SEPARABLE DETERMINATION OF (GENERALIZED-)LUSHNESS

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Abstract. We prove that every Asplund lush space is generalized-lush using the method of separable reduction. This gives a partial positive answer to a question by Jan-David Hardtke.

Introduction

Let us fix first some notations. X denotes a Banach space, X^* its dual, B_X its closed unit ball and S_X its unit sphere. All linear spaces are over the field $\mathbb{R}$. For $x^* \in S_{X^*}$ and $\varepsilon > 0$ we put $S(x^*, \varepsilon) := \{x \in B_X : \ x^*(x) > 1 - \varepsilon\}$. If A is a subset of X, we write $\operatorname{aco} A$ for the absolutely convex hull of A. Finally, we say that a Banach space X is *lush*, if for every $x, y \in S_X$ and every $\varepsilon > 0$ there exists a functional $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and $\operatorname{dist}(y, \operatorname{aco} S(x^*, \varepsilon)) < \varepsilon$.

The concept of lushness, introduced by K. Boyko, V. Kadets, M. Martín and D. Werner in [3], is a Banach space property, which ensures that the space has numerical index 1. It was used in [3] to solve a problem concerning the numerical index of a Banach space. Lushness was further investigated e.g. in [2] as a property of a Banach space. Later, D. Tan, X. Hunag and R. Liu in [7] proved that every lush space has "Mazur-Ulam property"; i.e., every isometry from a unit sphere of a lush space E onto a unit sphere of a Banach space F extends to a linear isometry of E and F. Up to our knowledge, it is still an open problem whether every Banach space has Mazur-Ulam property. In order to prove that lush spaces have Mazur-Ulam property, the authors of [7] introduced the notion of generalized-lushness.

A Banach space X is called *generalized lush* (GL) if for every $x \in S_X$ and every $\varepsilon > 0$ there is $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and, for every $y \in S_X$,

$$\operatorname{dist}(y, S(x^*, \varepsilon)) + \operatorname{dist}(y, -S(x^*, \varepsilon)) < 2 + \varepsilon.$$

It is proved in [7] that every separable lush space is (GL) and that every (GL) space has Mazur-Ulam property. Hence, every separable lush space has Mazur-Ulam property. Using certain refinements of this result and a kind of reduction to the separable case, it is deduced in [7] that every lush space has Mazur-Ulam property.

The concept of (GL) Banach spaces was further investigated as a property of a Banach space by J.-D. Hardtke in [6]. At the 43$^{\text{rd}}$ Winter School of Abstract Analysis, he presented his results and asked whether every nonseparable lush Banach space is (GL). In this note we give a partial positive answer to this question (recall that a Banach space is called *Asplund* if every separable subspace of it has separable dual).

Theorem 1. *Let X be an Asplund lush space. Then X is (GL).*

We prove Theorem 1 using the method of separable reduction. By a separable reduction we mean the possibility to extend the validity of a statement from separable spaces to the nonseparable setting without knowing the proof of the statement in the separable case. The proof of

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separable reduction theorems depends on a "separable determination": a statement ϕ concerning a nonseparable Banach space X is here considered to be *separably determined* if

$$\text{The statement } \phi \text{ holds in } X \iff \forall V \in \mathcal{R}: \text{ The statement } \phi \text{ holds in } V,$$

where $\mathcal{R}$ is a sufficiently large family of separable subspaces of X; typically, for any separable subspace of X there is a bigger subspace from $\mathcal{R}$. Although in applications one makes the final deduction using just one separable subspace, it is convenient to know that the family $\mathcal{R}$ is large in order to join finitely many arguments together. There are several approaches to this. One of them is the concept of *rich families* introduced in [1] by J. Borwein, W. Moors.

Definition 2. Let X be a Banach space. By $\mathcal{S}(X)$ we denote the family of all separable closed subspaces of X. Let us have $\mathcal{R} \subset S(X)$ and consider the following two conditions.

