ON THE WEAK CONTINUITY OF THE MOSER FUNCTIONAL IN LORENTZ-SOBOLEV SPACES

ROBERT ČERNÝ

ABSTRACT. Let $B(R) \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, be an open ball. By a result from [1], the Moser functional with the borderline exponent from the Moser-Trudinger inequality fails to be sequentially weakly continuous on the set of radial functions from the unit ball in $W_0^{1,n}(B(R))$ only in the exceptional case of sequences acting like a concentrating Moser sequence.

We extend this result into Lorentz-Sobolev space $W_0^1 L^{n,q}(B(R))$, where $q \in (1, n]$, equipped with the norm

$$||\nabla u||_{n,q} := ||t^{\frac{1}{n}-\frac{1}{q}} |\nabla u|^*(t)||_{L^q((0,|B(R)|))}.$$

We also consider the case of a nontrivial weak limit and the corresponding Moser functional with the borderline exponent from the Concentration-Compactness Alternative.

1. Introduction

Throughout the paper, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, $\mathcal{L}_n$ is the n-dimensional Lebesgue measure and $|\Omega|$ stands for $\mathcal{L}_n(\Omega)$. By ∇u we denote the generalized gradient of a function u and u^* is its non-increasing rearrangement. The space $W_0^{1,n}(\Omega)$ or $W_0^1 L^{n,q}(\Omega)$, $q \in (1, \infty)$, stands for the closure of $C_0^\infty(\Omega)$ in $W^{1,n}(\Omega)$ or $W^1 L^{n,q}(\Omega)$, respectively. We use the standard notation $q' = \frac{q}{q-1}$ (with the convention that $\infty' = 1$ and $1' = \infty$).

For functions from $W_0^{1,n}(\Omega)$ the famous Moser-Trudinger inequality [11] concerning a classical embedding theorem by Trudinger [13] states that

$$\sup_{||\nabla u||_{L^n(\Omega)} \leq 1} \int_\Omega \exp((K|u(x)|)^{n'})\, dx \begin{cases} \leq C(n, K, |\Omega|) & \text{when } K \leq n\omega_n^{\frac{1}{n}} \\ = \infty & \text{when } K > n\omega_n^{\frac{1}{n}}. \end{cases} \tag{1.1}$$

The proof in the case of $K > n\omega_n^{\frac{1}{n}}$ easily follows from the properties of the Moser functions $m_s \in W_0^{1,n}(B(R))$, $s \in (0, 1)$, defined by

$$m_s(x) = \begin{cases} n^{-\frac{1}{n}} \omega_n^{-\frac{1}{n}} \log^{\frac{1}{n'}}(\frac{1}{t}) & \text{for } |x| \in [0, sR] \\ n^{-\frac{1}{n}} \omega_n^{-\frac{1}{n}} \log^{-\frac{1}{n}}(\frac{1}{t}) \log(\frac{R}{|x|}) & \text{for } |x| \in [sR, R]. \end{cases} \tag{1.2}$$

From (1.1) and the Vitali Convergence Theorem (see e.g. [7, page 187]), it follows that if $p < 1$, then the functional

$$J_p(u) = \int_\Omega \exp((n\omega_n^{\frac{1}{n}} p|u(x)|)^{n'})\, dx$$

is sequentially weakly continuous on the unit ball in $W_0^{1,n}(\Omega)$. That is,

$$u_k \rightharpoonup u \quad \text{and} \quad ||\nabla u_k||_{L^n(\Omega)} \leq 1 \qquad \Longrightarrow \qquad J_p(u_k) \to J_p(u).$$

If $p \geq 1$, then it is well-known and easy to check that the above implication is not true. Indeed, if $p > 1$ and Ω contains the origin, we fix $R > 0$ such that $B(R) \subset \Omega$ and we obtain $J_p(m_s) \to \infty$ as $s \to 0$, while for every sequence $s_k \subset (0, 1)$, such that $s_k \to 0$, we have $m_{s_k} \rightharpoonup 0$ and $J_p(0) = \mathcal{L}_n(\Omega) < \infty$ (in the case of $0 \notin \Omega$, we use translated Moser functions). If $p = 1$, we fix $R > 0$, we set $\Omega = B(R)$ and it is easy to check that there are $C_0 > \mathcal{L}_n(B(R)) = J_1(0)$ and $t_0 \in (0, 1)$ such that $J_1(m_s) \geq C_0$ for every $s \in (0, t_0)$.

Date: January 8, 2015.

2000 *Mathematics Subject Classification.* 46E35, 46E30, 26D10.

Key words and phrases. Sobolev spaces, Lorentz-Sobolev spaces, Moser-Trudinger inequality, Concentration-Compactness Principle, Sharp constants.

In recent paper [1] the following characterization of the sequential weak continuity of the functional J_p concerning the case of $p = 1$ and $u_k \rightharpoonup 0$, where u_k are radial functions from $W_0^{1,n}(B(R))$, is given.

Theorem 1.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and $R > 0$. Suppose that $\{u_k\} \subset W_0^{1,n}(B(R))$ are radial functions such that $||\nabla u_k||_{L^n(B(R))} \leq 1$ and $u_k \rightharpoonup 0$ in $W_0^{1,n}(B(R))$. If*

$$\limsup_{k\to\infty} J_1(u_k) > J_1(0),$$

then there are $\{u_{k_m}\} \subset \{u_k\}$ and $\{s_m\} \subset (0,1)$, $s_m \to 0$, such that

$$u_{k_m} - m_{s_m} \overset{m\to\infty}{\rightarrow} 0 \qquad \text{in } W_0^{1,n}(B(R)).$$

In fact, Theorem 1.1 gives some information only in the case of $u = 0$ a.e. Otherwise (i.e. when u is nontrivial), Theorem 1.2 below and the Vitali Convergence Theorem imply $\lim_{k\to\infty} J_1(u_k) = J_1(u)$.

