ROUGH BILINEAR SINGULAR INTEGRALS

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ABSTRACT. We study the rough bilinear singular integral, introduced by Coifman and Meyer [7],

$$T_\Omega(f,g)(x) = \text{p.v.}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |(y,z)|^{-2n}\Omega((y,z)/|(y,z)|)f(x-y)g(x-z)dydz,$$

when Ω is a function in $L^q(\mathbb{S}^{2n-1})$ with vanishing integral and $2 \le q \le \infty$. When $q = \infty$ we obtain boundedness for T_Ω from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1 < p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. For $q = 2$ we obtain that T_Ω is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. For q between 2 and infinity we obtain the analogous boundedness on a set of indices around the point $(1/2, 1/2, 1)$. To obtain our results we introduce a new bilinear technique based on tensor-type wavelet decompositions.

CONTENTS

1. INTRODUCTION

Singular integral theory was initiated in the seminal work of Calderón and Zygmund [2]. The study of boundedness of rough singular integrals of convolution type has been an active area of research since the middle of the twentieth century. Calderón and Zygmund [3] first studied the rough

The first author was supported by the Simons Foundation. The third author was supported by the ERC CZ grant LL1203 of the Czech Ministry of Education.

2010 Mathematics Classification Number 42B20, 42B99.

singular integral

$$L_\Omega(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy$$

where Ω is in $L\log L(\mathbb{S}^{n-1})$ with mean value zero and showed that L_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The same conclusion under the less restrictive condition that Ω lies in $H^1(\mathbb{S}^{n-1})$ was obtained by Coifman and Weiss [8] and Connett [9]. The weak type $(1,1)$ boundedness of L_Ω when $n = 2$ was established by Christ and Rubio de Francia [5] and independently by Hofmann [17]. (In unpublished work, Christ and Rubio de Francia extended this result to all dimensions $n \le 7$.) The weak type $(1,1)$ property of L_Ω was proved by Seeger [25] in all dimensions and was later extended by Tao [27] to situations in which there is no Fourier transform structure. Several questions remain concerning the endpoint behavior of L_Ω, such as if the condition $\Omega \in L\log L(\mathbb{S}^{n-1})$ can be relaxed to $\Omega \in H^1(\mathbb{S}^{n-1})$, or merely $\Omega \in L^1(\mathbb{S}^{n-1})$ when Ω is an odd function. On the former there is a partial result of Stefanov [26] but not much is still known about the latter.

The bilinear counterpart of the rough singular integral linear theory is notably more intricate. To fix notation, we fix $1 < q \le \infty$ and we let Ω in $L^q(\mathbb{S}^{2n-1})$ with $\int_{\mathbb{S}^{2n-1}} \Omega\, d\sigma = 0$, where $\mathbb{S}^{2n-1}$ is the unit sphere in $\mathbb{R}^{2n}$. Coifman and Meyer [7] introduced the bilinear singular integral operator associated with Ω by

$$T_\Omega(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x-y, x-z) f(y) g(z)\, dy dz, \tag{1}$$

where f, g are functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$,

$$K(y,z) = \Omega((y,z)')/|(y,z)|^{2n},$$

and $x' = x/|x|$ for $x \in \mathbb{R}^{2n}$. General facts about bilinear operators can be found in [23, Chapter 13], [14, Chapter 7], and [24]. If Ω possesses some smoothness, i.e. if is a function of bounded variation on the circle, Coifman and Meyer [7, Theorem I] showed that T_Ω is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$. In higher dimensions, it was shown Grafakos and Torres [16], via a bilinear $T1$ condition, that if Ω a Lipschitz function on $\mathbb{S}^{2n-1}$, then T_Ω is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1 < p_1, p_2 < \infty$, $1/2 < p < \infty$, and $1/p = 1/p_1 + 1/p_2$. But if Ω is rough, the situation is significantly more complicated, and the boundedness of T_Ω remained unresolved until this work, except when in situations when it reduces to the uniform boundedness of bilinear Hilbert transforms. If Ω is merely integrable function on $\mathbb{S}^1$, but is odd, the operator T_Ω is intimately connected with the celebrated (directional) bilinear Hilbert transform

$$\mathcal{H}_{\theta_1,\theta_2}(f_1,f_2)(x) = \int_{-\infty}^{+\infty} f_1(x-t\theta_1)f_2(x-t\theta_2)\,\frac{dt}{t}$$

(in the direction (θ_1,θ_2)), via the relationship

$$T_\Omega(f_1,f_2)(x) = \frac{1}{2}\int_{\mathbb{S}^{2n-1}} \Omega(\theta_1,\theta_2)\mathcal{H}_{\theta_1,\theta_2}(f_1,f_2)(x)\,d(\theta_1,\theta_2)\,.$$

The boundedness of $\mathcal{H}_{\theta_1,\theta_2}$ was proved by Lacey and Thiele [19], [20] while the more relevant, for this problem, uniform in θ_1,θ_2 boundedness of $\mathcal{H}_{\theta_1,\theta_2}$ was addressed by Thiele [28], Grafakos and Li [15], and Li [21]. Exploiting the uniform boundedness of $\mathcal{H}_{\theta_1,\theta_2}$, Diestel, Grafakos, Honzík, Si, and Terwilleger [11] showed that if $n=2$ and the even part of Ω lies in $H^1(\mathbb{S}^1)$, then T_Ω is bounded from $L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1<p_1,p_2,p<\infty$, $1/p=1/p_1+1/p_2$, and the triple $(1/p_1,1/p_2,1/p)$ lies in the open hexagon described by the conditions:

$$\Big|\frac{1}{p_1}-\frac{1}{p_2}\Big|<\frac{1}{2}\,,\qquad \Big|\frac{1}{p_1}-\frac{1}{p'}\Big|<\frac{1}{2}\,,\qquad \Big|\frac{1}{p_2}-\frac{1}{p'}\Big|<\frac{1}{2}\,.$$

This is exactly the region in which the uniform boundedness of the bilinear Hilbert transforms is currently known. It is noteworthy to point out the T_Ω reduces itself to a bilinear Hilbert transform $\mathcal{H}_{\theta_1,\theta_2}$, if Ω is the sum of the pointmasses $\delta_{(\theta_1,\theta_2)}+\delta_{-(\theta_1,\theta_2)}$ on $\mathbb{S}^1$.

In this work we provide a proof of the boundedness of T_Ω on L^p for all indices with $p>1/2$ and for all dimensions. This breakthrough is a consequence of the novel technical ingredients we employ in this context. We build on the work of Duoandikoetxea and Rubio de Francia [13] but our key idea is to decompose the multiplier in terms of a tensor-type compactly-supported wavelet decomposition and to use combinatorial arguments to group the different pieces together, exploiting orthogonality.

The main result of this paper is the following theorem.

Theorem 1. *For all $n\ge 1$, if $\Omega\in L^\infty(\mathbb{S}^{2n-1})$, then for T_Ω defined in* (1)*, we have*

$$\|T_\Omega\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}<\infty \tag{2}$$

whenever $1<p_1,\,p_2<\infty$ and $1/p=1/p_1+1/p_2$.

In the remaining sections we focus on the proof of this result while in the last section we focus on extensions to the case where Ω lies in $L^q(\mathbb{S}^{2n-1})$ for $q<\infty$.

Some remarks about our notation in this paper: For $1<q<\infty$ we set $q'=q/(q-1)$ and for $q=\infty$, we set $\infty'=1$. We denote the the norm of a bounded bilinear operator T from $X\times Y$ to Z by

$$\|T\|_{X\times Y\to Z}=\sup_{\|f\|_X\le 1}\sup_{\|g\|_Y\le 1}\|T(f,g)\|_Z\,.$$

This notation was already used in (2). If x_1,x_2 are in $\mathbb{R}^n$, then we denote the point (x_1,x_2) in $\mathbb{R}^{2n}$ by $\vec{x}$. We denote the set of positive integers by $\mathbb{N}$ and we set $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. In the sequel, multiindices in $\mathbb{Z}^{2n}$ are elements of $\mathbb{N}_0^{2n}$. Finally, we adhere to the standard convention to denote by C a constant that depends only on inessential parameters of the problem.

2. Estimates of Fourier Transforms of the Kernels

Let us fix a q satisfying $1<q\le\infty$ and a function $\Omega\in L^q(\mathbb{S}^{n-1})$ with mean value zero. We fix a smooth function α in $\mathbb{R}^+$ such that $\alpha(t)=1$ for $t\in(0,1]$, $0<\alpha(t)<1$ for $t\in(1,2)$ and $\alpha(t)=0$ for $t\ge 2$. For $(y,z)\in\mathbb{R}^{2n}$ and $j\in\mathbb{Z}$ we introduce the function

$$\beta_j(y,z)=\alpha(2^{-j}|(y,z)|)-\alpha(2^{-j+1}|(y,z)|).$$

We write $\beta=\beta_0$ and we note that this is a function supported in $[1/2,2]$. We denote Δ_j the Littlewood-Paley operator $\Delta_j f=\mathcal{F}^{-1}(\beta_j\widehat{f})$. Here and throughout this paper $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, which is defined via $\mathcal{F}^{-1}(g)(x)=\int_{\mathbb{R}^n}g(\xi)e^{2\pi ix\cdot\xi}d\xi=\widehat{g}(-x)$, where $\widehat{g}$ is the Fourier transform of g. We decompose the kernel K as follows: we denote $K^i=\beta_iK$ and we set $K^i_j=\Delta_{j-i}K^i$ for $i,j\in\mathbb{Z}$. Then we write

$$K=\sum_{j=-\infty}^{\infty}K_j,$$

where

$$K_j=\sum_{i=-\infty}^{\infty}K^i_j.$$

We also denote $m_j=\widehat{K_j}$.

