum_{n=1}^{N} \left(\sum_{k=k_{(n-1)}}^{k_n - 1} 2^k U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \|g\|_{L^p(v)}\\
&\le \left[\sum_{n=1}^{N} \left(\int_{t_{k_{(n-2)}}}^{t_{k_n}} w(t) U^q(t, t_{k_n}) \, \mathrm{d}t \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \|g\|_{L^p(v)} \qquad (36)\\
&\lesssim \left[\sum_{n=1}^{N} \int_{t_{k_{(n-2)}}}^{t_{k_n}} \left(\int_{t_{k_{(n-2)}}}^{t} w(x) U^q(x, t_{k_n}) \, \mathrm{d}x \right)^{\frac{r}{p}} w(t) U^q(t, t_{k_n}) \, \mathrm{d}t \left(\int_{t_{k_n}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \|g\|_{L^p(v)}\\
&\lesssim \left[\sum_{n=1}^{N} \int_{t_{k_{(n-2)}}}^{t_{k_n}} \left(\int_{t_{k_{(n-2)}}}^{t} w(x) U^q(x, t) \, \mathrm{d}x \right)^{\frac{r}{p}} w(t) U^q(t, t_{k_n}) \, \mathrm{d}t \left(\int_{t_{k_n}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \|g\|_{L^p(v)}\\
&\quad + \left[\sum_{n=1}^{N} \int_{t_{k_{(n-2)}}}^{t_{k_n}} \left(\int_{t_{k_{(n-2)}}}^{t} w(x) \, \mathrm{d}x \right)^{\frac{r}{p}} w(t) U^r(t, t_{k_n}) \, \mathrm{d}t \left(\int_{t_{k_n}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \|g\|_{L^p(v)}\\
&\le \left[\sum_{n=1}^{N} \int_{t_{k_{(n-2)}}}^{t_{k_n}} \left(\int_{0}^{t} w(x) U^q(x, t) \, \mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z \in [t,\infty)} U^q(t, z) \left(\int_{z}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \mathrm{d}t \right]^{\frac{1}{r}} \|g\|_{L^p(v)}\\
&\quad + \left[\sum_{n=1}^{N} \int_{t_{k_{(n-2)}}}^{t_{k_n}} W^{\frac{r}{p}}(t) w(t) \sup_{z \in [t,\infty)} U^r(t, z) \, \mathrm{d}t \left(\int_{z}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \|g\|_{L^p(v)}\\
&= \left[\sum_{n=1}^{N} \int_{t_{k_{(n-2)}}}^{t_{k_n}} \left(\int_{0}^{t} w(x) U^q(x, t) \, \mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z \in [t,\infty)} u^q(z) \left(\int_{z}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \mathrm{d}t \right]^{\frac{1}{r}} \|g\|_{L^p(v)} \qquad (37)\\
&\quad + \left[\sum_{n=1}^{N} \int_{t_{k_{(n-2)}}}^{t_{k_n}} W^{\frac{r}{p}}(t) w(t) \sup_{z \in [t,\infty)} u^r(z) \left(\int_{z}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \mathrm{d}t \right]^{\frac{1}{r}} \|g\|_{L^p(v)}\\
&\lesssim \sum_{i=0}^{1} \left[\sum_{\substack{1 \le n \le N \\ n \bmod 2 = i}} \int_{t_{k_{(n-2)}}}^{t_{k_n}} \left(\int_{0}^{t} w(x) U^q(x, t) \, \mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z \in [t,\infty)} u^q(z) \left(\int_{z}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \mathrm{d}t \right]^{\frac{1}{r}} \|g\|_{L^p(v)} \qquad (38)\\
&\quad + \sum_{i=0}^{1} \left[\sum_{\substack{1 \le n \le N \\ n \bmod 2 = i}} \int_{t_{k_{(n-2)}}}^{t_{k_n}} W^{\frac{r}{p}}(t) w(t) \sup_{z \in [t,\infty)} u^r(z) \left(\int_{z}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \mathrm{d}t \right]^{\frac{1}{r}} \|g\|_{L^p(v)}\\
&\lesssim \left(A_{(8)} + A_{(7)} \right) \|g\|_{L^p(v)}.
\end{aligned}
$$

On the line (35) we applied Proposition 4(ii). Step (36) is based on (19) and the inequality $t_{k_{(n-1)}-1} \ge t_{k_{(n-2)}}$ which is valid for all $n \in \{1, \dots, N\}$. The identity on (37) follows by (34). On the line (38) we split the sums into sums over even and odd numbers n so that the intervals $[t_{k_{(n-2)}}, t_{k_n}]$ become disjoint. This manoeuver will be commonly used in the rest of the paper.

Omitting the details of it, now we perform the limit passes $\mu \to -\infty$ and $K \to \infty$ as in the final part of the proof of sufficiency in case (i). We obtain

$$
\left[\int_0^\infty \left(\sup_{x \in [t,\infty)} u(x) \int_x^\infty g \right)^q w(t) \, \mathrm{d}t \right]^{\frac{1}{q}} \lesssim \left(A_{(7)} + A_{(8)} \right) \|g\|_{L^p(v)}
$$

for our arbitrarily chosen $g \in L^p(v)$. Hence, (5) is valid for all $g \in \mathscr{M}_+$ and the optimal $C_{(5)}$ satisfies $C_{(5)} \lesssim A_{(7)} + A_{(8)}$. This completes the sufficiency part.

Necessity. Suppose that (5) holds for all $g \in \mathscr{M}_+$. Let $1 < p < \infty$ and let $q \in (0,\infty)$ be arbitrary. Let $x > 0$. By the duality of L^p-spaces there exists a function $\varphi \in L^p(v)$ such that $\varphi(t) = 0$ for all $t < x$,

$$\int_x^\infty \varphi^p v = \int_0^\infty \varphi^p v = 1 \qquad \text{and} \qquad \left(\int_x^\infty v^{1-p'}\right)^{\frac{1}{p'}} \le 2\int_x^\infty \varphi.$$

Then

$$\begin{aligned}
&\left(\int_0^x w(t)U^q(t,x)\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_x^\infty v^{1-p'}\right)^{\frac{1}{p'}} \lesssim \left(\int_0^x w(t)U^q(t,x)\,\mathrm{d}t\right)^{\frac{1}{q}}\int_x^\infty \varphi \\
&\le \left[\int_0^x \left(\sup_{y\in[t,x]} u(y)\int_y^\infty \varphi\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \le \left[\int_0^x \left(\sup_{y\in[t,\infty)} u(y)\int_y^\infty \varphi\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \\
&= \left[\int_0^x \left(\sup_{y\in[t,\infty)} u(y)\int_y^\infty \varphi\right)^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}} \le C_{(5)}\|\varphi\|_{L^p(v)} = C_{(5)}.
\end{aligned}$$

Taking the supremum over $x > 0$, we obtain

$$A_{(6)} \lesssim C_{(5)}. \tag{39}$$

This proves that the condition $A_{(6)}$ is in fact necessary in both cases (i) and (ii). The proof of case (i) is therefore complete.

In the rest of the proof we will deal with the case (ii). Thus, from now on assume that $1 < p < \infty$ and $0 < q < p$.