Let us note that, in paper [1], a more difficult version of Theorem 1.1 concerning the case of non-radial functions on an open set $\Omega \subset \mathbb{R}^2$ is given. In that case, one has to consider a translated Moser sequence. It is an open problem whether some analogue of the result as Theorem 1.1 for non-radial functions in the general dimension $n \geq 2$ holds.

If $p > 1$ and $u_k \rightharpoonup u$ (we do not mind whether u is trivial or not), then there are many sequences distant from $\{m_{s_k}\}$ such that $J_p(u_k) \to \infty$ while we always have $J_p(u) < \infty$ by the Trudinger embedding (for example, fix any $\varrho \in [1,p)$ and consider $u_k = \varrho^{-\frac{n-1}{n}} m_{s_k}$, with $s_k \to 0$, then one can observe that $u_k \rightharpoonup 0$ in $W_0^{1,n}(B(R))$).

A natural question to ask is what happens if the limit function u in Theorem 1.1 is nontrivial. This question was answered in paper [3]. The result is the following. If $0 \leq ||\nabla u||_{L^n(B(R))} < 1$, then there is $P > 1$ depending on $||\nabla u||_{L^n(B(R))}$ such that the functional J_P behaves in a similar way as the one in Theorem 1.1, while for every $p < P$ we have $J_p(u_k) \to J_p(u)$ and for every $p > P$ we generally do not have that $\{J_p(u_k)\}$ is a bounded sequence. On the other hand, if $||\nabla u||_{L^n(B(R))} = 1$, it is easy to see that $u_k \to u$ (in norm) and $J_p(u_k) \to J_p(u)$ for every $p \in \mathbb{R}$.

The above mentioned constant P is the borderline exponent corresponding to the following result from [5] and [9, Theorem I.6 and Remark I.18] which concerns one of the cases in the Concentration-Compactness Alternative for the Moser-Trudinger inequality.

Theorem 1.2. *Let $n \in \mathbb{N}$, $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a sequence satisfying*

$$||\nabla u_k||_{L^n(\Omega)} \leq 1, \qquad u_k \rightharpoonup u \quad \text{in } W_0^{1,n}(\Omega) \qquad \text{and} \qquad u_k \to u \quad \text{a.e. in } \Omega$$

for some non-trivial function $u \in W_0^{1,n}(\Omega)$. Let us set

$$\theta = ||\nabla u||^n_{L^n(\Omega)} \in (0,1] \qquad \text{and} \qquad P = (1-\theta)^{-\frac{1}{n}} \tag{1.3}$$

(where we read $P = \infty$ if $\theta = 1$). Then for every $p < P$ there is $C > 0$ such that

$$\int_\Omega \exp((n\omega_n^{\frac{1}{n}} p\,|u_k(x)|)^{n'})\,dx \leq C.$$

Moreover, such an upper bound for p is sharp.

In the version of Theorem 1.1 with a nontrivial weak limit, it is natural to work with the functional J_p where $p = P$. Indeed, if $p < P$, we can again use the Vitali Convergence Theorem. Furthermore, it is shown in [5], that if we take a suitable function $u \in W_0^{1,n}(B(3R))$ and if we set

$$u_k = u + (1-\theta)^{\frac{1}{n}} m_{\frac{1}{k}},$$

then we have $||\nabla u_k||_{L^n(B(3R))} = 1$, $u_k \rightharpoonup u$ and $J_p(u_k) \to \infty$ for every $p > P$. Hence for $p > P$, we can again construct many sequences such that $u_k \rightharpoonup u$ and $J_p(u_k) \to \infty$, while $J_p(u) < \infty$.

Now, let us recall the full statement of the main result of [3].

Theorem 1.3. *Let $n \in \mathbb{N}$, $n \geq 2$ and $R > 0$. Let $\{u_k\} \subset W_0^{1,n}(B(R))$ be radial functions such that $||\nabla u_k||_{L^n(B(R))} \leq 1$ and $u_k \rightharpoonup u$ in $W_0^{1,n}(B(R))$. Let $\theta \in [0,1]$ and $P \in [1,\infty]$ be defined by (1.3). If $\theta < 1$ and*

$$\limsup_{k\to\infty} J_P(u_k) > J_P(u),$$

then there are $\{u_{k_m}\} \subset \{u_k\}$ and $\{s_m\} \subset (0,1)$, $s_m \to 0$, such that

$$u_{k_m} - u - (1-\theta)^{\frac{1}{n}} m_{s_m} \overset{m\to\infty}{\to} 0 \qquad \text{in } W_0^{1,n}(B(R)).$$

Lorentz-Sobolev case. The aim of this paper is to extend Theorem 1.3 into Lorentz-Sobolev spaces $W_0^1 L^{n,q}(\Omega)$, where $q \in (1,n]$, equipped with the norm

$$||\nabla u||_{n,q} := ||t^{\frac{1}{n}-\frac{1}{q}} |\nabla u|^*(t)||_{L^q((0,|\Omega|))}. \tag{1.4}$$

Let us recall that the above quantity is not a norm for $q > n$ (it is a quasi-norm).

The Moser-type inequality for Lorentz-Sobolev spaces $W_0^1 L^{n,q}(\Omega)$ was obtained in [2] and it has the following form. If $q \in (1,\infty)$, then

$$\sup_{||\nabla u||_{n,q}\leq 1} \int_\Omega \exp((K|u(x)|)^{q'})\, dx \begin{cases} \leq C(n,K,q,|\Omega|) & \text{when } K \leq n\omega_n^{\frac{1}{n}} \\ = \infty & \text{when } K > n\omega_n^{\frac{1}{n}} \end{cases}$$

and if $q = \infty$, then

$$\sup_{||\nabla u||_{n,\infty}\leq 1} \int_\Omega \exp(K|u(x)|)\, dx \begin{cases} \leq C(n,K,|\Omega|) & \text{when } K < n\omega_n^{\frac{1}{n}} \\ = \infty & \text{when } K \geq n\omega_n^{\frac{1}{n}}. \end{cases}$$

Notice that, since $\infty' = 1$, the main difference between cases $q \in (1,\infty)$ and $q = \infty$ is the uniform boundedness of the integrals in the case $K = n\omega_n^{\frac{1}{n}}$ for $q \in (1,\infty)$. There is no Moser-type inequality for $q = 1$, since $W_0^1 L^{n,1}(\Omega)$ is embedded into $L^\infty(\Omega)$.