Then the operator can be written as

$$T_\Omega(f,g)(x)=\sum_j\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K_j(x-y,x-z)f(y)g(z)\,dydz=:\sum_j T_j(f,g)(x).$$

We have the following lemma whose proof is known (see for instance [12]) and is omitted.

Lemma 2. *Given $\Omega\in L^q(\mathbb{S}^{2n-1})$, $0<\delta<1/q'$ and $\vec{\xi}=(\xi_1,\xi_2)\in\mathbb{R}^{2n}$ we have we have*

$$|\widehat{K^0}(\vec{\xi}\,)|\le C\|\Omega\|_{L^q}\min(|\vec{\xi}\,|,|\vec{\xi}\,|^{-\delta})$$

and for all multiindices α in $\mathbb{Z}^{2n}$ with $\alpha \neq 0$ we have

$$|\partial^{\alpha}\widehat{K^0}(\vec{\xi}\,)| \leq C_{\alpha}\|\Omega\|_{L^q}\min(1,|\vec{\xi}\,|^{-\delta})\,.$$

The following proposition is a consequence of the preceding lemma.

Proposition 3. *Let $1 \leq p_1, p_2 < \infty$ and define p via $1/p = 1/p_1 + 1/p_2$. Let $\Omega \in L^q(\mathbb{S}^{2n-1})$, $1 < q \leq \infty$, $0 < \delta < 1/q'$, and for $j \in \mathbb{Z}$ consider the bilinear operator*

$$T_j(f,g)(x) = \int_{\mathbb{R}^{2n}} K_j(x-y, x-z) f(y) g(z)\, dy dz\,.$$

If both $p_1, p_2 > 1$, then T_j is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with norm at most $C\,\|\Omega\|_{L^q} 2^{(2n-\delta)j}$ if $j \geq 0$ and at most $C\,\|\Omega\|_{L^q} 2^{-|j|(1-\delta)}$ if $j < 0$. If at least one of p_1 and p_2 is equal to 1, then T_j maps $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ with a similar norm.

Proof. We prove the assertion by showing that the multiplier $\sigma_j = \widehat{K}_j$ associated with T_j satisfies the conditions of the Coifman-Meyer multiplier theorem [6], which was extended to the case $p < 1$ by Kenig and Stein [18] and by Grafakos and Torres [16]. To be able to use this theorem, we need to show that σ_j is a C^{∞} function on $\mathbb{R}^{2n} \setminus \{0\})$ that satisfies

$$|\partial^{\alpha}\sigma_j(\vec{\xi}\,)| \leq C\,Q(j)\|\Omega\|_{L^q}|\vec{\xi}\,|^{-|\alpha|}$$

for all multiindices α in $\mathbb{Z}^{2n}$ with $|\alpha| \leq 2n$ and all $\vec{\xi} \in \mathbb{R}^{2n} \setminus \{0\}$, where $Q(j) = 2^{(2n-\delta)j}$ if $j \geq 0$ and $Q(j) = 2^{-|j|(1-\delta)}$ if $j < 0$. Then we may use Theorem 7.5.3 in [14] to deduce the claimed boundedness. It is not hard to verify that

$$\sigma_j(\vec{\xi}\,) = \sum_{i=-\infty}^{\infty} \beta(2^{i-j}|\vec{\xi}\,|)\widehat{K^0}(2^i\vec{\xi}\,) \tag{3}$$

If $|\vec{\xi}\,| \approx 2^l$, then since β is supported in $[1/2, 2]$, 2^i must be comparable to 2^{j-l} in (3). Using Lemma 2 we have the estimate

$$|\sigma_j(\vec{\xi}\,)| \leq \sum_{i\in F} |\widehat{K^0}(2^i\vec{\xi}\,)| \leq C\|\Omega\|_{L^q} \sum_{i\in F} \min\left\{2^i|\vec{\xi}|, (2^i|\vec{\xi}|)^{-\delta}\right\} \leq C\|\Omega\|_{L^q} I(j),$$

where F is a finite set of i's near $j-l$ and $I(j) = 2^{-|j|\delta}$ if $j \geq 0$ whereas $I(j) = 2^{-|j|}$ if $j < 0$. For an αth derivative of σ_j with $1 \leq |\alpha| \leq 2n$, using that $|\partial^{\alpha}\widehat{K^0}(\vec{\xi}\,)| \leq C_{\alpha}\,\|\Omega\|_{L^q}|\vec{\xi}|^{-\delta}$, we obtain

$$\begin{aligned}
\sum_{i\in F} |\partial^{\alpha}(\widehat{K^0}(2^i\vec{\xi}\,)\Phi(2^i\vec{\xi}\,))| &\leq \|\Omega\|_{L^q} \sum_{i\in F} C_{\alpha} 2^{i|\alpha|}(2^i|\vec{\xi}|)^{-\delta} \\
&\leq C\|\Omega\|_{L^q} 2^{j(|\alpha|-\delta)}|\vec{\xi}|^{-|\alpha|}
\end{aligned}$$

and this is at most $C\|\Omega\|_{L^q} 2^{(2n-\delta)j}$ if $j \geq 0$ and at most $C\|\Omega\|_{L^q} 2^{-|j|(1-\delta)}$ if $j < 0$, since $\delta \in (0, 1/q')$. $\square$

The operators T_j associated with the multipliers $\widehat{K_j}$ are bounded with bounds that grow in j since the smoothness of the symbol is getting worse with j. We certainly have that

$$\|\widehat{K}_j\|_{L^\infty} \leq C2^{-|j|\delta},$$

but there is no good estimate available for the derivatives of $\widehat{K}_j$, and moreover, a good L^∞ estimate for the multiplier does not suffice to yield boundedness in the bilinear setting. The key argument of this article is to circumvent this obstacle and prove that the norms of the operators T_j indeed decay exponentially. Our proof is new in this context and is based on a suitable wavelet expansion combined with combinatorial arguments.

3. Boundedness: a good point

In this section we prove the following result which is a special case of Theorem 1:

Theorem 4. *Suppose $\Omega \in L^q(\mathbb{S}^{2n-1})$ with $2 \leq q \leq \infty$, then for f, g in $L^2(\mathbb{R}^n)$ we have*

$$\|T_\Omega(f,g)\|_{L^1(\mathbb{R}^n)} \leq C\|\Omega\|_{L^q}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$

In view of Proposition 3, Theorem 4 will be a consequence of the following proposition.

Proposition 5. *Given $2 \leq q \leq \infty$ and $0 < \delta < 1/(8q')$, then for any $j \geq 0$, the operator T_j associated with the kernel K_j maps $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with norm at most $C\|\Omega\|_{L^q} 2^{-\delta j}$.*

To obtain the proof of the proposition, we utilize wavelets with compact support. Their existence is due to Daubechies [10] and can also be found in Meyer's book [22]. For our purposes we need product type smooth wavelets with compact supports; the construction of such objects can be found in Triebel [29].

Lemma 6. *For any $k \in \mathbb{N}$ there are real compactly supported functions $\psi_F, \psi_M \in C^k(\mathbb{R})$ such that, if ψ^G is defined by*

$$\Psi^G(\vec{x}) = \psi_{G_1}(x_1) \cdots \psi_{G_{2n}}(x_{2n})$$

for $G = (G_1, \dots, G_{2n})$ in the set

$$\mathcal{I} := \Big\{(G_1, \dots, G_{2n}) : \; G_i \in \{F, M\}\Big\},$$

then the family of functions

$$\bigcup_{\vec{\mu}\in\mathbb{Z}^{2n}}\left[\left\{\Psi^{(F,\dots,F)}(\vec{x}-\vec{\mu})\right\}\cup\bigcup_{\lambda=0}^{\infty}\left\{2^{\lambda n}\Psi^{G}(2^{\lambda}\vec{x}-\vec{\mu}):\, G\in\mathcal{I}\setminus\{(F,\dots,F)\}\right\}\right]$$

forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$*, where* $\vec{x}=(x_1,\dots,x_{2n})$*.*

Proof of Proposition 5. To obtain the estimate, we first decompose the symbol into dyadic pieces, estimate them separately, and then use orthogonality arguments to put them back together. Let us take a look at the the symbol $\widehat{K_j^0}$ which we denote $m_{j,0}$. The classical estimates show that

$$\|m_{j,0}\|_{L^\infty}=\|\widehat{K_j^0}\|_{L^\infty}\le C\|\Omega\|_{L^q}2^{-\delta j},\qquad 0<\delta<1/q', \tag{4}$$

while for $2\le q\le\infty$

$$\|m_{j,0}\|_{L^2}=\|\beta_j(\widehat{\beta_0 K})\|_{L^2}\le C\|\widehat{\beta_0 K}\|_{L^2}\le C\|\Omega\|_{L^2}\le C\|\Omega\|_{L^q}. \tag{5}$$

We observe that for the case $i\neq 0$ we have the identity $m_{j,i}=\widehat{K}_j^i=m_{j,0}(2^i\cdot)$ from the homogeneity of the symbol, and thus $m_{j,i}$ also lies in L^2.