Since we assumed $C_{(5)} < \infty$, the inequality (39) implies

$$\int_x^\infty v^{1-p'}(s)\,\mathrm{d}s < \infty \quad \text{for all } x \in (0,\infty). \tag{40}$$

It may be checked as follows. Let $x > 0$. By the definition of a weight, it holds $\int_0^s w > 0$ and $\int_0^s u > 0$ for any $s > 0$. Hence, both u and w are positive a.e. on an interval $(0,\varepsilon)$ with $\varepsilon > 0$, which implies that $\int_0^x w(t)u^q(t)\,\mathrm{d}t > 0$. Using (39), we now get

$$\left(\int_x^\infty v^{1-p'}\right)^{\frac{1}{p'}} \lesssim C_{(5)}\left(\int_0^x w(t)U^q(t,x)\,\mathrm{d}t\right)^{-\frac{1}{q}} \le C_{(5)}\left(\int_0^x w(t)u^q(t)\,\mathrm{d}t\right)^{-\frac{1}{q}} < \infty.$$

Now assume again that $\int_0^\infty w = 2^K$, define the k-segments, choose $\mu \in \mathbb{Z}$ such that $\mu \le K-2$ and construct the n-blocks. Then we have

$$
\begin{aligned}
&\left[\int_{t_\mu}^\infty \left(\int_{t_\mu}^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \\
&= \left[\sum_{k\in\mathbb{Z}_\mu} \int_{\Delta_k} \left(\int_{t_\mu}^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{k\in\mathbb{Z}_\mu} 2^{\frac{kr}{q}} \sup_{z\in[t_k,\infty)} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (41)\\
&= \left[\sum_{k\in\mathbb{Z}_\mu} 2^{\frac{kr}{q}} \sup_{k\le j\le N} \sup_{z\in\Delta_j} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{k\in\mathbb{Z}_\mu} 2^{\frac{kr}{q}} \sup_{z\in\Delta_k} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (42)\\
&\lesssim \left[\sum_{k=\mu}^{K-1} 2^{\frac{kr}{q}} \sup_{z\in\Delta_k} u^r(z) \left(\int_z^{t_{(k+1)}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&+ \left[\sum_{k=\mu}^{K-2} 2^{\frac{kr}{q}} U^r(\Delta_k) \left(\int_{t_{(k+1)}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&=: B_4 + B_5.
\end{aligned}
$$

In (41) we applied (9) and step (42) follows from Corollary 3.

Using (9) and (39), we continue by a preliminary estimate.

$$
\begin{aligned}
B_4 &\le \left[\sum_{k=\mu}^{K-1} 2^{\frac{kr}{q}} \sup_{z\in\Delta_k} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \lesssim \left[\sum_{k=\mu}^{K-1} \left(\int_0^{t_k} w(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \sup_{z\in\Delta_k} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\le \left[\sum_{k=\mu}^{K-1} \sup_{z\in\Delta_k} \left(\int_0^z w(t) U^q(t,z)\,\mathrm{d}t\right)^{\frac{r}{q}} \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \le (K-\mu)^{\frac{1}{r}} A_{(6)} \lesssim (K-\mu)^{\frac{1}{r}} C_{(5)} < \infty.
\end{aligned}
$$

An attentive reader could now rightfully accuse the author of cheating. Indeed, the previous chain of inequalities provides an estimate of B_4 by $C_{(5)}$ and may thus seem to be what we want, but the estimate is not uniform. The problem is the term $K-\mu$ which depends on the auxiliary sum. To get the proper uniform bound we therefore need to do more work. However, by the previous estimate we managed to show that $B_4 < \infty$, which was the true reason why we made it. The information about the finiteness is needed in what follows.

For each $k \in \mathbb{Z}_\mu$ let $z_k \in \Delta_k$ be such number that

$$
2u^r(z_k) \left(\int_{z_k}^\infty v^{1-p'}\right)^{\frac{r}{p'}} \ge \sup_{z\in\Delta_k} u^r(z) \left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}. \qquad (43)
$$

Both sides of the inequality are finite, which follows from the finiteness of B_4.

Now, since by (40) one has $\int_{z_k}^{t_{(k+1)}} v^{1-p'} < \infty$ for all $k \in \mathbb{Z}_\mu$, duality of L^p-spaces yields that for each $k \in \mathbb{Z}_\mu$ there exists a nonnegative function h_k with support in the interval $[z_k, t_{(k+1)}]$ and such that

$$
\int_{\Delta_k} h_k^p v = \int_{z_k}^{t_{(k+1)}} h_k^p v = 1 \qquad \text{and} \qquad \left(\int_{z_k}^{t_{(k+1)}} v^{1-p'}\right)^{\frac{1}{p'}} \le 2\int_{z_k}^{t_{(k+1)}} h_k. \qquad (44)
$$

Then it holds

$$\sup_{z\in\Delta_k} u(z)\left(\int_z^{t_{(k+1)}} v^{1-p'}\right)^{\frac{1}{p'}} \lesssim u(z_k)\left(\int_{z_k}^{t_{(k+1)}} v^{1-p'}\right)^{\frac{1}{p'}} \lesssim u(z_k)\int_{z_k}^{t_{(k+1)}} h_k \le \sup_{z\in\Delta_k} u(z)\int_z^{t_{(k+1)}} h_k. \quad (45)$$

Furthermore, since $B_4 < \infty$, by Proposition 5 there exists a nonnegative sequence $\{a_k\}_{k\in\mathbb{Z}_\mu}$ such that $\sum_{k\in\mathbb{Z}_\mu} a_k^p = 1$ and

$$\left[\sum_{k=\mu}^{K-1} 2^{\frac{kr}{q}} \sup_{z\in\Delta_k} u^r(z)\left(\int_z^{t_{(k+1)}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \le 2\left[\sum_{k=\mu}^{K-1} 2^k \sup_{z\in\Delta_k} u^q(z)\left(\int_z^{t_{(k+1)}} v^{1-p'}\right)^{\frac{q}{p'}} a_k^q\right]^{\frac{1}{q}}. \quad (46)$$

Define the function $h := \sum_{k=1}^{K-1} a_k h_k$. Then it satisfies

$$\|h\|_{L^p(v)} = \left(\sum_{k\in\mathbb{Z}_\mu}\int_{\Delta_k} h^p v\right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{Z}_\mu} a_k^p \int_{\Delta_k} h_k^p v\right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{Z}_\mu} a_k^p\right)^{\frac{1}{p}} = 1.$$

We may finally derive the following estimate on B_4.

$$\begin{aligned}
B_4 &\lesssim \left[\sum_{k=\mu}^{K-1} 2^k \sup_{z\in\Delta_k} u^q(z)\left(\int_z^{t_{(k+1)}} v^{1-p'}\right)^{\frac{q}{p'}} a_k^q\right]^{\frac{1}{q}} && (47)\\
&\lesssim \left[\sum_{k=\mu}^{K-1} 2^k \sup_{z\in\Delta_k} u^q(z)\left(\int_z^{t_{(k+1)}} h_k\right)^q a_k^q\right]^{\frac{1}{q}} && (48)\\
&= \left[\sum_{k=\mu}^{K-1} 2^k \sup_{z\in\Delta_k} u^q(z)\left(\int_z^{t_{(k+1)}} h\right)^q\right]^{\frac{1}{q}}\\
&\lesssim \left[\sum_{k=\mu}^{K-1} \int_{\Delta_{(k-1)}} w(t)\,\mathrm{d}t \sup_{z\in\Delta_k} u^q(z)\left(\int_z^{t_{(k+1)}} h\right)^q\right]^{\frac{1}{q}} && (49)\\
&\le \left[\sum_{k=\mu}^{K-1} \int_{\Delta_{(k-1)}} w(t)\left(\sup_{z\in[t,\infty)} u(z)\int_z^\infty h(s)\,\mathrm{d}s\right)^q \mathrm{d}t\right]^{\frac{1}{q}}\\
&\le \left(\int_0^\infty w(t)\left(\sup_{z\in[t,\infty)} u(z)\int_z^\infty h(s)\,\mathrm{d}s\right)^q \mathrm{d}t\right)^{\frac{1}{q}}\\
&\le C_{(5)}\|h\|_{L^p(v)} = C_{(5)}.
\end{aligned}$$

Here in (47) we used (46) and in (48) we used (45). The inequality on (49) is, as usual, due to (9). Only now we obtained the "proper" estimate

$$B_4 \lesssim C_{(5)},$$

in which the constant behind the symbol "$\lesssim$" really depends only on p and q.