We define the following Moser functionals

$$J_p(u) = \int_\Omega \exp((n\omega_n^{\frac{1}{n}} p|u(x)|)^{q'})\, dx \tag{1.5}$$

and for $R > 0$ fixed and every $s \in (0,1)$, we define the Moser function $m_s \in W_0^1 L^{n,q}(B(R))$ by

$$m_s(x) = \begin{cases} n^{-\frac{1}{q}} \omega_n^{-\frac{1}{n}} \log^{\frac{q-1}{q}}(\frac{1}{s}) & \text{for } 0 \leq |x| \leq sR \\ n^{-\frac{1}{q}} \omega_n^{-\frac{1}{n}} \log^{-\frac{1}{q}}(\frac{1}{s}) \log(\frac{R}{|x|}) & \text{for } sR \leq |x| \leq R. \end{cases} \tag{1.6}$$

Now, let us recall the result from [4] concerning the improvement of the Moser-Trudinger inequality in the case of a nontrivial weak limit.

Theorem 1.4. *Let $n \in \mathbb{N}$, $n \geq 2$, $q \in (1,\infty)$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Let $u \in W_0^1 L^{n,q}(\Omega)$ be a non-trivial function and let $\{u_k\} \subset W_0^1 L^{n,q}(\Omega)$ be a sequence such that*

$$||\nabla u_k||_{n,q} \leq 1, \qquad u_k \rightharpoonup u \text{ in } W_0^1 L^{n,q}(\Omega) \qquad \text{and} \qquad u_k \to u \text{ a.e. in } \Omega.$$

Let us set

$$P := \begin{cases} \left(1 - ||\nabla u||_{n,q}^q\right)^{-\frac{1}{q}} & \text{for } ||\nabla u||_{n,q} < 1 \\ \infty & \text{for } ||\nabla u||_{n,q} = 1. \end{cases}$$

If $q \in (1,n]$, then for every $p < P$ there is $C > 0$ such that

$$\int_\Omega \exp\Big(\Big(n\omega_n^{\frac{1}{n}} p|u_k(x)|\Big)^{q'}\Big)\, dx \leq C \qquad \text{for every } k \in \mathbb{N}. \tag{1.7}$$

Moreover, the assumption $p < P$ is sharp.

If $q \in (n,\infty)$, then there is $\tilde{P} \in (1,P]$ such that (1.7) holds for every $p < \tilde{P}$, but we do not have $\tilde{P} = P$ in general.

Notice that in the case $q \in (1,n]$ (in this case the quantity (1.4) is a norm) the result is of the same type as Theorem 1.2. On the other hand, when $q \in (n,\infty)$, the fact that the quantity (1.4) is not weakly lower semicontinuous (see [4, Lemma 3.1]) entails some loss of integrability.

Now, let us state our new result concerning the sequential weak continuity of the functional J_P.

Theorem 1.5. *Let $n \in \mathbb{N}$, $n \geq 2$, $q \in (1,n]$ and let $R > 0$. Let $\{u_k\} \subset W_0^1 L^{n,q}(B(R))$ be a sequence of radial functions satisfying*

$$||\nabla u_k||_{n,q} \leq 1 \qquad \text{and} \qquad u_k \rightharpoonup u \quad \text{in } W_0^1 L^{n,q}(B(R))$$

for some $u \in W_0^1 L^{n,q}(B(R))$. Let us set

$$\theta := ||\nabla u||_{n,q}^q \in [0,1] \qquad \text{and} \qquad P = (1-\theta)^{-\frac{1}{q}} \in [1,\infty]. \tag{1.8}$$

If $\theta < 1$ and

$$\limsup_{k\to\infty} J_P(u_k) > J_P(u),$$

then there are $\{u_{k_m}\} \subset \{u_k\}$ and $\{s_m\} \subset (0,1)$, $s_m \to 0$, such that

$$u_{k_m} - u - (1-\theta)^{\frac{1}{q}} m_{s_m} \overset{m\to\infty}{\to} 0 \qquad \text{in } W_0^1 L^{n,q}(B(R)).$$

Again, it can be easily seen (with the aid of the Vitali Convergence Theorem) that if $\theta = 1$, then $J_p(u_k) \to J_p(u)$ for every $p \in \mathbb{R}$.

We do not study the case $q > n$ for two reasons. On one hand, we do not know the value of the borderline parameter $\tilde{P}$ from Theorem 1.4. On the other hand, for $q > n$ the quantity (1.4) is not a norm and we lose such tools as the uniform convexity.

The paper is organized as follows. After Preliminaries we show that if $q \leq n$, then the norm (1.4) is uniformly convex. Such a result was already proved by Halperin [8], however his definition of the uniform convexity slightly differs from the classical one by Clarkson [6] which is the definition which is useful for our purposes. In Section 4 we derive some properties of the Moser functions from (1.6).

Section 5 contains construction and properties of a collection of auxiliary linear functionals that are used to estimate the distance from Moser functions. Let us recall that in paper [1] (the Sobolev case) a suitable functionals were

$$L_s(u) = \int_{B(R)} |\nabla m_s|^{n-2} \nabla m_s \cdot \nabla u \, dx, \quad s \in (0,1)$$

(for $n = 2$ it is just a scalar product of the gradients) satisfying in addition an important property

$$L_s(u) = \frac{h(Rs)}{g_s(Rs)},$$

where $h, g_s \colon (0,R) \mapsto \mathbb{R}$ are the one-dimensional representatives of radial functions u and m_s, respectively (the above identity is easily obtained using the definition of the Moser functions (1.2) and the Newton formula). In the case of the Lorentz-Sobolev spaces we had to modify these functionals so that they are corresponding to the norm (1.4). The resulting functionals are given in (5.1) (we also had to overcome the fact that the weight $t \mapsto t^{\frac{q}{n}-1}$ has a bit wild behavior near the origin). Notice that our functionals do not use the non-increasing rearrangement (surprisingly, since it is involved in the norm (1.4)), but this defect is repaired by the fact that $-g_s'$ is positive and decreasing on (sR,R) and the Hardy-Littlewood inequality ensures that $L_s(u)$ is large only if $-h'$ behaves in a similar way.