(i) Each separable subspace of X is contained in an element of $\mathcal{R}$.
(ii) For every increasing sequence V_i in $\mathcal{R}$, $\overline{\bigcup_{i=1}^{\infty} V_i}$ belongs to $\mathcal{R}$.

If (i) holds, we say that $\mathcal{R}$ is *cofinal*. If (ii) holds, we say that $\mathcal{R}$ is *σ-closed*. If both (i) and (ii) hold, we say that $\mathcal{R}$ is *rich*.

The crucial property, which enables us to combine several results together, is the following fact.

Proposition 3 ([1, Proposition 1.1])**.** *The intersection of two (even of countably many) rich families of a given Banach space is (not only non-empty but again even) rich.*

The main step towards the proof of Theorem 1 is contained in the following two results.

Theorem 4. *Let X be a Banach space. Then there exists a rich family $\mathcal{A} \subset \mathcal{S}(X)$ such that for every $V \in \mathcal{A}$ we have*

$$X \textit{ is lush} \iff V \textit{ is lush.}$$

Theorem 5. *Let X be an Asplund space. Then there exists a rich family $\mathcal{A} \subset \mathcal{S}(X)$ such that for every $V \in \mathcal{A}$ we have*

$$V \textit{ is (GL)} \implies X \textit{ is (GL).}$$

We obtain Theorem 1 as an immediate corollary to Theorem 4 and Theorem 5.

Proof of Theorem 1. Let X be an Asplund lush space. By Theorem 4, Theorem 5 and Proposition 3 (we intersect rich families from Theorem 4 and Theorem 5 and pick one space from the intersection), there is a closed separable subspace $V \subset X$ such that

$$X \text{ is lush} \iff V \text{ is lush} \qquad \text{and} \qquad V \text{ is (GL)} \implies X \text{ is (GL).}$$

Since X is lush, V is separable and lush; hence, by [7, Example 2.5], V is (GL). Thus, X is (GL). □

In the remainder of this note we prove Theorem 4, Theorem 5 and we discuss the possibility of obtaining the reverse implication in Theorem 5. It is not known to the author whether Theorem 5 holds without the assumption that X is Asplund. Note that if it were so, it would easily follow that every lush space is (GL).

We recall the most relevant notions, definitions, notations and results: We denote by $\mathbb{Q}_+$ the set $(0,\infty) \cap \mathbb{Q}$. Let X be a Banach space. If $A \subset X$, the symbols $\overline{\text{sp}}\, A$ and $\text{sp}_{\mathbb{Q}}\, A$ mean the closed linear span of A and the set consisting of all finite linear combinations of elements in A with rational coefficients, respectively. For $A \subset X$ and $B \subset X^*$ we put $B|_A := \{x^*|_A : x^* \in B\}$; hence, if A is a subspace of X, then $B|_A$ is a subset of the dual space A^*. Let $[X]^{\mathbb{N}}$ and $[X^*]^{\mathbb{N}}$ denote the families of all countable subsets of X and X^* respectively. By $\mathcal{S}(X)$ we denote the family of all separable closed subspaces of X. By $\mathcal{S}_{\square}(X \times X^*)$ we denote the set $\{V \times Y : V \in \mathcal{S}(X), Y \in \mathcal{S}(X^*)\}$. We say that $\mathcal{R} \subset \mathcal{S}_{\square}(X \times X^*)$ is *rich* if every member of $\mathcal{S}_{\square}(X \times X^*)$ is contained in some $V \times Y \in \mathcal{R}$ and whenever we have an increasing sequence $(V_i \times Y_i)_{i\in\mathbb{N}}$ in $\mathcal{R}$, then $\overline{\bigcup_{i\in\mathbb{N}} V_i \times Y_i} = \overline{\bigcup_{i\in\mathbb{N}} V_i} \times \overline{\bigcup_{i\in\mathbb{N}} Y_i} \in \mathcal{R}$.

Finally, we recall the concept introduced in [4] which serves as a link between X and X^* (and, by [4, Theorem 2.3], exists right if and only if X is Asplund).