We utilize a wavelet transform of $m_{j,0}$. We take the product wavelets described above, with compact supports and M vanishing moments, where M is a large number to be determined later. Here we choose generating functions with support diameter approximately 1. The wavelets with the same dilation factor 2^λ have some bounded overlap N independent of λ. Since the inverse Fourier transform of $m_{j,0}$ is essentially supported in the dyadic annulus of radius 1, the symbol is smooth and the wavelet transform has a nice decay. Precisely with

$$\Psi^{\lambda,G}_{\vec{\mu}}(\vec{x})=2^{\lambda n}\Psi^G(2^\lambda\vec{x}-\vec{\mu}),\qquad \vec{x}\in\mathbb{R}^{2n},$$

we have the following result:

Lemma 7. *Using the preceding notation, for any* $j\in\mathbb{Z}$ *and* $\lambda\in\mathbb{N}_0$ *we have*

$$|\langle\Psi^{\lambda,G}_{\vec{\mu}},m_{j,0}\rangle|\le C\|\Omega\|_{L^q}2^{-\delta j}2^{-(M+1+n)\lambda}, \tag{6}$$

where M *is the number of vanishing moments of* ψ_M *and* δ *is as in* (4)*.*

Proof. Let $\lambda\ge 0$ and $G\in\mathcal{I}\setminus\{(F,\dots,F)\}$. We apply the smoothness-cancellation estimate in Appendix B.2 of [14] with Ψ being the function $\Psi^{\lambda,G}_{\vec{\mu}}$, $L=M+1$, and Φ being the function $m_{j,0}$. Then we have the properties

(i) $\int_{\mathbb{R}^{2n}}\Psi^{\lambda,G}_{\vec{\mu}}(\vec{x})\vec{x}^{\beta}\,d\vec{x}=0$ for $|\beta|\le L-1$,

(ii) $|\Psi^{\lambda,G}_{\vec{\mu}}(\vec{x})|\le\frac{C2^{\lambda n}}{(1+2^\lambda|\vec{x}-2^{-\lambda}\vec{\mu}|)^{M_1}}$,

(iii) For $|\alpha| = L$, $|\partial^\alpha(m_{j,0})(\vec{x})| \le \frac{C\|\Omega\|_{L^q}2^{-j\delta}}{(1+2^{-j}|\vec{x}|)^{M_2}}$. To verify this property we notice that since β_0 is a Schwartz function, we have

$$\begin{aligned}|\partial^\alpha(\beta_j\widehat{K^0})(\vec{x})| \le & \sum_{\gamma\le\alpha} 2^{-j|\gamma|}|\partial^\gamma\beta_0(2^{-j}\vec{x})\partial^{\alpha-\gamma}\widehat{K^0}(\vec{x})| \\ & \le C\|\Omega\|_{L^q}\sum_{\gamma\le\alpha} 2^{-j|\gamma|}\frac{2^{-j\delta}}{(1+2^{-j}|\vec{x}|)^{M_2}} \\ & \le C\|\Omega\|_{L^q}\frac{2^{-j\delta}}{(1+2^{-j}|\vec{x}|)^{M_2}},\end{aligned}$$

where we used Lemma 2, i.e. the property that $|\partial^\alpha\widehat{K^0}(\vec{x})| \le C\|\Omega\|_{L^q}|\vec{x}|^{-\delta}$ for all multiindices α.

Thus $\Psi^{\lambda,G}_{\vec{\mu}}$ has cancellation and $m_{j,0}$ has appropriate smoothness and so it follows that

$$|<\Psi^{\lambda,G}_{\vec{\mu}}, m_{j,0}>| \le C\|\Omega\|_{L^q}\frac{2^{-j\delta}2^{\lambda n}2^{-\lambda(L+2n)}}{(1+2^{-j-\lambda}|\vec{\mu}|)^{M_2}} \le C\|\Omega\|_{L^q}2^{-j\delta}2^{-\lambda(M+1+n)},$$

thus (6) holds. Notice that the constant C is independent of $\vec{\mu}$.

Next we consider the case $\lambda = 0$ and $G = (F,\dots,F)$. In this case we have $|\Psi^{\lambda,G}_{\vec{\mu}}(\vec{x})| \le \frac{C}{(1+|\vec{x}-\vec{\mu}|)^{M_1}}$ and $|m_{j,0}(\vec{x})| \le \frac{C\|\Omega\|_{L^q}2^{-j\delta}}{(1+2^{-j}|\vec{x}|)^{M_2}}$. Using the result in Appendix B1 in [14] we deduce that

$$|<\Psi^{\lambda,G}_{\vec{\mu}}, m_{j,0}>| \le C\|\Omega\|_{L^q}\frac{2^{-j\delta}}{(1+2^{-j}|\vec{\mu}|)^{M_2}} \le C\|\Omega\|_{L^q}2^{-j\delta}$$

and thus (6) follows in this case as well. □

The wavelets sharing the same generation index may be organized into $C_{n,M,N}$ groups so that members of the same group have disjoint supports and are of the same product type, i.e., they have the same index $G \in \mathcal{I}$. For $1 \le \kappa \le C_{n,M,N}$ we denote by $D_{\lambda,\kappa}$ one of these groups consisting of wavelets whose supports have diameters about $2^{-\lambda}$. We now have that the wavelet expansion

$$m_{j,0} = \sum_{\substack{\lambda\ge 0\\ 1\le\kappa\le C_{n,M,N}}}\sum_{\omega\in D_{\lambda,\kappa}} a_\omega\omega$$

and ω all have disjoint supports within the group $D_{\lambda,\kappa}$. For the sequence $a = \{a_\omega\}$ we get $\|a\|_{\ell^2} \le C$, in view of (5), because $\{\omega\}$ is an orthonormal basis. Since the ω are continuous functions and and bounded by $2^{\lambda n}$, if we set $b_\omega = \|a_\omega \omega\|_{L^\infty}$, we have

$$\|\{b_\omega\}_{\omega\in D_{\lambda,\kappa}}\|_{\ell^2} \le 2^{\lambda n}\Big(\sum_{\omega\in D_{\lambda,\kappa}} |a_\omega|^2\Big)^{1/2} \le C\|\Omega\|_{L^2} 2^{n\lambda}.$$

Clearly we also have

$$\|\{b_\omega\}_{\omega\in D_{\lambda,\kappa}}\|_{\ell^\infty} \le \|\{a_\omega\}_{\omega\in D_{\lambda,\kappa}}\|_{\ell^\infty} 2^{n\lambda} \le C\|\Omega\|_{L^q} 2^{-\delta j-(M+1)\lambda}. \tag{7}$$

Now, we split the group $D_{\lambda,\kappa}$ into three parts. Recall the fixed integer j in the statement of Proposition 5. We define sets

$$D^1_{\lambda,\kappa} = \Big\{\omega\in D_{\lambda,\kappa} : a_\omega \neq 0,\ \operatorname{supp}\omega \subset \{(\xi_1,\xi_2):\ 2^{-j}|\xi_1| \le |\xi_2| \le 2^j|\xi_1|\}\Big\},$$

$$D^2_{\lambda,\kappa} = \Big\{\omega\in D_{\lambda,\kappa} : a_\omega \neq 0,\ \operatorname{supp}\omega \cap \{(\xi_1,\xi_2):\ 2^{-j}|\xi_1| \ge |\xi_2|\} \neq \emptyset\Big\},$$

and

$$D^3_{\lambda,\kappa} = \Big\{\omega\in D_{\lambda,\kappa} : a_\omega \neq 0,\ \operatorname{supp}\omega \cap \{(\xi_1,\xi_2):\ 2^{-j}|\xi_2| \ge |\xi_1|\} \neq \emptyset\Big\}.$$

These groups are disjoint for large j. Notice that $D^1_{\lambda,\kappa}\cap D^2_{\lambda,\kappa} = \emptyset$ is obvious. For $D^2_{\lambda,\kappa}$ and $D^3_{\lambda,\kappa}$ the worst case is $\lambda = 0$ when we have balls of radius 1 centered at integers, and $D^2_{\lambda,\kappa}\cap D^3_{\lambda,\kappa} = \emptyset$ if j is sufficiently large, for instance $j \ge 100\sqrt{n}$ works, since if $a_\omega \neq 0$, then ω is supported in an annulus centered at the origin of size about 2^j. We are assuming here that $j \ge 100\sqrt{n}$ but notice that for $j < 100\sqrt{n}$, Proposition 5 is an easy consequence of Proposition 3.

We denote, for $\iota = 1,2,3$,

$$m^\iota_{j,0} = \sum_{\lambda,\kappa}\sum_{\omega\in D^\iota_{\lambda,\kappa}} a_\omega \omega,$$

and define

$$m^\iota_j = \sum_{k=-\infty}^{\infty} m^\iota_{j,k}$$

with $m^\iota_{j,k}(\vec\xi\,) = m^\iota_{j,0}(2^k\vec\xi\,)$. We prove boundedness for each piece m^1_j, m^2_j, m^3_j. We call m^1_j the diagonal part of m_j and m^2_j, m^3_j the off-diagonal parts of $m_j = \widehat{K_j}$.

4. The diagonal part

We first deal with the first group $D^1_{\lambda,\kappa}$. Each $\omega\in D^1_{\lambda,\kappa}$ is of tensor product type $\omega = \omega_1\omega_2$, therefore, we may index the sequences by two indices $k,l\in\mathbb{Z}^n$ according to the first and second variables. Thus $\omega_{k,l} = \omega_{1,k}\omega_{2,l}$.