In the last section we conclude the proof of Theorem 1.5. The basic strategy of the proof is inspired by [1], the problems arising when dealing with nontrivial limit functions are solved in the same way as in [3]. However, there also occurred some problems related to the non-increasing rearrangement involved in the norm (1.4) and the solution to these problems required some new ideas.

2. Preliminaries

Notation. If u is a measurable function on Ω, then by $u = 0$ (or $u \neq 0$) we mean that u is equal (or not equal) to the zero function a.e. on Ω.

By $B(x,R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with the radius $R > 0$. If $x = 0$, we simply write $B(R)$.

We write that $u_k \rightharpoonup u$ in $W_0^1 L^{n,q}(\Omega)$, $q \in (1,\infty)$, if

$$\int_\Omega \frac{\partial u_k}{\partial x_i} v\, dx \to \int_\Omega \frac{\partial u}{\partial x_i} v\, dx \qquad \text{for every } v \in L^{n',q'}(\Omega) \text{ and } i = 1,\dots,n.$$

By C we denote a generic positive constant which may depend on n, q, R and p. This constant may vary from expression to expression as usual. Sometimes we say that for every $\varepsilon > 0$ something is true. Then the constants C in such a case may depend also on fixed $\varepsilon > 0$.

Non-increasing rearrangement. The non-increasing rearrangement f^* of a measurable function f on Ω is

$$f^*(t) = \sup\Big\{s \geq 0 : \big|\{x \in \Omega : |f(x)| > s\}\big| > t\Big\} \qquad \text{for } t \in (0,\infty).$$

We are going to use the Hardy-Littlewood inequality for measurable functions

$$\int_\Omega |f(x)g(x)|\, dx \leq \int_0^{|\Omega|} f^*(t) g^*(t)\, dt.$$

When dealing with a radial function u on $B(R)$, it is often convenient for us to work with its one-dimensional representative $h : (0,R) \mapsto [0,\infty)$ defined by

$$h(|x|) := u(x) \qquad \text{for } 0 < |x| < R. \tag{2.1}$$

Remark 2.1. For every radial function $u \in W_0^{1,1}(\Omega)$, its one-dimensional representative h from (2.1) is locally absolutely continuous on $(0,R)$ (and thus differentiable almost everywhere).

Proof. The proof easily follows from the fact that every function from $W^{1,1}(\Omega)$ satisfies ACL, i.e. it is absolutely continuous on almost all lines parallel to coordinate axes (see [10, Section 1.1.3]). □

Finally, let us recall an inequality obtained in [12]. If Ω is open and $u \in W_0^{1,1}(\Omega)$, then

$$u^*(t) \leq \frac{1}{n\omega_n^{\frac{1}{n}}} \left(t^{-\frac{1}{n'}} \int_0^t |\nabla u|^*(s)\, ds + \int_t^{|\Omega|} |\nabla u|^*(s) s^{-\frac{1}{n'}}\, ds \right) \qquad \text{for every } t \in (0,|\Omega|). \tag{2.2}$$

If Ω is bounded, combining (2.2) with Hölder's inequality, $||\nabla u||_{n,q} \leq 1$ and $((\frac{1}{q} - \frac{1}{n})q' + 1)\frac{1}{q'} = \frac{n-1}{n}$ we obtain

$$\begin{aligned}
u^*(t) &\leq \frac{1}{n\omega_n^{\frac{1}{n}}} \left(t^{-\frac{1}{n'}} \int_0^t |\nabla u|^*(s) s^{\frac{1}{n}-\frac{1}{q}} s^{\frac{1}{q}-\frac{1}{n}}\, ds + \int_t^{|\Omega|} |\nabla u|^*(s) s^{\frac{1}{n}-\frac{1}{q}} s^{\frac{1}{q}-1}\, ds \right) \\
&\leq \frac{1}{n\omega_n^{\frac{1}{n}}} \left(t^{-\frac{1}{n'}} \left(\int_0^t (|\nabla u|^*(s) s^{\frac{1}{n}-\frac{1}{q}})^q\, ds \right)^{\frac{1}{q}} \left(\int_0^t (s^{\frac{1}{q}-\frac{1}{n}})^{q'}\, ds \right)^{\frac{1}{q'}} \right. \\
&\qquad \left. + \left(\int_t^{|\Omega|} (|\nabla u|^*(s) s^{\frac{1}{n}-\frac{1}{q}})^q\, ds \right)^{\frac{1}{q}} \left(\int_t^{|\Omega|} (s^{\frac{1}{q}-1})^{q'}\, ds \right)^{\frac{1}{q'}} \right) \\
&\leq \frac{1}{n\omega_n^{\frac{1}{n}}} \left(||\nabla u||_{n,q} t^{-\frac{n-1}{n}} \left(\frac{1}{(\frac{1}{q} - \frac{1}{n})q' + 1} \Big[s^{(\frac{1}{q}-\frac{1}{n})q'+1} \Big]_0^t \right)^{\frac{1}{q'}} + ||\nabla u||_{n,q} \left(\Big[\log(s) \Big]_t^{|\Omega|} \right)^{\frac{1}{q'}} \right) \\
&\leq C + \frac{1}{n\omega_n^{\frac{1}{n}}} \log^{\frac{1}{q'}} \Big(\frac{|\Omega|}{t} \Big).
\end{aligned} \tag{2.3}$$

Notice that (2.3) implies that for any $\varepsilon > 0$ we have

$$u^*(t) \leq \begin{cases} (1+\varepsilon) n^{-1} \omega_n^{-\frac{1}{n}} \log^{\frac{1}{q'}} (\frac{|\Omega|}{t}) & \text{for } t \text{ sufficiently small} \\ C & \text{otherwise.} \end{cases}$$

3. Uniform convexity

Clarkson [6] has defined the uniform convexity in the following way.