We proceed with B_5 as follows.

$$
\begin{aligned}
B_5 &= \left[\sum_{n=1}^{N} \sum_{k=k_n-1}^{k_{(n+1)}-2} 2^{\frac{kr}{q}} U^r(\Delta_k) \left(\int_{t_{(k+1)}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \\
&\le \left[\sum_{n=1}^{N} \sum_{k=k_n-1}^{k_{(n+1)}-2} 2^{\frac{kr}{q}} U^r(\Delta_k) \left(\int_{t_{k_n}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \\
&\le \left[\sum_{n=1}^{N} \left(\sum_{k=k_n-1}^{k_{(n+1)}-2} 2^{k} U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \qquad (50) \\
&\lesssim \left[\sum_{n=1}^{N} \left(\sum_{k=k_{(n-1)}}^{k_n-1} 2^{k} U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \qquad (51) \\
&\lesssim \left[\sum_{n=1}^{N} \left(\sum_{k=k_{(n-1)}}^{k_n-1} 2^{k} U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \qquad (52) \\
&=: B_6.
\end{aligned}
$$

Step (50) follows by Jensen inequality since $\frac{r}{q} > 1$. Step (51) follows from (13). In (52) one uses Corollary 2, considering also (12).

Before estimating further, let us first prove finiteness of B_6, as we did in case of B_4. By (19) and (39) we obtain

$$
B_6 \le \left[\sum_{n=1}^{N} \left(\int_0^{t_{k_n}} w(t) U^q(t, t_{k_n}) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \le N^{\frac{1}{r}} A_{(6)} \lesssim N^{\frac{1}{r}} C_{(5)} < \infty.
$$

Considering (40) and the L^p-duality, for each $n \in \mathbb{N}$ such that $n \le N$ we can find a function g_n supported in the interval $[t_{k_n}, t_{k_{(n+1)}}]$ and such that

$$
\int_{t_{k_n}}^{t_{k_{(n+1)}}} g_n^p v = 1 \qquad \text{and} \qquad \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'} \right)^{\frac{1}{p'}} \le 2 \int_{t_{k_n}}^{t_{k_{(n+1)}}} g_n. \tag{53}
$$

Furthermore, since we know that $B_6 < \infty$, by Proposition 5 we find a nonnegative sequence $\{c_n\}_{n=1}^N$ such that $\sum_{n=1}^N c_n^p = 1$ and

$$
B_6 = \left[\sum_{n=1}^{N} \left(\sum_{k=k_{(n-1)}}^{k_n-1} 2^{k} U^q(\Delta_k) \right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}} \le 2 \left[\sum_{n=1}^{N} \sum_{k=k_{(n-1)}}^{k_n-1} 2^{k} U^q(\Delta_k) \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'} \right)^{\frac{q}{p'}} c_n^q \right]^{\frac{1}{q}}. \tag{54}
$$

Define the function $g := \sum_{n=1}^{N} c_n g_n$. It is easy to verify that $\|g\|_{L^p(v)} = 1$. Moreover, it holds

$$
\begin{aligned}
B_6 &\lesssim \left[\sum_{n=1}^{N} \sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k) \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'} \right)^{\frac{q}{p'}} c_n^q \right]^{\frac{1}{q}} && (55)\\
&\lesssim \left[\sum_{n=1}^{N} \sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k) \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} g_n \right)^{q} c_n^q \right]^{\frac{1}{q}} && (56)\\
&= \left[\sum_{n=1}^{N} \sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k) \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} g \right)^{q} \right]^{\frac{1}{q}} \\
&\lesssim \left[\sum_{n=1}^{N} \sum_{k=k_{(n-1)}}^{k_n-1} 2^k U^q(\Delta_k) \left(\int_{t_{(k+1)}}^{\infty} g \right)^{q} \right]^{\frac{1}{q}} \\
&= \left[\sum_{k=\mu}^{k_N-1} 2^k U^q(\Delta_k) \left(\int_{t_{(k+1)}}^{\infty} g \right)^{q} \right]^{\frac{1}{q}} \\
&\lesssim \left[\sum_{k=\mu}^{k_N-1} \int_{\Delta_{(k-1)}} w(t)\,\mathrm{d}t \sup_{z\in[t_k,\infty)} U^q(t_k,z) \left(\int_z^{\infty} g(s)\,\mathrm{d}s \right)^{q} \right]^{\frac{1}{q}} && (57)\\
&= \left[\sum_{k=\mu}^{k_N-1} \int_{\Delta_{(k-1)}} w(t)\,\mathrm{d}t \sup_{z\in[t_k,\infty)} u^q(z) \left(\int_z^{\infty} g(s)\,\mathrm{d}s \right)^{q} \right]^{\frac{1}{q}} && (58)\\
&\le \left[\sum_{k=\mu}^{k_N-1} \int_{\Delta_{(k-1)}} w(t) \left(\sup_{z\in[t,\infty)} u(z) \int_z^{\infty} g(s)\,\mathrm{d}s \right)^{q} \mathrm{d}t \right]^{\frac{1}{q}} \\
&\le \left(\int_0^{\infty} w(t) \left(\sup_{z\in[t,\infty} u(z) \int_z^{\infty} g(s)\,\mathrm{d}s \right)^{q} \right]^{\frac{1}{q}} \\
&\le C_{(5)} \|g\|_{L^p(v)} = C_{(5)}.
\end{aligned}
$$

Here, (55) is the same as (54), inequality (56) follows from (53) and inequality (57) from (9). An argument analogous to (34) is used to establish the identity (58).

We have shown

$$B_5 \lesssim B_6 \lesssim C_{(5)},$$

hence, combining this with the other estimates, we get

$$\left[\int_{t_\mu}^{\infty} \left(\int_{t_\mu}^{t} w(x)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^r(z) \left(\int_z^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \mathrm{d}t \right]^{\frac{1}{r}} \lesssim B_4 + B_5 \lesssim B_4 + B_6 \lesssim C_{(5)}. \tag{59}$$

Passing $\mu \to -\infty$ and then $K \to \infty$ analogously as we did before, we obtain

$$A_{(7)} \lesssim C_{(5)} \tag{60}$$

for a general weight w.