Likewise, we index the sequence $b = \{b_{(k,l)}\}_{k,l}$. Now for $r \geq 0$ we define sets

$$U_r = \{(k,l) \in \mathbb{Z}^{2n} : 2^{-r-1}\|b\|_{\ell^\infty} < |b_{(k,l)}| \leq 2^{-r}\|b\|_{\ell^\infty}\}.$$

From the ℓ^2 norm of b, we find that the cardinality of this set is at most $C\|\Omega\|_{L^2}^2 2^{2n\lambda}2^{2r}\|b\|_{\ell^\infty}^{-2}$. Indeed, we have

$$|U_r| \leq 4\sum_{(k,l)\in U_r} |b_{(k,l)}|^2(\|b\|_{\ell^\infty}2^{-r})^{-2} \leq 4\|b\|_{\ell^2}^2\|b\|_{\ell^\infty}^{-2}2^{2r} \leq C\frac{\|\Omega\|_{L^2}^2}{\|b\|_{\ell^\infty}^2}2^{2n\lambda}2^{2r}.$$

We split each $U_r = U_r^1 \cup U_r^2 \cup U_r^3$, where

$$U_r^1 = \{(k,l) \in U_r : \mathrm{card}\{s : (k,s) \in U_r\} \geq 2^{(r+\delta j+M\lambda)/4}\},$$
$$U_r^2 = \{(k,l) \in U_r \setminus U_r^1 : \mathrm{card}\{s : (s,l) \in U_r \setminus U_r^1\} \geq 2^{(r+\delta j+M\lambda)/4}\}.$$

and the third set is the remainder. These three sets are disjoint. We notice that if the index k satisfies $\mathrm{card}\{s : (k,s) \in U_r\} \geq 2^{(r+\delta j+M\lambda)/4}$, then the pair (k,l) lies in U_r^1 for all $l \in \mathbb{Z}^n$ such that $(k,l) \in U_r$.

We observe that in the first set U_r^1, we have

$$N_1 := \mathrm{card}\{k : \text{there is } l \text{ s.t. } (k,l) \in U_r^1\} \leq C\frac{\|\Omega\|_{L^2}^2}{\|b\|_{\ell^\infty}^2}2^{(2n-M/4)\lambda+\frac{7r}{4}-\frac{j\delta}{4}}, \tag{8}$$

since $N_1 2^{(r+\delta j+M\lambda)/4} \leq C\|\Omega\|_{L^2}^2 2^{2n\lambda}2^{2r}\|b\|_{\ell^\infty}^{-2}$. We now write

$$m_j^{r,1} = \sum_{(k,l)\in U_r^1} a_{(k,l)}\omega_{1,k}\omega_{2,l}$$

and estimate the norm of $m_j^{r,1}$ as a bilinear multiplier as follows:

$$\begin{aligned}\|T_{m_j^{r,1}}(f,g)\|_{L^1} &\leq \Big\|\sum_{(k,l)\in U_r^1} a_{(k,l)}\mathcal{F}^{-1}(\omega_{1,k}\widehat{f})\mathcal{F}^{-1}(\omega_{2,l}\widehat{g})\Big\|_{L^1}\\ &\leq \sum_{k\in E}\|\omega_{1,k}\widehat{f}\|_{L^2}\Big\|\sum_{(k,l)\in U_r^1} a_{(k,l)}\omega_{2,l}\widehat{g}\Big\|_{L^2}.\end{aligned}$$

For fixed k, by the choice of $D_{\lambda,\kappa}$, the supports of $\omega_{k,l} = \omega_{1,k}\omega_{2,l}$ are disjoint, in particular, the supports of $\omega_{2,l}$ are disjoint. Since $\|\omega_{1,k}\|_{L^\infty} \approx 2^{\lambda n/2}$, we have the estimate

$$\Big\|\sum_{(k,l)\in U_r^1} a_{(k,l)}\omega_{2,l}\Big\|_{L^\infty} \leq C\Big\|\sum_{(k,l)\in U_r^1}|b_{(k,l)}|2^{-\lambda n/2}\chi_{E_l}\Big\|_{L^\infty} \leq C\|b\|_{\ell^\infty}2^{-r}2^{-\lambda n/2},$$

where $E_l \subset \mathbb{R}^n$ is the support of $\omega_{2,l}$. As a result,

$$\Big\|\sum_{(k,l)\in U_r^1} a_{(k,l)}\omega_{2,l}\widehat{g}\Big\|_{L^2} \leq C\|b\|_{\ell^\infty}2^{-r}2^{-\lambda n/2}\|g\|_{L^2}.$$

Now let $E = \{k : \exists\, l \; s.t.\; (k,l) \in U_r^1\}$ and note that $|E| = N_1$.

Notice that the $\omega_{k,l}$ in U_r^1 have the following property. If $(k,l)\neq(k',l')$, then the supports of $\omega_{1,k}$ and $\omega_{1,k'}$ are disjoint. Since the $\omega_{1,k}$ satisfy $\|\omega_{1,k}\|_{L^\infty}\approx 2^{\lambda n/2}$ and have disjoint supports, we have

$$\begin{aligned}
&\big\|T_{m_j^{r,1}}(f,g)\big\|_{L^1}\\
&\le \sum_{k\in E}\big\|\omega_{1,k}\widehat{f}\big\|_{L^2}2^{-\lambda n/2}2^{-r}\|b\|_{\ell^\infty}\|g\|_{L^2}\\
&\le \Big(\sum_{k\in E}1\Big)^{1/2}\Big(\sum_{k\in E}\big\|\omega_{1,k}\widehat{f}\big\|_{L^2}^2\Big)^{1/2}2^{-\lambda n/2}2^{-r}\|b\|_{\ell^\infty}\|g\|_{L^2}\\
&\le C\Big(\|\Omega\|_{L^2}^2 2^{(2n-M/4)\lambda+7r/4-\delta j/4}\|b\|_{\ell^\infty}^{-2}\Big)^{\frac12}2^{\lambda n/2}\|f\|_{L^2}2^{-\lambda n/2}2^{-r}\|b\|_{\ell^\infty}\|g\|_{L^2}\\
&\le C\|\Omega\|_{L^2}2^{(n-M/8)\lambda-r/8-\delta j/8}\|f\|_{L^2}\|g\|_{L^2},
\end{aligned}$$

where we used (8) and (7). This gives sufficient decay in j, r and λ if $M\ge 16n$. The set U_r^2 is handled the same way.

To estimate the set U_r^3, we further decompose it into at most $2^{(r+\delta j+M\lambda)/2}$ disjoint sets V_s, such that if $(k,l),(k',l')\in V_s$ then $(k,l)\neq(k',l')$ implies $k\neq k'$ and $l\neq l'$. Indeed, by the definition of U_r^3, for each (k,l) in it with k fixed there exist at most N_2 pairs (k,l') in U_r^3 with $N_2=2^{(r+\delta j+M\lambda)/4}$. Otherwise, it is in U_r^1 and therefore a contradiction. Similarly for each (k,l) in U_r^3 with l fixed we have at most N_2 pairs (k',l) in U_r^3. Therefore we have at most $N_2^2=C2^{(r+\delta j+M\lambda)/2}$ sets V_s satisfying the claimed property.

For each of these sets, since $|a_\omega|=C|b_\omega|2^{-\lambda n}$, for the multiplier

$$m_j^{V_s}=\sum_{(k,l)\in V_s}a_{(k,l)}\omega_{1,k}\omega_{2,l}$$

we have the following estimate

$$\begin{aligned}
\big\|T_{m_j^{V_s}}(f,g)\big\|_{L^1}&\le \sum_{(k,l)\in V_s}|b_{(k,l)}|2^{-\lambda n}\big\|\mathcal{F}^{-1}(\omega_{1,k}\widehat{f})\mathcal{F}^{-1}(\omega_{2,l}\widehat{g})\big\|_{L^1}\\
&\le C2^{-r}\|b\|_{\ell^\infty}2^{-\lambda n}\Big[\sum_{(k,l)\in V_s}\big\|\omega_{1,k}\widehat{f}\big\|_{L^2}^2\Big]^{\frac12}\Big[\sum_{(k,l)\in V_s}\big\|\omega_{2,l}\widehat{g}\big\|_{L^2}^2\Big]^{\frac12}\\
&\le C2^{-r}\|b\|_{\ell^\infty}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}$$

Summing over s and using estimate (7) and the fact that $N_2^2=C2^{r/2}\|b\|_{\ell^\infty}^{-1/2}$ yields

$$\begin{aligned}
\big\|T_{m_j^{r,3}}(f,g)\big\|_{L^1}&\le N_2^2 2^{-r}\|b\|_{\ell^\infty}\|f\|_{L^2}\|g\|_{L^2}\\
&\le C2^{(r+\delta j+M\lambda)/2}2^{-r}\|b\|_{\ell^\infty}\|f\|_{L^2}\|g\|_{L^2}\\
&\le C\|\Omega\|_{L^q}2^{(-r-\delta j-M\lambda)/2}\|f\|_{L^2}\|g\|_{L^2},
\end{aligned}$$

which is also a good decay. We then have

$$
\begin{aligned}
\big\|T_{m_j^r}(f,g)\big\|_{L^1} &\le \Big[\big\|T_{m_j^{r,1}}(f,g)\big\|_{L^1}+\big\|T_{m_j^{r,2}}(f,g)\big\|_{L^1}\Big]+\big\|T_{m_j^{r,3}}(f,g)\big\|_{L^1}\\
&\le C\|\Omega\|_{L^2}2^{(n-M/8)\lambda}2^{-r/8}2^{-\delta j/8}\|f\|_{L^2}\|g\|_{L^2}\\
&\qquad\qquad +C\|\Omega\|_{L^q}2^{(-r-\delta j-M\lambda)/2}\|f\|_{L^2}\|g\|_{L^2}\\
&\le C\|\Omega\|_{L^q}2^{(-2\delta j-M\lambda-r)/16}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}
$$