Definition 3.1. A Banach space is uniformly convex if for every $\varepsilon > 0$ there is $\delta > 0$ with the following property: if $||f|| = ||g|| = 1$ and $||f - g|| > \varepsilon$, then $||\frac{1}{2}(f + g)|| < 1 - \delta$.

In this paper we also use the uniform monotonicity.

Definition 3.2. A Banach space is uniformly monotone if for every $\varepsilon > 0$ there is $\eta > 0$ with the following property: if $0 \le g \le f$, $||f|| = 1$ and $||g|| > \varepsilon$, then $||f - g|| < 1 - \eta$.

It is an easy exercise to show that the uniform convexity implies the uniform monotonicity.
Halperin [8] has proved that Lorentz spaces have the following property.

Theorem 3.3. *Let $1 < q \le p < \infty$. For every $\varepsilon > 0$ and $\eta \in (0, 1)$ there is $\delta > 0$ with the following property: whenever two non-negative Lorentz functions satisfy $||u||_{p,q} = ||v||_{p,q} = 1$ and $(1 - \eta)u(x) \ge v(x)$ in some set G with $||u\chi_G||_{p,q} > \varepsilon$, then $||\frac{1}{2}(u + v)||_{p,q} < 1 - \delta$.*

Our aim is to prove that the Halperin property implies the uniform convexity.

Corollary 3.4. *If $1 < q \le p < \infty$, then the Lorentz norm is uniformly convex.*

Proof. Step 1. (uniform convexity for non-negative functions)
Fix $\varepsilon > 0$, set $\eta = \varepsilon$ and let $\delta > 0$ be the constant given by Theorem 3.3. Let u, v be two non-negative Lorentz functions satisfying $||u||_{p,q} = ||v||_{p,q} = 1$ and $||\frac{1}{2}(u + v)||_{p,q} \ge 1 - \delta$. Hence the set $G_u := \{(1 - \varepsilon)u \ge v\}$ satisfies $||u\chi_{G_u}||_{p,q} \le \varepsilon$ and the set $G_v := \{(1 - \varepsilon)v \ge u\}$ satisfies $||v\chi_{G_v}||_{p,q} \le \varepsilon$. Thus

$$\begin{aligned} ||u - v||_{p,q} &= ||(u - v)\chi_{G_u}||_{p,q} + ||(v - u)\chi_{G_v}||_{p,q} + ||(u - v)\chi_{\Omega\setminus(G_u\cup G_v)}||_{p,q} \\ &\le ||u\chi_{G_u}||_{p,q} + ||v\chi_{G_v}||_{p,q} + ||\varepsilon u||_{p,q} + ||\varepsilon v||_{p,q} \le 4\varepsilon. \end{aligned}$$

Step 2 (uniform monotonicity)
Fix $\varepsilon \in (0, 1)$, let $\delta > 0$ be the number corresponding to $\frac{\varepsilon}{2}$ in the definition of the uniform convexity (for non-negative functions) and let f, g be the same as in Definition 3.2. Let us set $u = f$ and $v = \frac{f-g}{||f-g||_{p,q}} \ge 0$. We can assume that $||f - g||_{p,q} > 1 - \frac{\varepsilon}{2}$, otherwise we are done setting $\eta = \frac{\varepsilon}{2}$.
Hence

$$\begin{aligned} ||u - v||_{p,q} &= ||(1 - \tfrac{1}{||f-g||_{p,q}})f + \tfrac{1}{||f-g||_{p,q}}g||_{p,q} = \tfrac{1}{||f-g||_{p,q}}||g - (1 - ||f - g||_{p,q})f||_{p,q} \\ &\ge ||g - (1 - ||f - g||_{p,q})f||_{p,q} \ge ||g||_{p,q} - (1 - ||f - g||_{p,q})||f||_{p,q} > \varepsilon - \tfrac{1}{2}\varepsilon = \tfrac{1}{2}\varepsilon. \end{aligned}$$

Thus, by the uniform convexity for non-negative functions

$$\begin{aligned} 1 - \delta > ||\tfrac{1}{2}(u + v)||_{p,q} &= \tfrac{1}{||f-g||_{p,q}}||\tfrac{||f-g||_{p,q}+1}{2}f - \tfrac{1}{2}g||_{p,q} \ge \tfrac{1}{||f-g||_{p,q}}||\tfrac{||f-g||_{p,q}+1}{2}f - \tfrac{||f-g||_{p,q}+1}{2}g||_{p,q} \\ &= \tfrac{||f-g||_{p,q}+1}{2}. \end{aligned}$$

Therefore $||f - g||_{p,q} < 1 - 2\delta$ and we can set $\eta = 2\delta$.
Step 3. (uniform convexity for general functions)
Fix $\varepsilon > 0$ and let $\eta > 0$ be the number from Step 2 corresponding to $\frac{1}{3}\varepsilon$. Let u, v be two Lorentz functions satisfying $||u||_{p,q} = ||v||_{p,q} = 1$ and $||u - v||_{p,q} > \varepsilon$. We distinguish two cases.
Case 1. If $||\,|u| - |v|\,||_{p,q} > \max\{\frac{\varepsilon}{6}, \frac{\eta}{2}\}$, then, by Step 1, we can use the uniform convexity for non-negative functions $|u|, |v|$ to obtain $\delta > 0$ such that

$$1 - \delta > \|\tfrac{1}{2}(|u| + |v|)\|_{p,q} \ge \|\tfrac{1}{2}(u + v)\|_{p,q}$$

and we are done.
Case 2. Otherwise, we can plainly suppose that $u \ge 0$ and $v = v_+ - v_-$, where $v_+, v_- \ge 0$. To simplify our notation, let us write $u = u_1 + u_2$, where $u_1 = u\chi_{\{v\ge0\}}$ and $u_2 = u\chi_{\{v<0\}}$.
Now, as $||\,|u| - |v|\,||_{p,q} \le \frac{\varepsilon}{6}$, we have

$$\begin{aligned} \varepsilon < ||u - v||_{p,q} &= ||u_1 + u_2 - v_+ + v_-||_{p,q} \le ||u_1 - v_+||_{p,q} + ||u_2 - v_-||_{p,q} + 2||v_-||_{p,q} \\ &\le \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + 2||v_-||_{p,q}. \end{aligned}$$