In the rest we will focus on the condition $A_{(8)}$. At first, observe that for any $0 < a < t < \infty$ it holds

$$U^{\frac{rq}{p}}(a,t) \sup_{z\in[t,\infty)} u^q(z) \left(\int_z^{\infty} v^{1-p'} \right)^{\frac{r}{p'}} \le \sup_{z\in[t,\infty)} u^r(z) \left(\int_z^{\infty} v^{1-p'} \right)^{\frac{r}{p'}}. \tag{61}$$

Indeed, one has

$$
\begin{aligned}
&\sup_{s\in[a,t)} u^{\frac{rq}{p}}(s) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \le \sup_{s\in[a,t)} u^{\frac{rq}{p}}(s) \sup_{z\in[t,\infty)} \sup_{\tau\in[t,z)} u^q(\tau)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\\
&= \sup_{z\in[t,\infty)} \sup_{s\in[a,t)} u^{\frac{rq}{p}}(s) \sup_{\tau\in[t,z)} u^q(\tau)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \le \sup_{z\in[t,\infty)} \sup_{s\in[a,z)} u^r(s)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\\
&\le \sup_{z\in[a,\infty)} \sup_{s\in[a,z)} u^r(s)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} = \sup_{z\in[a,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}.
\end{aligned}
$$

The identity (34) implies the last step.

The starting point for estimating $A_{(8)}$ is the following decomposition.

$$
\begin{aligned}
&\left[\int_{t_\mu}^\infty \left(\int_{t_\mu}^t w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
&\approx \left[\int_{\Delta_\mu} \left(\int_{t_\mu}^t w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}} \qquad (62)\\
&\quad + \left[\sum_{n=1}^N \int_{\Delta_{k_{(n+1)}-1}} \left(\int_{t_\mu}^t w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
&\quad + \left[\sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} \int_{\Delta_k} \left(\int_{t_\mu}^t w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
&=: B_7 + B_8 + B_9.
\end{aligned}
$$

For B_7 one has

$$
\begin{aligned}
B_7 &\le \left[\int_{\Delta_\mu} \left(\int_{t_\mu}^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)U^{\frac{rq}{p}}(t_\mu,t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}}\\
&\le \left[\int_{\Delta_\mu} \left(\int_{t_\mu}^t w(x)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t)\,\mathrm{d}t \sup_{z\in[t_\mu,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (63)\\
&\lesssim \left[2^{\frac{\mu r}{q}} \sup_{z\in[t_\mu,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (64)\\
&\lesssim \left[\int_0^{t_\mu} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \sup_{z\in[t_\mu,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}} \qquad (65)\\
&\le A_{(7)} \lesssim C_{(5)}.
\end{aligned}
$$

We used (61) to get (63), and (9) was used for (64) and (65). The very last inequality was obtained in (60).

Next, for B_8 we get

$$\begin{aligned}
B_8 &\lesssim \left[\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1}2^jU^q(\Delta_j)\right)^{\frac{r}{p}}\int_{\Delta_{k_{(n+1)}-1}}w(t)\sup_{z\in[t,\infty)}u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\,\mathrm{d}t\right]^{\frac{1}{r}}\\
&\quad+\left[\sum_{n=1}^{N}2^{k_{(n+1)}\frac{r}{p}}\int_{\Delta_{k_{(n+1)}-1}}w(t)U^{\frac{rq}{p}}(t_{k_{(n+1)}-1},t)\sup_{z\in[t,\infty)}u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\,\mathrm{d}t\right]^{\frac{1}{r}}\\
&\lesssim \left[\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1}2^jU^q(\Delta_j)\right)^{\frac{r}{p}}2^{k_{(n+1)}}\sup_{z\in\left[t_{k_{(n+1)}-1},\infty\right)}u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\quad+\left[\sum_{n=1}^{N}2^{k_{(n+1)}\frac{r}{q}}\sup_{z\in\left[t_{k_{(n+1)}-1},\infty\right)}u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&=: B_{10}+B_{11}.
\end{aligned}$$

The first step follows by (18). In the second step we used (9) to estimate the first summand, and (61) and (9) to estimate the second one.

Now formally define $k_{-1} := k_0 - 1 = \mu - 1$ and $t_{k_{(N+2)}-1} := \infty$. Furthermore, observe that, by (12), it holds

$$\left(\sum_{j=k_{(n-1)}}^{k_n-1}2^jU^q(\Delta_j)\right)^{\frac{r}{p}}2^{k_{(n+1)}}\ge 2^{\frac{r}{p}}\left(\sum_{j=k_{(n-2)}}^{k_{(n-1)}-1}2^jU^q(\Delta_j)\right)^{\frac{r}{p}}2^{k_{(n+1)}}\ge 2^{\frac{r}{q}}\left(\sum_{j=k_{(n-2)}}^{k_{(n-1)}-1}2^jU^q(\Delta_j)\right)^{\frac{r}{p}}2^{k_n}$$

for every $n\in\mathbb{N}$ such that $2\le n\le N$. Therefore, since it holds $2^{\frac{r}{q}}>1$, the sequence $\{b_n\}_{n=1}^N$ with $b_n := \left(\sum_{j=k_{(n-1)}}^{k_n-1}2^jU^q(\Delta_j)\right)^{\frac{r}{p}}2^{k_{(n+1)}}$ is strongly increasing.

For B_{10} we then obtain

$$\begin{aligned}
B_{10} &= \left[\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1}2^jU^q(\Delta_j)\right)^{\frac{r}{p}}2^{k_{(n+1)}}\sup_{n+1\le i\le N+1}\ \sup_{z\in\left[t_{k_i-1},t_{k_{(i+1)}-1}\right)}u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\lesssim \left[\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1}2^jU^q(\Delta_j)\right)^{\frac{r}{p}}2^{k_{(n+1)}}\sup_{z\in\left[t_{k_{(n+1)}-1},t_{k_{(n+2)}-1}\right)}u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\lesssim \left[\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1}2^jU^q(\Delta_j)\right)^{\frac{r}{p}}2^{k_{(n+1)}}\sup_{z\in\left[t_{k_{(n+1)}-1},t_{k_{(n+2)}-1}\right)}u^q(z)\left(\int_z^{t_{k_{(n+2)}-1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\quad+\left[\sum_{n=1}^{N-1}\left(\sum_{j=k_{(n-1)}}^{k_n-1}2^jU^q(\Delta_j)\right)^{\frac{r}{p}}2^{k_{(n+1)}}U^q(t_{k_{(n+1)}-1},t_{k_{(n+2)}-1})\left(\int_{t_{k_{(n+2)}-1}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&=: B_{12}+B_{13}.
\end{aligned}$$

The second step follows from Corollary 3.