Set $f^j=\mathcal{F}^{-1}(\widehat{f}\chi_{\{c_1\le|\xi_1|\le c_2 2^{j+1}\}})$ and $g^j=\mathcal{F}^{-1}(\widehat{g}\chi_{\{c_1\le|\xi_2|\le c_2 2^{j+1}\}})$ for some suitable constants $c_1,c_2>0$. In view of the preceding estimate for the piece $m^1_{j,0}=\sum_{\lambda,\kappa}\sum_{\omega\in D^1_{\lambda,\kappa}}a_\omega\omega$, we have

$$
\begin{aligned}
\|T_{m^1_{j,0}}(f,g)\|_{L^1} &= \|T_{m^1_{j,0}}(f^j,g^j)\|_{L^1}\\
&\le C\|\Omega\|_{L^q}\sum_{\kappa=1}^{C_{n,M,N}}\sum_{\lambda\ge0}\sum_{r\ge0}2^{(-2\delta j-M\lambda-r)/16}\|f^j\|_{L^2}\|g^j\|_{L^2}\\
&\le C\|\Omega\|_{L^q}2^{-\delta j/8}\|f^j\|_{L^2}\|g^j\|_{L^2}\,.
\end{aligned}
$$

The first equality was obtained from the support properties of $m^1_{j,0}$, which comes from the observation that $m_{j,0}(\vec{\xi}\,)\neq0$ only if $|\vec{\xi}\,|\approx 2^j$, and that $2^{-j}|\xi_1|\le|\xi_2|\le 2^j|\xi_1|$. Now recall that $m^1_{j,k}(\vec{\xi}\,)=m^1_{j,0}(2^k\vec{\xi}\,)$, so

$$
\begin{aligned}
&T_{m^1_{j,k}}(f,g)(x)\\
&= \int_{\mathbb{R}^{2n}}m^1_{j,0}(2^k\vec{\xi}\,)\widehat{f}(\xi_1)\widehat{g}(\xi_2)e^{2\pi ix\cdot(\xi_1+\xi_2)}d\xi_1d\xi_2\\
&= \int_{\mathbb{R}^{2n}}m^1_{j,0}(\vec{\eta}\,)\widehat{f}(2^{-k}\eta_1)\widehat{g}(2^{-k}\eta_2)e^{2\pi i2^{-k}x\cdot(\eta_1+\eta_2)}2^{-2kn}d\eta_1d\eta_2.
\end{aligned}
$$

Denote by f_k the function whose Fourier transform is $\widehat{f}(2^{-k}\xi_1)$ and $E_{j,k}=\{\xi_1\in\mathbb{R}^n:\,c_12^{-k}\le|\xi_1|\le c_22^{j-k}\}$, then

$$
\begin{aligned}
\|T_{m^1_{j,k}}(f,g)\|_{L^1} &= 2^{-2kn}\|T_{m^1_{j,0}}(f_k,g_k)(2^{-k}\cdot)\|_{L^1}\\
&= 2^{-kn}\|T_{m^1_{j,0}}(f_k,g_k)\|_{L^1}\\
&\le C\|\Omega\|_{L^q}2^{-kn}2^{-\delta j/8}\|\widehat{f}(2^{-k}\cdot)\chi_{E_{j,0}}\|_{L^2}\|\widehat{g}(2^{-k}\cdot)\chi_{E_{j,0}}\|_{L^2}\\
&= C\|\Omega\|_{L^q}2^{-\delta j/8}\|\widehat{f}\|_{L^2(E_{j,k})}\|\widehat{g}\|_{L^2(E_{j,k})}\,.
\end{aligned}
$$

Using this estimate and applying the Cauchy-Schwarz inequality we obtain for the diagonal part $m^1_j=\sum_{k\in\mathbb{Z}}m^1_{j,k}$ the estimate

$$
\|T_{m^1_j}(f,g)\|_{L^1}\le\sum_{k=-\infty}^{\infty}\|T_{m^1_{j,k}}(f,g)\|_{L^1}
$$

$$
\begin{aligned}
&\le C\|\Omega\|_{L^q} 2^{-\delta j/8} \sum_{k=-\infty}^{\infty} \|\widehat{f}\|_{L^2(E_{j,k})} \|\widehat{g}\|_{L^2(E_{j,k})} \\
&\le C\|\Omega\|_{L^q} 2^{-\delta j/8} \Big(\sum_k \|\widehat{f}\|^2_{L^2(E_{j,k})}\Big)^{\frac12} \Big(\sum_k \|\widehat{g}\|^2_{L^2(E_{j,k})}\Big)^{\frac12} \\
&\le C\|\Omega\|_{L^q} 2^{-\delta j/8} j^{1/2} \|f\|_{L^2} j^{1/2} \|g\|_{L^2} \\
&= C\|\Omega\|_{L^q} j 2^{-\delta j/8} \|f\|_{L^2} \|g\|_{L^2}
\end{aligned}
$$

since $\sum_{k=-\infty}^{\infty} \chi_{E_{j,k}} \le j$. This completes the decay of the first piece m_j^1.

5. The off-diagonal parts

We now estimate the off-diagonal parts of the operator, namely $T_{m_j^2}$ and $T_{m_j^3}$. To control these two operators, we need the following inequality,

$$\|T_{m_j^2}(f,g) + T_{m_j^3}(f,g)\|_{L^1} \le C\,\|\Omega\|_{L^q}\, 2^{-j\delta} \|f\|_{L^2} \|g\|_{L^2}. \tag{9}$$

which will be discussed in Lemma 8.

Now we show that the right hand side of (9) is finite. Let us select a group $D^2_{\lambda,\kappa}$ for some κ. For $\omega \in D^2_{\lambda,\kappa}$ we have the estimate $\|b_\omega\|_{L^\infty} \le C\|\Omega\|_{L^q} 2^{-j\varepsilon} 2^{-M\lambda}$. We further divide the group $D^2_{\lambda,\kappa}$ into columns $D^{2,a}_{\lambda,\kappa}$ such that all wavelets in a given column have the form $\omega = \omega_1 \omega_2^a$ with the same ω_2^a, where $a = (\mu_{n+1}, \dots, \mu_{2n}) \in \mathbb{Z}^n$. Notice that $\omega \in D^2_{\lambda,\kappa}$ implies that $|\xi_2| \le 2$, and each ω_2^a is supported in the cube

$$Q = [2^{-\lambda}(\mu_{n+1} - c), 2^{-\lambda}(\mu_{n+1} + c)] \times \cdots \times [2^{-\lambda}(\mu_{2n} - c), 2^{-\lambda}(\mu_{2n} + c)]$$

for some $c \approx 1$. Therefore, we have at most $C2^{\lambda n}$ choices of $(\mu_{n+1}, \cdots, \mu_{2n})$, i.e. there exist at most $C2^{\lambda n}$ different ω_2^a and $C2^{\lambda n}$ different columns.

For the multiplier $m^{2,a}_{\lambda,\kappa}$ related to the column of ω_2^a, we then get

$$
\begin{aligned}
&\int_{\mathbb{R}^{2n}} \sum_{\omega_1} a_\omega \omega_1(\xi_1) \omega_2^a(\xi_2) \widehat{f}(\xi_1) \widehat{g}(\xi_2) e^{2\pi i x\cdot(\xi_1+\xi_2)} d\xi_1 d\xi_2 \\
&\quad = \Big[\sum_{\omega_1} a_\omega T_{\omega_1}(f)(x)\Big] \Big[T_{\omega_2^a}(g)(x)\Big]
\end{aligned}
$$

with $\omega_2^a(\xi_2) = 2^{\lambda n/2} \omega_2(2^\lambda \xi_2 - a)$. Notice that

$$
\begin{aligned}
|T^a_{\omega_2}(g)(x)| &= \left| 2^{-\lambda n/2} \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\omega_2)(2^{-\lambda}(x-y)) g(y) e^{2\pi i 2^{-\lambda}(x-y)\cdot a} dy \right| \\
&\le 2^{-\lambda n/2} \int_{\mathbb{R}^n} \frac{g(y)}{(1 + 2^{-\lambda}|x-y|)^M} dy \\
&\le 2^{\lambda n/2} \mathcal{M}(g)(x),
\end{aligned}
$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function. We define

$$m_{a,\lambda}(\xi_1)=\frac{\sum_{\omega_1}a_\omega\omega_1(\xi_1)\chi_{2^{j-1}\le|\xi_1|\le 2^{j+1}}}{2^{-j\delta}2^{-(M+1+\frac{n}{2})\lambda}},$$

and then we have

$$\sum_{\omega_1}a_\omega T_{\omega_1}(f)(x)=2^{-j\delta}2^{-(M+1+\frac{n}{2})\lambda}\int_{\mathbb{R}^n}m_{a,\lambda}(\xi_1)\widehat{f}(\xi_1)e^{2\pi i x\cdot\xi_1}d\xi_1\,,$$

since the supports of ω_1's are disjoint and are all contained in the annulus $\{\xi_1:\,2^{j-1}\le|\xi_1|\le 2^{j+1}\}$. In view of (6) in Lemma 7 we have $|a_\omega|\le C_M\|\Omega\|_{L^q}2^{-j\delta}2^{-(M+1+n)\lambda}$ and this combined with $\|\omega_1\|_{L^\infty}\le C2^{n\lambda/2}$ implies that $|m_{a,\lambda}|\le C\|\Omega\|_{L^q}\chi_{2^{j-1}\le|\xi_1|\le 2^{j+1}}$. Therefore for the multiplier $m=\sum_\lambda 2^{-M\lambda}\sum_a m_{a,\lambda}$ we have

$$\|T_m(f)\|_{L^2}\le C\|\Omega\|_{L^q}\|\widehat{f}\chi_{2^{j-1}\le|\xi_1|\le 2^{j+1}}\|_{L^2},$$

since for each fixed λ there exist at most $C2^{\lambda n}$ indices a.