Definition 6. By an *Asplund generator in a Banach space X we understand any correspondence $G : [X]^{\mathbb{N}} \longrightarrow [X^*]^{\mathbb{N}}$ such that*

(a) $\left(\overline{\mathrm{sp}}\, C\right)^* \subset \overline{G(C)|_{\overline{\mathrm{sp}}\, C}}$ for every $C \in [X]^{\mathbb{N}}$;

(b) if $C_1,\ C_2,\ \dots$ is an increasing sequence in $[X]^{\mathbb{N}}$, then $G(C_1 \cup C_2 \cup \cdots) = G(C_1) \cup G(C_2) \cup \cdots$;

(c) $\bigcup\{G(C) :\ C \in [X]^{\mathbb{N}}\}$ is a dense subset in X^*; and

(d) if $C_1, C_2 \in [X]^{\mathbb{N}}$ are such that $\overline{\mathrm{sp}}\, C_1 = \overline{\mathrm{sp}}\, C_2$, then $\overline{\mathrm{sp}}\, G(C_1) = \overline{\mathrm{sp}}\, G(C_2)$.

1. Separable determination of (generalized-)lushness

The fact that "lushness is separably determined property" was in some sense proved in [2, Theorem 4.2]. However, we need to prove this result in the language of rich families in order to further combine it; see Theorem 4. Similarly as in [2], we will use the following result.

Theorem 7 ([2, Theorem 4.1])**.** *Let X be a real Banach space and $D \subset X$ a dense subspace. Then the following conditions are equivalent.*

(i) *X is lush*

(ii) *For every $x, y \in S_X$ and $\varepsilon > 0$, there are $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ and $x_1, x_2 \in B_X$ such that $\|x + x_1 + x_2\| > 3 - \varepsilon$ and $\|y - (\lambda_1 x_1 - \lambda_2 x_2)\| < \varepsilon$.*

(iii) *For every $x, y \in S_X \cap D$ and $\varepsilon > 0$, there are $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ and $x_1, x_2 \in B_X$ such that $\|x + x_1 + x_2\| > 3 - \varepsilon$ and $\|y - (\lambda_1 x_1 - \lambda_2 x_2)\| < \varepsilon$.*

Proof. (i)⇔(ii) is proved in [2, Theorem 4.1]. The equivalence (ii)⇔(iii) is evident. □

Proof of Theorem 4. First, we will find a rich family $\mathcal{R}_1 \subset \mathcal{S}(X)$ such that for every $V \in \mathcal{R}_1$ we have

$$X \text{ is lush} \implies V \text{ is lush.}$$

If X is not lush, we put $\mathcal{R}_1 := S(X)$. Otherwise, define $\mathcal{R}_1 \subset \mathcal{S}(X)$ as the family consisting of all $V \in \mathcal{S}(X)$ such that V is lush. We shall show that $\mathcal{R}_1$ is a rich family. This is obvious if X is not lush; hence, let us assume that X is lush. By [2, Theorem 4.2], $\mathcal{R}_1$ is cofinal. For checking the σ-completeness of $\mathcal{R}_1$, consider any increasing sequence $(V_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{R}_1$. We need to prove that $V := \overline{\bigcup_{i=1}^{\infty} V_i}$ is lush. Since $(V_i)_{i\in\mathbb{N}}$ is increasing, by Theorem 7 (i) $\implies$ (ii), condition (iii) in Theorem 7 is satisfied with $D = \bigcup_{i=1}^{\infty} V_i$ and $X = V$. Hence, V is lush.