Hence $||v_-||_{p,q} > \frac{1}{3}\varepsilon$ and thus the uniform monotonicity implies $||v_+||_{p,q} \le 1-\eta$ (see the first line of Step 3). Therefore, as $||\,|u|-|v|\,||_{p,q} \le \frac{\eta}{2}$,

$$\begin{aligned}||\tfrac{1}{2}(u+v)||_{p,q} &= \tfrac{1}{2}||u_1+u_2+v_+-v_-||_{p,q} \le \tfrac{1}{2}\Big(||u_1-v_+||_{p,q}+2||v_+||_{p,q}+||u_2-v_-||_{p,q}\Big)\\ &\le \tfrac{1}{2}\Big(\tfrac{\eta}{2}+2(1-\eta)+\tfrac{\eta}{2}\Big) = 1-\tfrac{1}{2}\eta\end{aligned}$$

and we are done also in the second case. □

Remark 3.5. If $1 \le p < q \le \infty$, then the Lorentz quasi-norm is not uniformly convex.

Proof. Fix $\delta > 0$ very small. Let us define $u, v : [0,1) \mapsto [0,\infty)$ by

$$u(t) = \begin{cases} 1+\delta & \text{for } t \in [0,\frac{1}{2}) \\ 1-\delta & \text{for } t \in [\frac{1}{2},1) \end{cases}$$

and

$$v(t) = \begin{cases} 1-\delta & \text{for } t \in [0,\frac{1}{2}) \\ 1+\delta & \text{for } t \in [\frac{1}{2},1). \end{cases}$$

Then we plainly have $\frac{u+v}{2} \equiv 1$. Now, let us estimate the quasi-norms.

Case $q \in (p,\infty)$**.** We have

$$||\tfrac{u+v}{2}||^q_{p,q} = \int_0^1 t^{\frac{q}{p}-1}\,dt = \frac{p}{q}\Big[t^{\frac{q}{p}}\Big]_0^1 = \frac{p}{q}$$

and

$$\begin{aligned}||u||^q_{p,q} = ||v||^q_{p,q} &= \int_0^{\frac{1}{2}} t^{\frac{q}{p}-1}(1+\delta)^q\,dt + \int_{\frac{1}{2}}^1 t^{\frac{q}{p}-1}(1-\delta)^q\,dt\\ &= \frac{p}{q}(1+\delta)^q\Big[t^{\frac{q}{p}}\Big]_0^{\frac{1}{2}} + \frac{p}{q}(1-\delta)^q\Big[t^{\frac{q}{p}}\Big]_{\frac{1}{2}}^1\\ &= \frac{p}{q}\Big((1+\delta)^q\Big(\frac{1}{2}\Big)^{\frac{q}{p}} + (1-\delta)^q\Big(1-\Big(\frac{1}{2}\Big)^{\frac{q}{p}}\Big)\Big).\end{aligned}$$

Hence, if $\delta > 0$ is small enough, we have

$$\begin{aligned}\frac{q}{p}\Big(||u||^q_{p,q} - ||\tfrac{u+v}{2}||^q_{p,q}\Big) &= (1+\delta)^q\Big(\frac{1}{2}\Big)^{\frac{q}{p}} + (1-\delta)^q\Big(1-\Big(\frac{1}{2}\Big)^{\frac{q}{p}}\Big) - 1\\ &= \Big(\frac{1}{2}\Big)^{\frac{q}{p}}\Big(1+q\delta-1+q\delta+\mathrm{o}(\delta)\Big) + 1 - q\delta + \mathrm{o}(\delta) - 1\\ &= q\delta\Big(\Big(\frac{1}{2}\Big)^{\frac{q}{p}-1} - 1 + \mathrm{o}(1)\Big).\end{aligned}$$

Now, since $(\frac{1}{2})^{\frac{q}{p}-1} < 1$, setting $\tilde{u} = \frac{u}{||u||_{p,q}}$ and $\tilde{v} = \frac{v}{||v||_{p,q}}$ we obtain $\|\tilde{u}\| = \|\tilde{v}\| = 1$ and $||\frac{\tilde{u}+\tilde{v}}{2}|| > 1$. Nevertheless, the uniform convexity requires the last number to be bounded away from 1 from below.

Case $q = \infty$**.** In this case we easily see that

$$||\tfrac{u+v}{2}||_{p,\infty} = \sup_{t\in(0,1)} t^{\frac{q}{p}} = 1$$

and, if δ is small enough, we obtain

$$||u||_{p,\infty} = ||v||_{p,\infty} = \max\Big\{\sup_{t\in(0,\frac{1}{2})} t^{\frac{1}{p}}(1+\delta), \sup_{t\in(\frac{1}{2},1)} t^{\frac{1}{p}}(1-\delta)\Big\} = 1-\delta.$$

Thus, we are done. □

It is a well-known fact that if a sequence converges weakly in a uniformly convex Banach space, that is $u_k \rightharpoonup u$, and $||u_k|| \to ||u||$ (where $\|\cdot\|$ is a norm in this space), then then $u_k \to u$ (strong convergence in norm). We shall need a slight modification of this property.