We now can control $T_{m^2_{j,0}}(f,g)(x)$ by $C2^{-j\varepsilon}\mathcal{M}(g)(x)T_m(f)(x)$. Recall that $m^2_{j,k}(\vec{\xi}\,)=m^2_{j,0}(2^k\vec{\xi}\,)$, then if f_k is the function whose Fourier transform is $\widehat{f}(2^{-k}\xi_1)$, we have

$$|T_{m^2_{j,k}}(f,g)(x)|\le C2^{-j\delta}2^{-2kn}\mathcal{M}(g_k)(2^{-k}x)|T_m(f_k)(2^{-k}x)|\,.$$

As a result

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in 5\mathbb{Z}}|T_{m^2_{j,k}}(f,g)|^2\Big)^{1/2}\Big\|_{L^1}\\
\le\ &\int_{\mathbb{R}^n}\Big(\sum_{k\in 5\mathbb{Z}}|2^{-j\varepsilon}2^{-kn}\mathcal{M}(g)(x)T_m(f_k)(2^{-k}x)|^2\Big)^{1/2}dx\\
\le\ &C2^{-j\delta}\|\mathcal{M}(g)\|_{L^2}\bigg(\int_{\mathbb{R}^n}\sum_{k\in 5\mathbb{Z}}|2^{-kn}T_m(f_k)(2^{-k}x)|^2dx\bigg)^{1/2}\\
\le\ &C\|\Omega\|_{L^q}2^{-j\delta}\|g\|_{L^2}\bigg(\sum_{k\in 5\mathbb{Z}}\int_{\mathbb{R}^n}\chi_{\{2^{j+k-1}\le|\xi_1|\le 2^{j+k+1}\}}|\widehat{f}(\xi_1)|^2d\xi_1\bigg)^{1/2}\\
\le\ &C\|\Omega\|_{L^q}2^{-j\delta}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}$$

The estimate for $T_{m^3_j}$ is similar. Thus the proof of Proposition 5 will be finished once we establish (9). The preceding estimate implies that for f,g in L^2 we have

$$\Big\|\Big(\sum_{k\in 5\mathbb{Z}}|T_{m^2_{j,k}}(f,g)|^2\Big)^{1/2}\Big\|_{L^1}<\infty \tag{10}$$

a fact that will be useful in the sequel.

Lemma 8. *There is a constant C such that* (9) *holds for all f,g in $L^2(\mathbb{R}^n)$.*

Proof. We first show that there exists a polynomial Q_1 of n variables such that $T_{m_j^2}(f,g)-Q_1\in L^1$.

Let $\psi\in\mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\psi}\geq 0$ with $\operatorname{supp}\widehat{\psi}\subset\{\xi : 1/2\leq|\xi|\leq 2\}$ and $\sum_{j=-\infty}^{\infty}\widehat{\psi}(2^{-j}\xi)=1$ for $\xi\neq 0$. Set $\widehat{\Phi}=\sum_{j=-2}^{2}\widehat{\psi}(2^{-j}\xi)$ and define $\Delta_k(f)=\mathcal{F}^{-1}(\widehat{\Phi}(2^{-k}\cdot)\widehat{f})$.

For $r=0,1,2,3,4$, define $m_j^{(r)}=\sum_{k\in 5\mathbb{Z}+r}m_{j,k}^2$. We will show that there exists a polynomial Q_j^r such that

$$\|T_{m_j^{(r)}}(f,g)-Q_j^r\|_{L^1}\leq\Big\|\Big(\sum_{k\in 5\mathbb{Z}+r}|T_{m_{j,k}^2}(f,g)|^2\Big)^{1/2}\Big\|_{L^1}. \tag{11}$$

We prove this assertion only in the case $r=0$ as the remaining cases are similar. By Corollary 2.2.10 in [14] there is a polynomial Q_1^0 such that

$$\|T_{m_j^{(0)}}(f,g)-Q_1^0\|_{L^1}\leq C\Big\|\Big(\sum_{k\in 5\mathbb{Z}}|\Delta_k(T_{m_j^{(0)}}(f,g))|^2\Big)^{1/2}\Big\|_{L^1}.$$

Notice that

$$\begin{aligned}&\Delta_k(T_{m_j^{(0)}}(f,g))(x)=\\ &\sum_{l\in 5\mathbb{Z}}\int_{\mathbb{R}^{2n}}\widehat{\Phi}(2^{-k}(\xi_1+\xi_2))m_{j,0}^2(2^l\xi)\widehat{f}(\xi_1)\widehat{g}(\xi_2)e^{2\pi i x\cdot(\xi_1+\xi_2)}d\xi_1 d\xi_2.\end{aligned} \tag{12}$$

Observe that $m_{j,0}^2(\vec{\xi}\,)$ is supported in the set

$$\{(\xi_1,\xi_2):\ 2^{j-1}\leq|(\xi_1,\xi_2)|\leq 2^{j+1},|\xi_1|\geq 2^j|\xi_2|\}$$

which is a subset of

$$\{(\xi_1,\xi_2):\ 2^{j-2}\leq|\xi_1+\xi_2|\leq 2^{j+2}\},$$

so $m_{j,0}^2(2^l\vec{\xi}\,)$ is supported in $\{(\xi_1,\xi_2): 2^{j-l-2}\leq|\xi_1+\xi_2|\leq 2^{j-l+2}\}$. The integrand in (12) is nonzero only if $k=j-l$, when $\widehat{\Phi}(2^{-k}\vec{\xi}\,)m_{j,0}^2(2^l\vec{\xi}\,)=m_{j,0}^2(2^l\vec{\xi}\,)$, otherwise the product is 0. In summary we obtained

$$\sum_{k\in 5\mathbb{Z}}|\Delta_k(T_{m_j^{(0)}}(f,g))|^2=\sum_{k\in 5\mathbb{Z}}|T_{m_{j,k}^2}(f,g)|^2. \tag{13}$$

Now (11) is a consequence of (10) and (13). Thus, there exist polynomials Q_1,Q_2 such that $T_{m_j^2}(f,g)-Q_1, T_{m_j^3}(f,g)-Q_2\in L^1$. Given f,g in $L^2(\mathbb{R}^n)$, we have already shown that $T_{m_j^1}(f,g)$ lies in L^1. Moreover, we showed in Proposition 3 that $T_j(f,g)$ lies in L^1. These facts imply that $T_{m_j^2}(f,g)+T_{m_j^3}(f,g)$ lies in L^1, and therefore $Q_1+Q_2=0$. Hence

$$\begin{aligned}\|T_{m_j^2}(f,g)+T_{m_j^3}(f,g)\|_{L^1} \le\ & \|T_{m_j^2}(f,g)-Q_1\|_{L^1}+\|T_{m_j^3}(f,g)-Q_2\|_{L^1}\\ \le\ & C\|\Omega\|_{L^q}2^{-j\delta}\|f\|_{L^2}\|g\|_{L^2}.\end{aligned}$$

□

This completes the proof of Proposition 5. □

6. Boundedness everywhere when $q=\infty$

Proposition 9. *Let $\Omega\in L^\infty(\mathbb{S}^{2n-1})$, $1<p_1,p_2<\infty$, $1/p=1/p_1+1/p_2$. Then for any given $0<\varepsilon<1$ there is a constant $C_{n,\varepsilon}$ such that*

$$\|T_j\|_{L^{p_1}\times L^{p_2}\to L^p}\le C_{n,\varepsilon}\|\Omega\|_{L^\infty}2^{j\varepsilon}$$

for all $j\ge 0$.

To prove Proposition 9 we use Theorem 3 of [16] and Proposition 5. To apply the result in [16] we need to know that the kernel of T_j is of bilinear Calderón-Zygmund type with bound $A\le C_{n,\varepsilon}\|\Omega\|_{L^\infty}2^{j\varepsilon}$ for any $\varepsilon\in(0,1)$. This is proved in Lemma 10 below. Assuming this lemma, it follows that

$$\|T_j\|_{L^{p_1}\times L^{p_2}\to L^p}\le C(A+\|T_j\|_{L^2\times L^2\to L^1})\le C_{n,\varepsilon}\|\Omega\|_{L^\infty}2^{j\varepsilon},$$

which yields the claim in Proposition 9.

Recall that a bilinear Calderón-Zygmund kernel is a function L defined away from the diagonal in $\mathbb{R}^{3n}$ which, for some $A>0$, satisfies the size estimate

$$|L(u,v,w)|\le\frac{A}{(|u-v|+|u-w|+|v-w|)^{2n}}$$

and the smoothness estimate

$$|L(u,v,w)-L(u',v,w)|\le\frac{A|u-u'|^\varepsilon}{(|u-v|+|u-w|+|v-w|)^{2n+\varepsilon}}$$

when

$$|u-u'|\le\frac{1}{3}(|u-v|+|u-w|)$$

(with analogous conditions in v and w). Such a kernel is associated with the bilinear operator

$$(f,g)\mapsto T_L(f,g)(u)=\text{p.v.}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(v)g(w)L(u,v,w)\,dvdw\,.$$

For the theory of such class of operators we refer to [16]. Thus we need to prove the following:

Lemma 10. *Given $\Omega \in L^\infty(\mathbb{S}^{2n-1})$ and any $j \in \mathbb{Z}$, for any $0 < \varepsilon < 1$ there is a constant $C_{n,\varepsilon}$ such that*

$$L(u,v,w) = K_j(u-v,u-w) = \sum_{i\in\mathbb{Z}} K_j^i(u-v,u-w)$$

is a bilinear Calderón-Zygmund kernel with constant $A \le C_{n,\varepsilon}\|\Omega\|_{L^\infty} 2^{|j|\varepsilon}$.