Now, we will find a rich family $\mathcal{R}_2 \subset \mathcal{S}(X)$ such that for every $V \in \mathcal{R}_2$ we have

$$X \text{ is not lush} \implies V \text{ is not lush.}$$

If X is lush, we put $\mathcal{R}_2 := S(X)$. Otherwise, by Theorem 7 (ii) $\implies$ (i), there are $x, y \in S_X$ and $\varepsilon > 0$ such that for every $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ and $x_1, x_2 \in B_X$ we have $\|x + x_1 + x_2\| \leq 3 - \varepsilon$ or $\|y - (\lambda_1 x_1 - \lambda_2 x_2)\| \geq \varepsilon$. Hence, the family $\mathcal{R}_2 := \{V \in \mathcal{S}(X) :\ x, y \in V\}$ is a rich family such that each member of the family is not lush.

Finally, it remains to put $\mathcal{A} := \mathcal{R}_1 \cap \mathcal{R}_2$. This is a rich family because, by the construction above, we have $\mathcal{A} = \mathcal{R}_1$ or $\mathcal{A} = \mathcal{R}_2$ depending on the "lushness" of X. It is obvious that for every $V \in \mathcal{A}$, V is lush if and only if X is lush. □

Before proving Theorem 5, let us note that in the definition of (GL) spaces we may work only with a dense subset of X^*. This is the content of the following Lemma. Since the proof is straightforward and easy, we omit it.

Lemma 8. *Let X be a Banach space and let $G \subset X^*$ be a dense subset of X^*. Let us assume that there are $x \in S_X$ and $\varepsilon > 0$ such that for every $x^* \in G$ with $x \in S\Big(\frac{x^*}{\|x^*\|}, \varepsilon\Big)$ there exists $y \in S_X$ such that*

$$\operatorname{dist}\left(y, S\Big(\frac{x^*}{\|x^*\|}, \varepsilon\Big)\right) + \operatorname{dist}\left(y, -S\Big(\frac{x^*}{\|x^*\|}, \varepsilon\Big)\right) \geq 2 + \varepsilon.$$

Then X is not (GL).

Proof of Theorem 5. If X is (GL), it suffices to put $\mathcal{A} := \mathcal{S}(X)$. Therefore, we may assume that X is not (GL). Let $G : [X]^{\mathbb{N}} \to [X^*]^{\mathbb{N}}$ be an Asplund generator in X. Since X is not (GL), there are $x_0 \in S_X$ and $\varepsilon_0 > 0$ such that

$$(1) \qquad \forall x^* \in S_{X^*} : x_0 \in S(x^*, \varepsilon_0) \quad \exists y \in S_X : \operatorname{dist}(y, S(x^*, \varepsilon_0)) + \operatorname{dist}(y, -S(x^*, \varepsilon_0)) \geq 2 + \varepsilon_0.$$

By Lemma 8 and the definition of an Asplund generator, it suffices to find a rich family $\mathcal{A} \subset \mathcal{S}(X)$ such that for every $V \in \mathcal{A}$ we have $x_0 \in V$ and there exists $C \subset V$ with $\overline{\operatorname{sp}}\, C = V$ satisfying the following property.

$$(2) \qquad \begin{aligned} &\forall x^* \in G(C) : x_0 \in S\Big(\frac{x^*|_V}{\|x^*|_V\|}, \varepsilon_0\Big) \quad \exists y \in S_V \\ &\qquad \operatorname{dist}\left(y, S\Big(\frac{x^*|_V}{\|x^*|_V\|}, \varepsilon_0\Big)\right) + \operatorname{dist}\left(y, -S\Big(\frac{x^*|_V}{\|x^*|_V\|}, \varepsilon_0\Big)\right) \geq 2 + \varepsilon_0. \end{aligned}$$

Define $\mathcal{R}' \subset \mathcal{S}_{\square}(X \times X^*)$ as the family consisting of all rectangles $\overline{\operatorname{sp}}\, C \times \overline{\operatorname{sp}}\, G(C)$, with $C \in [X]^{\mathbb{N}}$, such that the assignment

$$\overline{\operatorname{sp}}\, G(C) \ni x^* \longmapsto x^*|_{\overline{\operatorname{sp}}\, C} \in (\overline{\operatorname{sp}}\, C)^*$$

is a surjective isometry. It is proved in [4, proof of Theorem 2.3 (ii) $\Longrightarrow$ (iii)] that $\mathcal{R}'$ is a rich family and whenever we have $V_1 \times Y_1$, $V_2 \times Y_2$ in $\mathcal{R}'$ such that $V_1 \subset V_2$, then $Y_1 \subset Y_2$. Consequently, the family $\mathcal{R}_1 := \{V : \exists Y : V \times Y \in \mathcal{R}'\} \subset \mathcal{S}(X)$ is rich.