Let us proceed with B_{12}. We get

$$
\begin{aligned}
B_{12} &\le \left[\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} 2^{k_{(n+1)}} \sup_{z\in\left[t_{k_{(n+1)}-1},t_{k_{(n+2)}-1}\right)} U^q(t_{k_{(n+1)}-1},z)\left(\int_z^{t_{k_{(n+2)}-1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\lesssim \left[\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \int_{t_{k_{(n+1)}-2}}^{t_{k_{(n+1)}-1}} w(t)\,\mathrm{d}t \sup_{z\in\left[t_{k_{(n+1)}-1},t_{k_{(n+2)}-1}\right)} U^q(t_{k_{(n+1)}-1},z)\left(\int_z^{t_{k_{(n+2)}-1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (66)\\
&\lesssim \left[\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sup_{z\in\left[t_{k_{(n+1)}-1},t_{k_{(n+2)}-1}\right)} \int_{t_{k_{(n+1)}-2}}^{z} w(t)U^q(t,z)\,\mathrm{d}t\left(\int_z^{t_{k_{(n+2)}-1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\lesssim \left[\sum_{n=1}^{N}\left(\int_{t_{k_{(n-1)}-1}}^{t_{k_n}} w(t)U^q(t,t_{k_n})\,\mathrm{d}t\right)^{\frac{r}{p}} \sup_{z\in\left[t_{k_{(n+1)}-1},t_{k_{(n+2)}-1}\right)} \int_{t_{k_{(n+1)}-2}}^{z} w(t)U^q(t,z)\,\mathrm{d}t\left(\int_z^{t_{k_{(n+2)}-1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (67)\\
&\le \left[\sum_{n=1}^{N} \sup_{z\in\left[t_{k_{(n+1)}-1},t_{k_{(n+2)}-1}\right)} \left(\int_{t_{k_{(n-1)}-1}}^{z} w(t)U^q(t,z)\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_z^{t_{k_{(n+2)}-1}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&=: B_{14}.
\end{aligned}
$$

In (66) one uses (9) and (67) follows from (19) and the relation $t_{k_{(n+1)}-1} \ge t_{k_n}$.

Let us check finiteness of B_{14}. It holds

$$
B_{14} \le \left[\sum_{n=1}^{N} \sup_{z\in\left[t_{k_{(n+1)}-1},t_{k_{(n+2)}-1}\right)} \left(\int_0^z w(t)U^q(t,z)\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \le N^{\frac{1}{r}} A_{(6)} \lesssim N^{\frac{1}{r}} C_{(5)} < \infty.
$$

Again we made use of the already proved estimate (39).

Now, for each $n\in\mathbb{N}$ such that $2\le n\le N+1$ find a number $z_n'\in[t_{k_n-1},t_{k_{(n+1)}-1})$ such that

$$
2\left(\int_{t_{k_{(n-1)}-1}}^{z_n'} w(t)U^q(t,z_n')\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_{z_n'}^\infty v^{1-p'}\right)^{\frac{r}{p'}} \ge \sup_{z\in\left[t_{k_n-1},t_{k_{(n+1)}-1}\right)} \left(\int_{t_{k_{(n-1)}-1}}^{z} w(t)U^q(t,z)\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}. \qquad (68)
$$

This is possible since the right term is finite, which fact in turn follows from the finiteness of B_{14}. Following (40) and the L^p-duality, for each $n\in\mathbb{N}$ such that $2\le n\le N+1$ there exists a nonnegative function f_n supported in $[z_n',t_{k_{(n+1)}-1}]$ and such that

$$
\int_{t_{k_n-1}}^{t_{k_{(n+1)}-1}} f_n^p v = \int_{z_n'}^{t_{k_{(n+1)}-1}} f_n^p v = 1 \qquad \text{and} \qquad \left(\int_{t_{k_n-1}}^{t_{k_{(n+1)}-1}} v^{1-p'}\right)^{\frac{1}{p'}} \le 2\int_{t_{k_n-1}}^{t_{k_{(n+1)}-1}} f_n.
$$

An argument analogous to that of (45) then yields

$$
\sup_{z\in\left[t_{k_n-1},t_{k_{(n+1)}-1}\right)} \left(\int_{t_{k_{(n-1)}-1}}^{z} w(t)U^q(t,z)\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_z^\infty v^{1-p'}\right)^{\frac{1}{p'}} \lesssim \sup_{z\in\left[t_{k_n-1},t_{k_{(n+1)}-1}\right)} \left(\int_{t_{k_{(n-1)}-1}}^{z} w(t)U^q(t,z)\,\mathrm{d}t\right)^{\frac{1}{q}}\int_z^\infty f_n \qquad (69)
$$

Next, since $B_{14} < \infty$, by Proposition 5 there exists a nonnegative sequence $\{d_n\}_{n=2}^{N+1}$ such that $\sum_{n=2}^{N+1} d_n^p = 1$ and

$$B_{14} = \left[\sum_{n=1}^{N} \sup_{z \in \left[t_{k_{(n+1)}-1}, t_{k_{(n+2)}-1} \right)} \left(\int_{t_{k_{(n-1)}-1}}^{z} w(t) U^q(t,z)\,\mathrm{d}t \right)^{\frac{r}{q}} \left(\int_z^{t_{k_{(n+2)}-1}} v^{1-p'} \right)^{\frac{r}{p'}} \right]^{\frac{1}{r}}$$

$$\leq 2 \left[\sum_{n=1}^{N} \sup_{z \in \left[t_{k_{(n+1)}-1}, t_{k_{(n+2)}-1} \right)} \int_{t_{k_{(n-1)}-1}}^{z} w(t) U^q(t,z)\,\mathrm{d}t \left(\int_z^{t_{k_{(n+2)}-1}} v^{1-p'} \right)^{\frac{q}{p'}} d_n^q \right]^{\frac{1}{q}}. \tag{70}$$

As expected, now we define the function $f := \sum_{n=2}^{N+1} d_n f_n$. An easy check confirms that $\|f\|_{L^p(v)} = 1$. Before continuing, let us make one more observation. Let $n \in \mathbb{N}$ be such that $2 \leq n \leq N+1$ and let $z \in \left[t_{k_{(n+1)}-1}, t_{k_{(n+2)}-1} \right)$. Then

$$\begin{aligned}
\int_{t_{k_{(n-1)}-1}}^{z} w(t) U^q(t,z) \left(\int_z^{\infty} f(s)\,\mathrm{d}s \right)^q \mathrm{d}t &\leq \int_{t_{k_{(n-1)}-1}}^{z} w(t) \sup_{x \in [t,\infty)} U^q(t,x) \left(\int_x^{\infty} f(s)\,\mathrm{d}s \right)^q \mathrm{d}t \\
&= \int_{t_{k_{(n-1)}-1}}^{z} w(t) \sup_{x \in [t,\infty)} u^q(x) \left(\int_x^{\infty} f(s)\,\mathrm{d}s \right)^q \mathrm{d}t \\
&\leq \int_{t_{k_{(n-1)}-1}}^{t_{k_{(n+2)}-1}} w(t) \left(\sup_{x \in [t,\infty)} u^q(x) \int_x^{\infty} f(s)\,\mathrm{d}s \right)^q \mathrm{d}t.
\end{aligned}$$

The second step is an analogy to (34). Taking supremum over $z \in \left[t_{k_{(n+1)}-1}, t_{k_{(n+2)}-1} \right)$, we get

$$\sup_{z \in \left[t_{k_{(n+1)}-1}, t_{k_{(n+2)}-1} \right)} \int_{t_{k_{(n-1)}-1}}^{z} w(t) U^q(t,z) \left(\int_z^{\infty} f \right)^q \mathrm{d}t \leq \int_{t_{k_{(n-1)}-1}}^{t_{k_{(n+2)}-1}} w(t) \left(\sup_{x \in [t,\infty)} u^q(x) \int_x^{\infty} f \right)^q \mathrm{d}t. \tag{71}$$