Lemma 3.6. *In every uniformly convex Banach space the following assertion holds. For every $\varepsilon > 0$ there is $\delta \in (0,1)$ such that*

$$u_k \rightharpoonup u,\ ||u|| = 1,\ ||u_k|| \leq 1+\delta \text{ for every } k \quad \Longrightarrow \quad ||u_k - u|| < \varepsilon \text{ for every } k \text{ sufficiently large.}$$

Proof. The proof is standard. □

4. Moser functions

In this section we study properties of the Moser functions defined by (1.6). Let us start with the estimate of the Dirichlet norm. We have

$$|\nabla m_s|(x) = \begin{cases} 0 & \text{for } 0 \leq |x| < sR \\ n^{-\frac{1}{q}} \omega_n^{-\frac{1}{n}} \log^{-\frac{1}{q}}(\frac{1}{s})\frac{1}{|x|} & \text{for } sR < |x| < R \end{cases} \tag{4.1}$$

and thus

$$|\nabla m_s|^*(t) = \begin{cases} n^{-\frac{1}{q}} \omega_n^{-\frac{1}{n}} \log^{-\frac{1}{q}}(\frac{1}{s})\frac{1}{(\frac{t}{\omega_n}+s^nR^n)^{\frac{1}{n}}} & \text{for } 0 < t < \omega_n R^n - \omega_n s^n R^n \\ 0 & \text{for } \omega_n R^n - \omega_n s^n R^n < t < \omega_n R^n \end{cases} \tag{4.2}$$

(indeed, the value of $|\nabla m_s|^*(t)$ corresponds to the value of $|\nabla m_s|$ on the sphere $\partial B(\varrho)$, where $\varrho > 0$ satisfies $t = |B(\varrho)| - |B(sR)| = \omega_n(\varrho^n - s^nR^n)$). Hence

$$\begin{aligned} \|\nabla m_s\|^q_{n,q} &= \int_0^{\omega_n R^n} t^{\frac{q}{n}} (|\nabla m_s|^*(t))^q \frac{dt}{t} \\ &= \frac{1}{n}\log^{-1}\Big(\frac{1}{s}\Big) \int_0^{\omega_n R^n - \omega_n s^n R^n} t^{\frac{q}{n}} \Big(\frac{1}{t+\omega_n s^n R^n}\Big)^{\frac{q}{n}} \frac{dt}{t}. \end{aligned}$$

Applying the change of variables $t = \omega_n s^n R^n y$ we infer for $s > 0$ so small that $\log(\log(\frac{1}{s})) > 0$

$$\begin{aligned} \|\nabla m_s\|^q_{n,q} &= \log^{-1}\Big(\frac{1}{s^n}\Big) \int_0^{s^{-n}-1} \Big(\frac{y}{y+1}\Big)^{\frac{q}{n}} \frac{dy}{y} \\ &= \log^{-1}\Big(\frac{1}{s^n}\Big) \Big(\int_0^{\log(\log(\frac{1}{s}))} + \int_{\log(\log(\frac{1}{s}))}^{s^{-n}-1}\Big) = \log^{-1}\Big(\frac{1}{s^n}\Big)\Big(I_1 + I_2\Big). \end{aligned} \tag{4.3}$$

Next

$$0 < I_1 < \int_0^{\log(\log(\frac{1}{s}))} y^{\frac{q}{n}-1}\, dy = \frac{n}{q} \log^{\frac{q}{n}}\Big(\log\Big(\frac{1}{s}\Big)\Big)$$

and thus $I_1 \log^{-1}(\frac{1}{s^n}) \to 0$ as $s \to 0_+$. For the second integral we have

$$I_2 \leq \int_{\log(\log(\frac{1}{s}))}^{s^{-n}-1} \frac{dy}{y} = \log\Big(s^{-n}-1\Big) - \log\Big(\log\Big(\log\Big(\frac{1}{s}\Big)\Big)\Big)$$

and contrariwise

$$\begin{aligned} I_2 &\geq \Big(\frac{\log(\log(\frac{1}{s}))}{\log(\log(\frac{1}{s}))+1}\Big)^{\frac{q}{n}} \int_{\log(\log(\frac{1}{s}))}^{s^{-n}-1} \frac{dy}{y} \\ &= \Big(1 - \frac{1}{\log(\log(\frac{1}{s}))+1}\Big)^{\frac{q}{n}} \Bigg(\log\Big(s^{-n}-1\Big) - \log\Big(\log\Big(\log\Big(\frac{1}{s}\Big)\Big)\Big)\Bigg) \end{aligned}$$

Thus, $I_2 \log^{-1}(\frac{1}{s^n}) \to 1$ as $s \to 0_+$. Hence we obtain from (4.3)

$$\|\nabla m_s\|_{n,q} \overset{s\to 0_+}{\to} 1. \tag{4.4}$$

Notice that by a minor modification of the above procedure it can be shown that

$$\begin{aligned} &\int_{\omega_n s^n \log^n(\frac{1}{s})R^n - \omega_n s^n R^n}^{\omega_n R^n - \omega_n s^n R^n} t^{\frac{q}{n}} (|\nabla m_s|^*(t))^q \frac{dt}{t} \\ &\qquad = \log^{-1}\Big(\frac{1}{s^n}\Big) \int_{\log^n(\frac{1}{s})-1}^{s^{-n}-1} \Big(\frac{y}{y+1}\Big)^{\frac{q}{n}} \frac{dy}{y} \overset{s\to 0_+}{\to} 1. \end{aligned} \tag{4.5}$$

It can be also seen from (4.1) and (1.6) that the Moser functions concentrate at the origin in the following sense

$$\eta > 0 \quad \Longrightarrow \quad \sup_{\eta<|x|<R} |\nabla m_s(x)| \overset{s\to 0_+}{\to} 0 \quad \text{and} \quad \sup_{\eta<|x|<R} |m_s(x)| \overset{s\to 0_+}{\to} 0.$$