For every $x^* \in X^*$, we pick, if it exists, a point $I(x^*) \in S_X$ such that

$$(3) \qquad \operatorname{dist}\left(I(x^*), S\Big(\frac{x^*}{\|x^*\|}, \varepsilon_0\Big)\right) + \operatorname{dist}\left(I(x^*), -S\Big(\frac{x^*}{\|x^*\|}, \varepsilon_0\Big)\right) \geq 2 + \varepsilon_0.$$

Define $\mathcal{R}_2 \subset \mathcal{S}(X)$ as the family consisting of all $V \in \mathcal{S}(X)$ with $x_0 \in V$ such that there is a countable set $C \subset V$ with $\overline{\operatorname{sp}}\, C = V$ and

$$(4) \qquad \forall x^* \in G(C) : \quad \big(I(x^*) \text{ is defined } \implies I(x^*) \in V\big).$$

We shall show that $\mathcal{R}_2$ is a rich family.

As regards the cofinality of $\mathcal{R}_2$, fix any countable set $S \subset X$. Put $C_0 := S \cup \{x_0\}$. Assume that for some $m \in \mathbb{N}$ we already found countable sets $C_0 \subset C_1 \subset \ldots \subset C_{m-1} \subset X$. Then we find $C_m \supset C_{m-1}$ such that, for every $x^* \in G(C_{m-1})$, we have $I(x^*) \in C_m$ whenever it is defined. Do so for every $m \in \mathbb{N}$ and put finally $C := \bigcup_{i=0}^{\infty} C_i$. It remains to see that $V := \overline{\operatorname{sp}}\ C \in \mathcal{R}_2$, which follows immediately from the construction because we have $G(C) = \bigcup_{i=0}^{\infty} G(C_i)$.

For checking the σ-completeness of $\mathcal{R}_2$, consider any increasing sequence $(V_i)_{i \in \mathbb{N}}$ of elements in $\mathcal{R}_2$. Let, for every $i \in \mathbb{N}$, be $C_i \subset V_i$ a set with $\overline{\operatorname{sp}}\, C_i = V_i$ satisfying (4) for C_i and V_i. We may assume that $C_1 \subset C_2 \subset \ldots$ (if not, we replace it by $C_1, C_1 \cup C_2, C_1 \cup C_2 \cup C_3, \ldots$). Then $V = \overline{V_1 \cup V_2 \cup \ldots}$ contains x_0 and we put $C := C_1 \cup C_2 \ldots$. Then $\overline{\operatorname{sp}}\, C = V$. Moreover, since $C_1, C_2, \ldots$ is an increasing sequence, (4) is satisfied.

Finally, we put $\mathcal{A} := \mathcal{R}_1 \cap \mathcal{R}_2$. It remains to prove that our $\mathcal{A}$ "works"; i.e., no member of $\mathcal{A}$ is (GL). So, pick any $V \in \mathcal{A}$. We need to show that (2) holds. Fix a set C with $\overline{\operatorname{sp}}\, C = V$ from the definition of the family $\mathcal{R}_2$. Fix $x^* \in G(C)$ with $x_0 \in S\Big(\frac{x^*|_V}{\|x^*|_V\|}, \varepsilon_0\Big)$. By the definiton of $\mathcal{R}_1$, there is a countable set $C' \subset V$ such that $\overline{\operatorname{sp}}\, C' = V$ and, for every $y^* \in \overline{\operatorname{sp}}\, G(C')$, we have $\|y^*\| = \|y^*|_V\|$. By the definition of an Asplund generator, $\overline{\operatorname{sp}}\, G(C) = \overline{\operatorname{sp}}\, G(C')$; thus, we have