Now we estimate

$$B_{14} \lesssim \left[\sum_{n=1}^{N} \sup_{z \in \left[t_{k_{(n+1)}-1}, t_{k_{(n+2)}-1} \right)} \int_{t_{k_{(n-1)}-1}}^{z} w(t) U^q(t,z)\,\mathrm{d}t \left(\int_z^{t_{k_{(n+2)}-1}} v^{1-p'} \right)^{\frac{q}{p'}} d_n^q \right]^{\frac{1}{q}} \tag{72}$$

$$\lesssim \left[\sum_{n=1}^{N} \sup_{z \in \left[t_{k_{(n+1)}-1}, t_{k_{(n+2)}-1} \right)} \int_{t_{k_{(n-1)}-1}}^{z} w(t) U^q(t,z)\,\mathrm{d}t \left(\int_z^{t_{k_{(n+2)}-1}} f_n \right)^q d_n^q \right]^{\frac{1}{q}} \tag{73}$$

$$= \left[\sum_{n=1}^{N} \sup_{z \in \left[t_{k_{(n+1)}-1}, t_{k_{(n+2)}-1} \right)} \int_{t_{k_{(n-1)}-1}}^{z} w(t) U^q(t,z)\,\mathrm{d}t \left(\int_z^{t_{k_{(n+2)}-1}} f(s)\,\mathrm{d}s \right)^q \right]^{\frac{1}{q}}$$

$$= \left[\sum_{n=1}^{N} \sup_{z \in \left[t_{k_{(n+1)}-1}, t_{k_{(n+2)}-1} \right)} \int_{t_{k_{(n-1)}-1}}^{z} w(t) U^q(t,z)\,\mathrm{d}t \left(\int_z^{\infty} f(s)\,\mathrm{d}s \right)^q \right]^{\frac{1}{q}}$$

$$\leq \left[\sum_{n=1}^{N} \int_{t_{k_{(n-1)}-1}}^{t_{k_{(n+2)}-1}} w(t) \left(\sup_{x \in [t,\infty)} u^q(x) \int_x^{\infty} f(s)\,\mathrm{d}s \right)^q \mathrm{d}t \right]^{\frac{1}{q}} \tag{74}$$

$$\lesssim \sum_{i=0}^{2} \left[\sum_{\substack{1 \leq n \leq N \\ n \bmod 3 = i}} \int_{t_{k_{(n-1)}-1}}^{t_{k_{(n+2)}-1}} w(t) \left(\sup_{x \in [t,\infty)} u^q(x) \int_x^{\infty} f(s)\,\mathrm{d}s \right)^q \mathrm{d}t \right]^{\frac{1}{q}}$$

$$\lesssim \left[\int_0^{\infty} w(t) \left(\sup_{x \in [t,\infty)} u^q(x) \int_x^{\infty} f(s)\,\mathrm{d}s \right)^q \mathrm{d}t \right]^{\frac{1}{q}}$$

$$\leq C_{(5)} \|f\|_{L^p(v)} = C_{(5)}.$$

The inequality (72) is taken from (70), and step (73) follows from (69). In (74) we used (71).

Let us now return to B_{13}. We have

$$\begin{aligned}
B_{13} &\lesssim \left[\sum_{n=1}^{N-1}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} 2^{k_{(n+1)}-1} U^q(t_{k_{(n+1)}-1}, t_{k_{(n+2)}-1})\left(\int_{t_{k_{(n+2)}-1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{n=1}^{N-1}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_{(n+1)}-1}^{k_{(n+2)}-2} 2^k U^q(t_k, t_{k_{(n+2)}-1})\left(\int_{t_{k_{(n+2)}-1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{n=1}^{N-1}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_{(n+1)}-1}^{k_{(n+2)}-2} 2^k U^q(\Delta_k)\left(\int_{t_{k_{(n+2)}-1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (75) \\
&\lesssim \left[\sum_{n=1}^{N-1}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_n}^{k_{(n+1)}-1} 2^k U^q(\Delta_k)\left(\int_{t_{k_{(n+2)}-1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (76) \\
&\lesssim \left[\sum_{n=1}^{N-1}\left(\sum_{k=k_n}^{k_{(n+1)}-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}} \left(\int_{t_{k_{(n+2)}-1}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (77) \\
&\le \left[\sum_{n=1}^{N}\left(\sum_{k=k_n}^{k_{(n+1)}-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
&\lesssim \left[\sum_{n=1}^{N}\left(\sum_{k=k_n}^{k_{(n+1)}-1} 2^k U^q(\Delta_k)\right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (78) \\
&= B_6 \lesssim C_{(5)}.
\end{aligned}$$

The estimate (75) follows from Corollary 2, step (76) is due to (13) and step (77) due to (12). Inequality (78) is implied by Corollary 2. The final estimate $B_6 \lesssim C_{(5)}$ was obtained in an earlier stage of the proof.

Now we have

$$B_{10} \lesssim B_{12} + B_{13} \lesssim B_{14} + B_{13} \lesssim C_{(5)}.$$

Next term to proceed with is B_{11}. It holds

$$\begin{aligned}
B_{11} &\lesssim \left[\sum_{n=1}^{N}\int_{t_{k_{(n+1)}-2}}^{t_{k_{(n+1)}-1}} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \sup_{z\in[t_{k_{(n+1)}-1},\infty)} u^r(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (79) \\
&\le \left[\sum_{n=1}^{N}\int_{t_{k_{(n+1)}-2}}^{t_{k_{(n+1)}-1}} W^{\frac{r}{p}}(t)w(t) \sup_{z\in[t,\infty)} u^r(z)\left(\int_z^{\infty} v^{1-p'}\right)^{\frac{r}{p'}}\,\mathrm{d}t\right]^{\frac{1}{r}} \\
&\le A_{(7)} \lesssim C_{(5)}.
\end{aligned}$$

In (79) we used (9). Recall also the earlier result (60).

At this point we completed the estimate

$$B_8 \lesssim B_{10} + B_{11} \lesssim C_{(5)}.$$

We return even deeper to the term B_9. By (18), it holds

$$\begin{aligned}
B_9 \lesssim & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_n}^{k_{(n+1)}-2} \int_{\Delta_k} w(t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \\
& + \left[\sum_{n\in\mathbb{A}} \sum_{k=k_n}^{k_{(n+1)}-2} 2^{\frac{kr}{p}} \int_{\Delta_k} w(t) U^{\frac{rq}{p}}(t_k,t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \\
& =: B_{15} + B_{16}.
\end{aligned}$$

Next, one has

$$\begin{aligned}
B_{15} \lesssim & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_n}^{k_{(n+1)}-2} \int_{\Delta_k} w(t) \sup_{z\in[t,t_{k_{(n+1)}-1})} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}} \mathrm{d}t\right]^{\frac{1}{r}} \\
& + \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_n}^{k_{(n+1)}-2} \int_{\Delta_k} w(t)\,\mathrm{d}t \sup_{z\in[t_{k_{(n+1)}-1},\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
\lesssim & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_n}^{k_{(n+1)}-2} 2^k U^q(t_k,t_{k_{(n+1)}-1})\left(\int_{t_{k_n}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (80) \\
& + \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} 2^{k_{(n+1)}} \sup_{z\in[t_{k_{(n+1)}-1},\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \\
\le & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_n}^{k_{(n+1)}-2} 2^k U^q(t_k,t_{k_{(n+1)}-1})\left(\int_{t_{k_n}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} + B_{10} \\
\lesssim & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{p}} \sum_{k=k_n}^{k_{(n+1)}-2} 2^k U^q(\Delta_k)\left(\int_{t_{k_n}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} + B_{10} \qquad (81) \\
\lesssim & \left[\sum_{n\in\mathbb{A}}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} + B_{10} \qquad (82) \\
\le & \left[\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} + B_{10} \\
\lesssim & \left[\sum_{n=1}^{N}\left(\sum_{j=k_{(n-1)}}^{k_n-1} 2^j U^q(\Delta_j)\right)^{\frac{r}{q}} \left(\int_{t_{k_n}}^{t_{k_{(n+1)}}} v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} + B_{10} \qquad (83) \\
= & B_6 + B_{10} \lesssim C_{(5)}.
\end{aligned}$$

In step (80) we used (9) and step (81) follows from Corollary 2. Step (82) is due to (13). To get (83) recall (12) and use Corollary 2. The estimate $B_6 + B_{10} \lesssim C_{(5)}$ was obtained earlier.