Proof. We begin by showing that for given $x,y \in \mathbb{R}^{2n}$ with $|x| \ge 3|y|/2$ we have

$$\sum_{i\in\mathbb{Z}} |K_j^i(x-y) - K_j^i(x)| \le C_{n,\varepsilon}\|\Omega\|_{L^\infty} \frac{2^{|j|\varepsilon}|y|^\varepsilon}{|x|^{2n+\varepsilon}} \tag{14}$$

Assuming (14), we deduce the smoothness of $K_j(u-v,u-w)$ as follows:

(a) For $u,\ v,\ v',\ w \in \mathbb{R}^n$ satisfying $|v-v'| \le \frac{1}{3}(|u-v|+|u-w|)$ we obtain

$$\begin{aligned}
&|K_j(u-v,u-w) - K_j(u-v',u-w)| \\
&\qquad \le \sum_{i\in\mathbb{Z}} |K_j^i(u-v',u-w) - K_j^i(u-v,u-w)| \\
&\qquad \le C_{n,\varepsilon}\|\Omega\|_{L^\infty} \frac{2^{|j|\varepsilon}|v-v'|^\varepsilon}{(|u-v|+|u-w|)^{2n+\varepsilon}} \\
&\qquad \le C_{n,\varepsilon}\|\Omega\|_{L^\infty} \frac{2^{|j|\varepsilon}|v-v'|^\varepsilon}{(|u-v|+|u-w|+|v-w|)^{2n+\varepsilon}}
\end{aligned}$$

since $|u-v|+|u-w|+|v-w| \le 2(|u-v|+|u-w|)$.

(b) For $u,\ u',\ v,\ w \in \mathbb{R}^n$ satisfying $|u-u'| \le \frac{1}{3}(|u-v|+|u-w|)$ we take $x = (u-v,u-w)$ and $y = (u-u',u-u')$ in (14) to deduce the claimed smoothness.

(c) For $u,\ v,\ w,\ w' \in \mathbb{R}^n$ satisfying $|w-w'| \le \frac{1}{3}(|u-v|+|u-w|)$ we take $x = (u-v,u-w)$ and $y = (0,w'-w)$.

We may therefore focus on (14). This will be a consequence of the following estimate

$$|K_j^i(x-y) - K_j^i(x)| \le C_{n,\varepsilon}\|\Omega\|_{L^\infty} \min\left(1, \frac{|y|}{2^{i-j}}\right) \frac{1}{2^{-i\varepsilon}2^{\min(j,0)\varepsilon}|x|^{2n+\varepsilon}} \tag{15}$$

when $|x| \ge 3|y|/2$. Assuming (15) we prove (14) as follows: We pick an integer N_3 such that $(\log_2|y|) + j \le N_3 < (\log_2|y|) + j + 1$.

If $j \ge 0$, then for i such that $2^{i-j} \le |y|$, i.e., $i \le N_3$, we have

$$\begin{aligned}
\sum_{i\le N_3} |K_j^i(x-y) - K_j^i(x)| \le&\ C_{n,\varepsilon}2^{2(2n+\varepsilon)}\|\Omega\|_{L^\infty} \sum_{i\le N_3} \frac{1}{2^{-i\varepsilon}|x|^{2n+\varepsilon}} \\
=&\ C_{n,\varepsilon}2^{2(2n+\varepsilon)}\|\Omega\|_{L^\infty}|x|^{-2n-\varepsilon}(2^j|y|)^\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&= C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \frac{2^{j\varepsilon}|y|^\varepsilon}{|x|^{2n+\varepsilon}} \\
&= C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \frac{2^{|j|\varepsilon}|y|^\varepsilon}{|x|^{2n+\varepsilon}}.
\end{aligned}
$$

For $j \geq 0$ and $i > N_3$, i.e. $2^{i-j} > |y|$,

$$
\begin{aligned}
\sum_{i>N_3} |K_j^i(x-y) - K_j^i(x)| &\leq C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \sum_{i>N_3} \frac{|y|}{2^{i-j}} \frac{1}{2^{-i\varepsilon}|x|^{2n+\varepsilon}} \\
&\leq C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} |y||x|^{-2n-\varepsilon} 2^j \sum_{i>N_3} 2^{i(\varepsilon-1)} \\
&= C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} |y||x|^{-2n-\varepsilon} 2^j (2^j|y|)^{\varepsilon-1} \\
&= C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \frac{2^{|j|\varepsilon}|y|^\varepsilon}{|x|^{2n+\varepsilon}}.
\end{aligned}
$$

If $j < 0$, then for $i \leq N_3$,

$$
\begin{aligned}
\sum_{i\leq N_3} |K_j^i(x-y) - K_j^i(x)| &\leq C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \sum_{i\leq N_3} \frac{1}{2^{-i\varepsilon}2^{j\varepsilon}|x|^{2n+\varepsilon}} \\
&= C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \frac{2^{-j\varepsilon}}{|x|^{2n+\varepsilon}} \sum_{i\leq N_3} 2^{i\varepsilon} \\
&\leq C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \frac{2^{-j\varepsilon}}{|x|^{2n+\varepsilon}} (2^j|y|)^\varepsilon \\
&= C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \frac{|y|^\varepsilon}{|x|^{2n+\varepsilon}}.
\end{aligned}
$$

If $j < 0$ and $i > N_3$, then

$$
\begin{aligned}
\sum_{i>N_3} |K_j^i(x-y) - K_j^i(x)| &\leq C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \sum_{i>N_3} \frac{|y|}{2^{i-j}} \frac{1}{2^{-i\varepsilon}2^{j\varepsilon}|x|^{2n+\varepsilon}} \\
&\leq C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} |y||x|^{-n-\varepsilon} 2^{j(1-\varepsilon)} \sum_{i>N_3} 2^{i(\varepsilon-1)} \\
&= C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} |y||x|^{-n-\varepsilon} 2^{j(1-\varepsilon)} (2^j|y|)^{\varepsilon-1} \\
&= C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \frac{|y|^\varepsilon}{|x|^{2n+\varepsilon}}.
\end{aligned}
$$

And for $j < 0$

$$
\frac{|y|^\varepsilon}{|x|^{2n+\varepsilon}} \leq \frac{2^{|j|\varepsilon}|y|^\varepsilon}{|x|^{2n+\varepsilon}}.
$$

This concludes the proof of (14) assuming (15). Finally we prove (15).

We have a decreasing estimate of $K^i(x)$, i.e. for $\varepsilon \in (0,1)$ and $i \in \mathbb{Z}$

$$
\begin{aligned}
|K^i(x)| \leq\ & \|\Omega\|_{L^\infty} 2^{-2in} \chi_{\frac{1}{2}\leq \frac{|x|}{2^i} \leq 2}(x) \\
\leq\ & \|\Omega\|_{L^\infty} 2^{2n+\varepsilon} \frac{2^{-2in}}{(1+2^{-i}|x|)^{2n+\varepsilon}} \chi_{\frac{1}{2}\leq \frac{|x|}{2^i} \leq 2}(x) \\
\leq\ & 2^{2n+\varepsilon} \|\Omega\|_{L^\infty} \frac{2^{-2in}}{(1+2^{-i}|x|)^{2n+\varepsilon}}. \qquad (16)
\end{aligned}
$$

Then recall the lemma from Appendix B1 of [14], by defining $\Psi(x) = \frac{1}{(1+|x|)^{2n+1}}$ we have that for $t \in [0,1]$

$$
\begin{aligned}
&(|K^i| * \Psi_{i-j})(x-ty) \\
&\leq\ 2^{2n+\varepsilon}\|\Omega\|_{L^\infty} \int_{\mathbf{R}^n} \frac{2^{-2in}}{(1+2^{-i}|z|)^{2n+\varepsilon}} \frac{2^{-2(i-j)n}}{(1+2^{-(i-j)}|x-ty-z|)^{2n+1}} dz \\
&\leq\ C_{n,\varepsilon} 2^{2n+\varepsilon} \|\Omega\|_{L^\infty} \frac{2^{\min(-i,-2(i-j))n}}{(1+2^{\min(-i,-(i-j))}|x-ty|)^{2n+\varepsilon}} \\
&\leq\ C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \frac{2^{-2in} 2^{2\min(j,0)n}}{(2^{-i}2^{\min(j,0)}|x|)^{2n+\varepsilon}} \\
&\leq\ C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \frac{2^{i\varepsilon}}{2^{\min(j,0)\varepsilon}|x|^{2n+\varepsilon}},
\end{aligned}
$$

which gives the first part of (15) by taking $t=0$ and 1 since

$$
|\mathcal{F}^{-1}(\beta_j)(x)| \leq C_\beta 2^{2jn}(1+2^j|x|)^{-2n-1} = C_\beta \Psi_j(x).
$$

The other part follows from the previous estimate in the following way

$$
\begin{aligned}
&|K^i_j(x-y) - K^i_j(x)| \\
&\quad = \left| \int_{\mathbb{R}^{2n}} K^i(z) (\mathcal{F}^{-1}(\beta_{i-j})(x-y-z) - \mathcal{F}^{-1}(\beta_{i-j})(x-z)) dz \right| \\
&\quad = \left| \int_{\mathbb{R}^{2n}} K^i(z) \int_0^1 2^{-2(i-j)n} 2^{-(i-j)} (\nabla(\mathcal{F}^{-1}\beta))(\frac{x-ty-z}{2^{i-j}}) \cdot y \, dt dz \right| \\
&\quad \leq \frac{C_{n,\varepsilon}|y|}{2^{i-j}} \int_0^1 \int_{\mathbb{R}^{2n}} |K^i(z)| \frac{2^{-(i-j)n}}{(1+2^{j-i}|x-ty-z|)^{2n+1}} dz dt \\
&\quad \leq C_{n,\varepsilon} \frac{|y|}{2^{i-j}} \int_0^1 (|K^i| * \Psi_{i-j})(x-ty) dt \\
&\quad \leq C_{n,\varepsilon} 2^{2(2n+\varepsilon)} \|\Omega\|_{L^\infty} \frac{|y|}{2^{i-j}} \frac{C_{n,\varepsilon}}{2^{-i\varepsilon} 2^{\min(j,0)\varepsilon}|x|^{2n+\varepsilon}}.
\end{aligned}
$$