$\|x^*\| = \|x^*|_V\|$. Hence, we have $x_0 \in S(x^*, \varepsilon_0)$ and, by (1), $I(x^*)$ is defined; hence, by (4), we have $y := I(x^*) \in S_V$. Consequently,

$$\begin{aligned}
&\operatorname{dist}\left(y, S\Big(\tfrac{x^*|_V}{\|x^*|_V\|}, \varepsilon_0\Big)\right) + \operatorname{dist}\left(y, -S\Big(\tfrac{x^*|_V}{\|x^*|_V\|}, \varepsilon_0\Big)\right)\\
&\qquad = \operatorname{dist}\left(y, S\Big(\tfrac{x^*}{\|x^*\|}, \varepsilon_0\Big) \cap B_V\right) + \operatorname{dist}\left(y, -S\Big(\tfrac{x^*}{\|x^*\|}, \varepsilon_0\Big) \cap B_V\right)\\
&\qquad \geq \operatorname{dist}\left(y, S\Big(\tfrac{x^*}{\|x^*\|}, \varepsilon_0\Big)\right) + \operatorname{dist}\left(y, -S\Big(\tfrac{x^*}{\|x^*\|}, \varepsilon_0\Big)\right) \overset{(3)}{\geq} 2 + \varepsilon_0.
\end{aligned}$$

Thus, (2) holds and V is not (GL). $\square$

Finally, we will show that the reverse implication holds in some sense in an arbitrary Banach space; see Proposition 10. Before proving it, let us notice that in the definition of (GL) spaces we may work only with a dense subset of X. This is the content of the following Lemma. Since the proof is straightforward and easy, we omit it.

Lemma 9. *Let X be a Banach space and let $D \subset X$ be a dense subsets of X. Let us assume that for every $x \in D$ and every $\varepsilon \in \mathbb{Q}_+$ there is $x^* \in S_{X^*}$ such that $\frac{x}{\|x\|} \in S(x^*, \varepsilon)$ and, for every $y \in D$,*

$$\operatorname{dist}\left(\tfrac{y}{\|y\|}, S(x^*, \varepsilon)\right) + \operatorname{dist}\left(\tfrac{y}{\|y\|}, -S(x^*, \varepsilon)\right) < 2 + \varepsilon.$$

Then X is (GL).

Proposition 10. *Let X be a Banach space. Then there exists a* `cofinal` *family $\mathcal{A} \subset \mathcal{S}(X)$ such that for every $V \in \mathcal{A}$ we have*

$$X \text{ is (GL)} \implies V \text{ is (GL)}.$$

Proof. If X is not (GL), we may put $\mathcal{A} = \mathcal{S}(X)$. Let us assume that X is (GL). Define $\mathcal{A} \subset \mathcal{S}(X)$ as the family consisting of all $V \in \mathcal{S}(X)$ such that V is (GL). We shall show that $\mathcal{A}$ is a `cofinal` family.

For every $x \in X$ and $\varepsilon > 0$, we pick a point $I_1(x, \varepsilon) \in S_{X^*}$ such that $\frac{x}{\|x\|} \in S\big(I_1(x, \varepsilon), \varepsilon\big)$ and, for every $y \in X$,

$$\operatorname{dist}\left(\tfrac{y}{\|y\|}, S\big(I_1(x, \varepsilon), \varepsilon\big)\right) + \operatorname{dist}\left(\tfrac{y}{\|y\|}, -S\big(I_1(x, \varepsilon), \varepsilon\big)\right) < 2 + \varepsilon.$$

Now, for every $x, y \in X$ and $\varepsilon > 0$, we pick two points $I_2(x, y, \varepsilon), I_3(x, y, \varepsilon) \in S\big(I_1(x, \varepsilon), \varepsilon\big)$ with

$$\left\|\tfrac{y}{\|y\|} - I_2(x, y, \varepsilon)\right\| + \left\|\tfrac{y}{\|y\|} + I_3(x, y, \varepsilon)\right\| < 2 + \varepsilon.$$

Note that, since X is (GL), for every $x, y \in X$ and $\varepsilon > 0$ the points $I_2(x, y, \varepsilon)$ and $I_3(x, y, \varepsilon)$ exist.