The term B_{16} is the last remaining one. We get

$$
\begin{aligned}
B_{16} &\le \left[\sum_{n\in\mathbb{A}}\sum_{k=k_n}^{k_{(n+1)}-2} 2^{\frac{kr}{p}}\int_{\Delta_k} w(t)\,\mathrm{d}t \sup_{z\in[t_k,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (84)\\
&\lesssim \left[\sum_{n\in\mathbb{A}}\sum_{k=k_n}^{k_{(n+1)}-2} 2^{\frac{kr}{q}} \sup_{z\in[t_k,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\le \left[\sum_{k=1}^{K} 2^{\frac{kr}{q}} \sup_{z\in[t_k,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\lesssim \left[\sum_{k=1}^{K} \int_{\Delta_{k-1}} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \sup_{z\in[t_k,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}} \qquad (85)\\
&\le \left[\sum_{k=1}^{K} \int_{\Delta_{k-1}} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \sup_{z\in[t,\infty)} u^r(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\right]^{\frac{1}{r}}\\
&\le A_{(7)} \lesssim C_{(5)}.
\end{aligned}
$$

To get the inequality (84) we used (61). Step (85) follows from (9). For the final estimate see (60).

We have shown

$$B_9 \lesssim B_{15} + B_{16} \lesssim C_{(5)}.$$

Now, collecting all our estimates and returning all the way back to the starting decomposition (62), we check that we have proved

$$\left[\int_{t_\mu}^\infty \left(\int_{t_\mu}^t w(x)U^q(x,t)\,\mathrm{d}x\right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^q(z)\left(\int_z^\infty v^{1-p'}\right)^{\frac{r}{p'}}\mathrm{d}t\right]^{\frac{1}{r}} \lesssim C_{(5)}.$$

The limit pass $\mu \to -\infty$ followed by the second one $K \to \infty$ in the same manner as done previously then finally yields

$$A_{(8)} \lesssim C_{(5)}.$$

Therefore, necessity of the conditions $A_{(7)}$ and $A_{(8)}$ in case (ii) is verified and the proof is finished.

The previous theorem has, not surprisingly, its analogy for $p = 1$. It may be proved by a similar technique as Theorem 6. Given the length of the previous proof, the reader will hopefully excuse omitting the proof the theorem below, which is the aforementioned version for $p = 1$.

Theorem 7 *Let v, w be weights and let u be a continuous weight. Consider the inequality*

$$\left(\int_0^\infty \left[\sup_{x\in[t,\infty)} u(x)\int_x^\infty g(s)\,\mathrm{d}s\right]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_{(86)}\int_0^\infty g(t)v(t)\,\mathrm{d}t. \qquad (86)$$

(i) *Let $1 \le q < \infty$. Then (86) holds for all $g \in \mathscr{M}_+$ if and only if*

$$A_{(87)} := \sup_{t\in(0,\infty)}\left(\int_0^t w(x)\sup_{z\in[x,t]} u^q(z)\,\mathrm{d}x\right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{s\in[t,\infty)}\frac{1}{v(s)} < \infty. \qquad (87)$$

Moreover, the least constant $C_{(86)}$ such that (86) holds for all $g \in \mathscr{M}_+$ satisfies $C_{(86)} \approx A_{(87)}$.

(ii) *Let $0 < q < 1$. Then* (86) *holds for all $g \in \mathscr{M}_+$ if and only if*

$$A_{(88)} := \left(\int_0^\infty W^{\frac{q}{1-q}}(t) w(t) \sup_{z\in[t,\infty)} u^{\frac{q}{1-q}}(z) \left(\operatorname*{ess\,sup}_{s\in[z,\infty)} \frac{1}{v(s)} \right)^{\frac{q}{1-q}} \mathrm{d}t \right)^{\frac{1-q}{q}} < \infty \tag{88}$$

and

$$A_{(89)} := \left(\int_0^\infty \left(\int_0^t w(x) \sup_{y\in[x,t]} u^q(y)\,\mathrm{d}x \right)^{\frac{q}{1-q}} w(t) \sup_{z\in[t,\infty)} u^q(z) \left(\operatorname*{ess\,sup}_{s\in[z,\infty)} \frac{1}{v(s)} \right)^{\frac{q}{1-q}} \mathrm{d}t \right)^{\frac{1-q}{q}} < \infty. \tag{89}$$

Moreover, the least constant $C_{(86)}$ such that (86) *holds for all $g \in \mathscr{M}_+$ satisfies $C_{(86)} \approx A_{(88)} + A_{(89)}$.*

As it was forecast in the introduction, the results which are now at our disposal, namely those of Theorem 7, allow us to find the missing integral condition characterizing boundedness of the supremal operator R_u acting on $\mathscr{M}_+^\downarrow$. Case (i) in the theorem below was proved in [6, Theorem 3.2(i)] and is listed here for the sake of completeness. Case (ii) is the new result containing the integral condition for $0 < q < p < \infty$. The proof in fact covers both cases.

Theorem 8 *Let v, w be weights and let u be a continuous weight.*

(i) *Let $0 < p \le q < \infty$. Then the inequality*

$$\left(\int_0^\infty \left[\sup_{s\in[t,\infty)} u(s) f(s) \right]^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \le C_{(90)} \left(\int_0^\infty f^p(t) v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} \tag{90}$$

holds for all $f \in \mathscr{M}_+^\downarrow$ if and only if

$$A_{(91)} := \sup_{t\in(0^\infty)} \left(\int_0^t w(x) \sup_{y\in[x,t)} u^q(y)\,\mathrm{d}x \right)^{\frac{1}{q}} V^{-\frac{1}{p}}(t) < \infty. \tag{91}$$

Moreover, the least constant $C_{(90)}$ such that (90) *holds for all $f \in \mathscr{M}_+^\downarrow$ satisfies*

$$C_{(90)} \approx A_{(91)}.$$

(ii) *Let $0 < q < p < \infty$ and $r = \frac{pq}{p-q}$. Then* (90) *holds for all $f \in \mathscr{M}_+^\downarrow$ if and only if*

$$A_{(92)} := \left(\int_0^\infty W^{\frac{r}{p}}(t) w(t) \sup_{z\in[t,\infty)} u^r(z) \left(\int_0^z v(s)\,\mathrm{d}s \right)^{-\frac{r}{p}} \mathrm{d}t \right)^{\frac{1}{r}} < \infty \tag{92}$$

and

$$A_{(93)} := \left(\int_0^\infty \left(\int_0^t w(x) \sup_{y\in[x,t]} u^q(y)\,\mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{z\in[t,\infty)} u^q(z) \left(\int_0^z v(s)\,\mathrm{d}s \right)^{-\frac{r}{p}} \mathrm{d}t \right)^{\frac{1}{r}} < \infty. \tag{93}$$