To prove the size condition, notice that by the decreasing estimate (16) we have

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} |K_j^i(v,w)| &\leq \sum_i \Big|\int K^i(v_1,w_1)\beta_{i-j}(v-v_1,w-w_1)\,dv_1dw_1\Big| \\
&\leq \sum_i C_{n,\varepsilon}\frac{2^{-2in}}{(1+c_k 2^{-i}|(v,w)|)^{2n+\varepsilon}} \\
&\leq C_{n,\varepsilon}\sum_{i>N^*} 2^{-2in} + C(c_j|(v,w)|)^{-(2n+\varepsilon)}\sum_{i\leq N^*} 2^{j\varepsilon} \\
&\leq C_{n,\varepsilon}|(v,w)|^{-2n}
\end{aligned}$$

where $c_j = 2^{\min(0,j)}$ and N^* is the number such that $2^{N^*} \approx c_j|(v,w)|$. Hence

$$|K_j(u-v,u-w)| \leq \frac{C_{n,\varepsilon}}{(|u-v|+|u-w|)^{2n}} \leq \frac{C_{n,\varepsilon}}{(|u-v|+|u-w|+|v-w|)^{2n}}.$$

□

We improve Proposition 9 by giving a necessary decay via interpolation. Once this is proved, Theorem 1 follows trivially.

Lemma 11. *Let $\Omega \in L^\infty(\mathbb{S}^{2n-1})$, $1 < p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$, then there exist constants $\varepsilon_0 > 0$ and C_{n,ε_0} such that for all $j \geq 0$ we have*

$$\|T_j\|_{L^{p_1}\times L^{p_2}\to L^p} \leq C_{n,\varepsilon_0}\|\Omega\|_{L^\infty}2^{-j\varepsilon_0}.$$

Proof. For any triple $(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p})$ with $1/p = 1/p_1 + 1/p_2$, we can choose two triples $\vec{P}_1 = (\frac{1}{p_{1,1}},\frac{1}{p_{1,2}},\frac{1}{q_1})$ and $\vec{P}_2 = (\frac{1}{p_{2,1}},\frac{1}{p_{2,2}},\frac{1}{q_2})$ such that $\vec{P}_1$, $\vec{P}_2$ and $(\frac{1}{2},\frac{1}{2},1)$ are not collinear and the point $(\frac{1}{p_1},\frac{1}{p_2},\frac{1}{p})$ is in the convex hull of them. By Proposition 9 and Proposition 5, T_j is bounded at $\vec{P}_1$, $\vec{P}_2$ with bound $C_{n,\varepsilon}\|\Omega\|_{L^\infty}2^{j\varepsilon}$ for any $\varepsilon \in (0,1)$ and at $(\frac{1}{2},\frac{1}{2},1)$ with bound $C_n\|\Omega\|_{L^\infty}2^{-j\delta}$ for some fixed $\delta < 1/8$. Applying Theorem 7.2.2 in [14] we obtain that

$$\|T_j(f,g)\|_{L^p} \leq C_{n,\varepsilon_0}\|\Omega\|_{L^\infty}2^{-j\varepsilon_0}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$$

for some constant ε_0 depending on p_1, p_2, p. □

As an application of Theorem 1 we derive the boundedness of the Calderón commutator in the full range of exponents $1 < p_1, p_2 < \infty$, a fact proved in [4]. The Calderón commutator is defined in [1] as

$$\mathcal{C}(a,f)(x) = p.v.\int_{\mathbb{R}} \frac{A(x)-A(y)}{(x-y)^2} f(y)dy,$$

where a is the derivative of A. It is a well known fact [7] that this operator can be written as

$$p.v. \int_{\mathbb{R}} \int_{\mathbb{R}} K(x-y, x-z) f(y) a(z) dy dz$$

with $K(y,z) = \frac{e(z)-e(z-y)}{y^2} = \frac{\Omega((y,z)/|(y,z)|)}{|(y,z)|^2}$, where $e(t)=1$ if $t>0$ and $e(t)=0$ if $t<0$. $K(y,z)$ is odd and homogeneous of degree -2 whose restriction on $\mathbb{S}^1$ is $\Omega(y,z)$. It is easy to check that Ω is odd, bounded and thus it satisfies the hypothesis of Theorem 1.

Corollary 12. *Given $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$ there is a constant C such that*

$$\|\mathcal{C}(a,f)\|_{L^p} \le C\|a\|_{L^{p_1}}\|f\|_{L^{p_2}}$$

is valid for all functions f and a on the line.

7. Boundedness of T_Ω when $\Omega \in L^q(\mathbb{S}^{2n-1})$ with $2 \le q < \infty$

Let $\mathcal{R}$ be the rhombus of all points $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ with $1 \le p_1, p_2 \le \infty$ and $1/p = 1/p_1 + 1/p_2$. We let $\mathcal{B}$ be the set of all points $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ such that either p_1 or p_2 are equal to 1 or ∞, i.e. $\mathcal{B}$ is the boundary of $\mathcal{R}$.

Theorem 13. *Given any dimension $n \ge 1$, there is a constant C_n and there exists a neighborhood $\mathcal{S}$ of the point $(\frac{1}{2}, \frac{1}{2}, 1)$ in $\mathcal{R}$, whose size is at least $C_n(q')^{-2}$, such that if Ω lies in $L^q(\mathbb{S}^{2n-1})$ with $2 \le q \le \infty$, then*

$$\|T_\Omega\|_{L^{p_1} \times L^{p_2} \to L^p} < \infty$$

for $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p}) \in \mathcal{S}$.

Proof. In Proposition 5 we showed that $\|T_j\|_{L^2 \times L^2 \to L^1} \le C\|\Omega\|_{L^q} 2^{-j\delta}$ with $\delta \approx \frac{1}{q'}$. Consider the point $(\frac{1}{2}, \frac{1}{2}, 1)$. Find two other points $(\frac{1}{p_{11}}, \frac{1}{p_{12}}, \frac{1}{q_1})$ and $(\frac{1}{p_{21}}, \frac{1}{p_{22}}, \frac{1}{q_2})$ in the interior of $\mathcal{R}$ such that these three points are not colinear.

Then if $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ lies in the open convex hull of these three points, precisely, if $\frac{1}{p_i} = \frac{1}{p_{1i}}\eta_1 + \frac{1}{p_{2i}}\eta_2 + \frac{1}{2}\eta_3$ for $i = 1,2$, and $\eta_1 + \eta_2 + \eta_3 = 1$, then multilinear interpolation (Theorem 7.2.2 in [14]) yields that

$$\|T_j\|_{L^{p_1} \times L^{p_2} \to L^p} \le C\|\Omega\|_{L^q} 2^{j(2n(\eta_1+\eta_2)-\delta\eta_3)}.$$

Moreover, if $2n(\eta_1+\eta_2) - \delta\eta_3 < 0$, then $\sum_{j \ge 0} \|T_j\|_{L^{p_1} \times L^{p_2} \to L^p} \le C\|\Omega\|_{L^q}$.

If $(\frac{1}{p_{11}}, \frac{1}{p_{12}}, \frac{1}{q_1})$ and $(\frac{1}{p_{21}}, \frac{1}{p_{22}}, \frac{1}{q_2})$ are close and let $\eta_1 = \eta_2 = \eta$, we roughly have $4n\eta - \delta(1-2\eta) < 0$, from which we get $\eta < \frac{\delta}{4n+2\delta}$. In particular, all points $\vec{P} = (\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ in the set

$$\left\{ \vec{P} = (1-t)(\tfrac{1}{2}, \tfrac{1}{2}, 1) + t\vec{B} :\ 0 \le t \le \delta/16n,\ \vec{B} \in \mathcal{B} \right\}$$

are contained in the claimed neighborhood, whose size is comparable to $(q')^{-2}$. $\square$

Remark 1. Theorem 13 is sharp in the following sense. Let $\vec{A} = (\frac{1}{2}, \frac{1}{2}, 1)$ and $\vec{B}_0 = (1,1,2)$. By Theorem 13, the smallest p such that $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ lies in $\mathcal{S}$ satisfies

$$\frac{1}{p} = 2\cdot\frac{2\delta}{16n} + (1-\frac{2\delta}{16n}) = 1+\frac{\delta}{8n},$$

from which $\frac{1}{p} - 1 = \frac{\delta}{8n} \approx \frac{1}{q'}$. For the case $n = 1$, by the example in [11], we have the requirement $\frac{1}{p} + \frac{1}{q} \leq 2$, which implies that $\frac{1}{p} - 1 \leq \frac{1}{q'}$.

We end this paper by stating two related open problems:

(a) Given $\Omega \in L^q(\mathbb{S}^{2n-1})$ with $2 \leq q < \infty$, find the full range of p_1, p_2, p such that T_Ω maps $L^{p_1} \times L^{p_2} \to L^p$.

(b) Is T_Ω bounded when $\Omega \in L^q(\mathbb{S}^{2n-1})$ for $q < 2$?

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