In order to show that $\mathcal{A}$ is cofinal, fix any countable set $S \subset X$. Put $C_0 := S$. Assume that for some $m \in \mathbb{N}$ we already found countable sets $C_0 \subset C_1 \subset \ldots \subset C_{m-1} \subset X$. Then we find $C_m \supset C_{m-1}$ such that, for every $x, y \in \operatorname{sp}_{\mathbb{Q}} C_{m-1}$ and $\varepsilon \in \mathbb{Q}_+$, we have $\{I_2(x, y, \varepsilon), I_3(x, y, \varepsilon)\} \subset C_m$. Do so for every $m \in \mathbb{N}$ and put finally $C := \bigcup_{i=0}^{\infty} C_i$. It remains to see that $V := \overline{\operatorname{sp}}\, C$ is (GL). It follows from the construction of C that we have

$$\forall x, y \in \operatorname{sp}_{\mathbb{Q}} C\ \forall \varepsilon \in \mathbb{Q}_+ : \quad \{I_2(x, y, \varepsilon), I_3(x, y, \varepsilon)\} \subset V. \tag{5}$$

We will verify the assumption of Lemma 9 with $D = \operatorname{sp}_{\mathbb{Q}} C$ for the space V. Fix $x \in \operatorname{sp}_{\mathbb{Q}} C$ and $\varepsilon \in \mathbb{Q}_+$ and consider $x^* := \frac{I_1(x,\varepsilon)|_V}{\|I_1(x,\varepsilon)|_V\|}$. Then $\frac{x}{\|x\|} \in S\big(I_1(x, \varepsilon), \varepsilon\big) \cap V \subset S(x^*, \varepsilon)$. Fix any $y \in \operatorname{sp}_{\mathbb{Q}} C$. Then we have $\{I_2(x, y, \varepsilon), I_3(x, y, \varepsilon)\} \subset S\big(I_1(x, \varepsilon), \varepsilon\big) \cap V \subset S(x^*, \varepsilon)$ and

$$\left\|\tfrac{y}{\|y\|} - I_2(x, y, \varepsilon)\right\| + \left\|\tfrac{y}{\|y\|} + I_3(x, y, \varepsilon)\right\| < 2 + \varepsilon.$$

Hence,

$$\operatorname{dist}\left(\frac{y}{\|y\|}, S\big(x^*, \varepsilon\big)\right) + \operatorname{dist}\left(\frac{y}{\|y\|}, -S\big(x^*, \varepsilon\big)\right) < 2 + \varepsilon,$$

which shows that the assumption of Lemma 9 is satisfied for the space V and so V is (GL). $\square$

Remark 11. It is known to the author that it is possible to prove Proposition 10 using the method of suitable models (more precisely, the family $\mathcal{R}$ is not only cofinal, but it consists of all the spaces of the form $\overline{X \cap M}$ where M is a suitable countable model); see [4, Section 4] and references therein for a description of the method. Since, by [5, Theorem 2.7] and [4, Theorem 5.5], this method is equivalent to the method of rich families in many classes of Banach spaces (e.g. in Asplund spaces), it is possible to obtain a rich family instead of a cofinal one in Proposition 10 for many classes of Banach spaces. In particular, in Theorem 5 we have an equivalence. However, the proof involving suitable models would require quite a deep understanding of the method and we do not know any application of such a result; hence, we decided to present here only a weaker statement, Proposition 10, which has the advantage that the proof does not require any knowledge of set theory or logic.

We do not know whether we can get a rich family instead of a cofinal one in Proposition 10 for an arbitrary Banach space.

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