Moreover, the least constant $C_{(90)}$ such that (90) *holds for all $f \in \mathscr{M}_+^\downarrow$ satisfies*

$$C_{(90)} \approx A_{(92)} + A_{(93)}.$$

Proof Since $p > 0$, the function $f \in \mathscr{M}$ is nonincreasing if and only if the function $g := f^{\frac{1}{p}}$ is nonincreasing. Hence, (90) holds for all $f \in \mathscr{M}_+^{\downarrow}$ if and only if

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u(s) g^{\frac{1}{p}}(s)\Big]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_{(90)} \left(\int_0^\infty g(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

holds for all $f \in \mathscr{M}_+^{\downarrow}$. By a standard argument (see e.g. [14, Lemma 1.2]), this is equivalent to the inequality

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u(s)\left(\int_s^\infty h(x)\,\mathrm{d}x\right)^{\frac{1}{p}}\Big]^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_{(90)} \left(\int_0^\infty \int_t^\infty h(x)\,\mathrm{d}x\; v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

being satisfied for all $h \in \mathscr{M}_+$. By taking the p-th power and applying Fubini theorem, this is true if and only if

$$\left(\int_0^\infty \Big[\sup_{s\in[t,\infty)} u^p(s)\int_s^\infty h(x)\,\mathrm{d}x\Big]^{\frac{q}{p}} w(t)\,\mathrm{d}t\right)^{\frac{p}{q}} \le C_{(90)}^p \int_0^\infty h(t)V(t)\,\mathrm{d}t$$

holds for all $h \in \mathscr{M}_+$. The result now follows from Theorem 7.

4 Comparison of the conditions

The paper [5] lists a variety of reduction theorems for weighted inequalities. These results, in general, allow for an equivalent reformulation of a weighted inequality in form of another weighted inequality, often on a different cone of functions. A particular case [5, Corollary 3.5] then offers an equivalent representation of the inequality (1), involving the operator S_u, by an analogous inequality with the operator $T_{\tilde{u}}$ (and with different weights). Hence, by using [5, Corollary 3.5], [6, Theorems 4.1 and 4.4] and after a careful recalculating of exponents, one can show that validity of (5) for all $g \in \mathscr{M}_+$ is characterized by the following conditions.

(i) If $1 < p \le q < \infty$, then (5) holds for all $g \in \mathscr{M}_+$ if and only if

$$\begin{aligned} A_{(94)} := &\sup_{t\in(0,\infty)} u(t)W^{\frac{1}{q}}(t)\left(\int_t^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} \\ &+ \sup_{t\in(0,\infty)}\left(\int_t^\infty w(x)\sup_{y\in[x,\infty)} u^q(y)\left(\int_y^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{2q}{p'+1}}\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_t^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{-1}{p(p'+1)}} < \infty. \end{aligned} \tag{94}$$

(ii) If $1 < p < \infty$ and $0 < q < p$, then (5) holds for all $g \in \mathscr{M}_+$ if and only if

$$\begin{aligned} A_{(95)} := &\left(\int_0^\infty W^{\frac{r}{p}}(t)w(t)\sup_{y\in[t,\infty)} u^r(y)\left(\int_y^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\mathrm{d}x\right)^{\frac{1}{r}} \\ &+\left(\int_0^\infty\left(\int_t^\infty w(x)\sup_{y\in[x,\infty)} u^q(y)\left(\int_y^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{2q}{p'+1}}\mathrm{d}x\right)^{\frac{r}{q}}\left(\int_t^\infty v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{p'}-\frac{2r}{p'+1}-1} v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{r}} < \infty. \end{aligned} \tag{95}$$

Observe that these conditions are different from those presented in Theorem 6. In case (i), it is easily shown that the first term in $A_{(94)}$ is dominated by $A_{(6)}$. In (ii), the first half of $A_{(95)}$ is in fact $A_{(7)}$, but the second term in $A_{(95)}$ is different from the condition $A_{(8)}$. Notice, in particular, the "flipped" interval of integration in the term involving w in the second part of the condition $A_{(95)}$ (and the same in $A_{(94)}$). This difference can be traced back to the "flip" from S_u to $T_{\tilde{u}}$ in the reduction technique of [5].

It can be said that the conditions $A_{(6)}, A_{(7)}$ and $A_{(8)}$ belong to one "class" (that may be called "classical conditions"), and $A_{(94)}$, $A_{(95)}$ belong to another one ("flipped conditions"). Existence of such equivalent classes of conditions is a rather common phenomenon, see e.g. [4, 7, 8].

The "classical" conditions are simpler than their "flipped" counterparts and, moreover, are compatible to older results, as these mostly have the "classical" form as well. Such matching issues are important in situations when combining of conditions is needed. That is often the case in problems concerning the iterated inequalities and more complicated function spaces based on them.

References

1. C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, 129. Academic Press, Boston, 1988.
2. A. Cianchi, R. Kerman, B. Opic, and L. Pick, *A sharp rearrangement inequality for fractional maximal operator*, Studia Math. **138** (2000), 277–284.
3. A. Gogatishvili, M. Křepela, L. Pick and F. Soudský, *Embeddings of classical Lorentz spaces involving weighted integral means*, preprint.
4. A. Gogatishvili, A. Kufner and L.-E. Persson, *Some new scales of weight characterizations of the class B_p*, Acta Math. Hungar. **123** (2009), 365–377.
5. A. Gogatishvili and R. Mustafayev, *Weighted iterated Hardy-type inequalities*, preprint.
6. A. Gogatishvili, B. Opic and L. Pick, *Weighted inequalities for Hardy-type operators involving suprema*, Collect. Math. **57** (2006), 227–255.
7. A. Gogatishvili, L.-E. Persson, V. D. Stepanov and P. Wall, *On scales of equivalent conditions characterizing weighted Stieltjes inequality*, Doklady Math. **86(3)** (2012), 738–739.
8. A. Gogatishvili, L.-E. Persson, V. D. Stepanov and P. Wall, *Some scales of equivalent conditions to characterize the Stieltjes inequality: the case $q < p$*, Math. Nachr. **287** (2014), 242–253.
9. A. Gogatishvili and L. Pick, *Duality principles and reduction theorems*, Math. Inequal. Appl. **43** (2000), 539–558.
10. A. Gogatishvili and L. Pick, *A reduction theorem for supremum operators*, J. Comput. Appl. Math. **208** (2007), 270–279.
11. A. Gogatishvili and V.D. Stepanov, *Reduction theorems for operators on the cones of monotone functions*, J. Math. Anal. Appl. **405** (2013), 156–172.
12. M.L. Goldman, H.P. Heinig and V.D. Stepanov, *On the principle of duality in Lorentz spaces*, Canad. J. Math. **48(5)** (1996), 959–979.
13. K.-G. Grosse-Erdmann, *The blocking technique, weighted mean operators and Hardy's inequality*, Lecture Notes in Mathematics, 1679. Springer-Verlag, Berlin, 1998.
14. G. Sinnamon, *Transferring monotonicity in weighted norm inequalities*, Collect. Math. **54** (2003), 181–216.