We also have for any sequence $s_k \in (0,1)$, $s_k \to 0$

$$m_{s_k} \to 0 \quad \text{on } B(R)\setminus\{0\} \qquad \text{and} \qquad m_{s_k} \rightharpoonup 0 \quad \text{in } W_0^1 L^{n,q}(B(R)).$$

5. Linear functionals

In this section we use the following notation. The function $h\colon (0,R) \mapsto \mathbb{R}$ is the one-dimensional representative of a radial function $u \in W_0^1 L^{n,q}(B(R))$ and functions $g_s\colon (0,R) \mapsto \mathbb{R}$, $s \in (0,1)$, represent m_s. That is

$$h(|x|) = u(x) \quad \text{and} \quad g_s(|x|) = m_s(x) \qquad \text{for } x \in B(R)\setminus\{0\}.$$

For every $s \in (0, \frac{1}{e})$, we define a linear functional L_s acting on a radial function $u \in W_0^1 L^{n,q}(B(R))$ by

(5.1)
$$L_s(u) = \int_{\omega_n s^n \log^n(\frac{1}{s})R^n - \omega_n s^n R^n}^{\omega_n R^n - \omega_n s^n R^n} t^{\frac{q}{n}} \Big|g_s'\Big(\Big(\frac{t}{\omega_n}+s^nR^n\Big)^{\frac{1}{n}}\Big)\Big|^{q-2} g_s'\Big(\Big(\frac{t}{\omega_n}+s^nR^n\Big)^{\frac{1}{n}}\Big) h'\Big(\Big(\frac{t}{\omega_n}+s^nR^n\Big)^{\frac{1}{n}}\Big)\,\frac{dt}{t}.$$

Changing the variables so that $z = (\frac{t}{\omega_n} + s^nR^n)^{\frac{1}{n}}$ (hence $t = \omega_n z^n - \omega_n s^n R^n$ and $\frac{dt}{dz} = n\omega_n z^{n-1}$) and using (4.1) and (1.6) we infer

$$(5.2)\qquad \begin{aligned} L_s(u) &= \int_{s\log(\frac{1}{s})R}^{R} (\omega_n z^n - \omega_n s^n R^n)^{\frac{q}{n}-1} |g_s'(z)|^{q-2} g_s'(z) h'(z) n\omega_n z^{n-1}\,dz \\ &= -\int_{s\log(\frac{1}{s})R}^{R} (\omega_n z^n - \omega_n s^n R^n)^{\frac{q}{n}-1} \Big(n^{-\frac{1}{q}}\omega_n^{-\frac{1}{n}} \log^{-\frac{1}{q}}\Big(\frac{1}{s}\Big)\frac{1}{z}\Big)^{q-1} h'(z) n\omega_n z^{n-1}\,dz \\ &= -\omega_n^{\frac{1}{n}} n^{\frac{1}{q}} \log^{-\frac{q-1}{q}}\Big(\frac{1}{s}\Big) \int_{s\log(\frac{1}{s})R}^{R} \Big(1-\frac{s^nR^n}{z^n}\Big)^{\frac{q}{n}-1} h'(z)\,dz \\ &= -\frac{1}{g_s(sR)} \int_{s\log(\frac{1}{s})R}^{R} \Big(1-\frac{s^nR^n}{z^n}\Big)^{\frac{q}{n}-1} h'(z)\,dz. \end{aligned}$$

Hence we have (recall $q \le n$ and $h(z) \to 0$ as $z \to R_-$)

$$(5.3)\qquad h\Big(s\log\Big(\frac{1}{s}\Big)R\Big) = -\int_{s\log(\frac{1}{s})R}^{R} h'(z)\,dz = L_s(u)g_s(sR) = L_s(u) n^{-\frac{1}{q}}\omega_n^{-\frac{1}{n}} \log^{\frac{q-1}{q}}\Big(\frac{1}{s}\Big).$$

On the other hand, we also have from (5.2)

$$(5.4)\qquad h\Big(s\log\Big(\frac{1}{s}\Big)R\Big) \ge \Big(1-\frac{s^nR^n}{(s\log(\frac{1}{s})R)^n}\Big)^{1-\frac{q}{n}} L_s(u)g_s(sR) \ge \Big(1-\frac{1}{\log^n(\frac{1}{s})}\Big) L_s(u)g_s(sR).$$

Let ψ be the inverse function to $s \mapsto s\log(\frac{1}{s})$ on $(0,\frac{1}{e})$. From (5.3) we obtain

$$(5.5)\qquad \begin{aligned} h(sR) &\le L_{\psi(s)}(u) n^{-\frac{1}{q}}\omega_n^{-\frac{1}{n}} \log^{\frac{q-1}{q}}\Big(\frac{1}{\psi(s)}\Big) \\ &\le L_{\psi(s)}(u) n^{-\frac{1}{q}}\omega_n^{-\frac{1}{n}} \log^{\frac{q-1}{q}}\Big(\frac{1}{s}\Big)\Big(1+\frac{C}{\log^{\frac{1}{2}}(\frac{1}{s})}\Big). \end{aligned}$$

Indeed, for $s > 0$ very small we have $s\log^{-2}(\frac{1}{s}) < \psi(s) < s\log^{-1}(\frac{1}{s})$ and thus

$$\begin{aligned} \log\Big(\frac{1}{\psi(s)}\Big) &\le \log\Big(\frac{1}{s\log^{-2}(\frac{1}{s})}\Big) = \log\Big(\frac{1}{s}\Big) + \log\Big(\log^2\Big(\frac{1}{s}\Big)\Big) \\ &= \log\Big(\frac{1}{s}\Big) + 2\log\Big(\log\Big(\frac{1}{s}\Big)\Big) = \log\Big(\frac{1}{s}\Big)\Big(1+\frac{2\log(\log(\frac{1}{s}))}{\log(\frac{1}{s})}\Big) \le \log\Big(\frac{1}{s}\Big)\Big(1+\frac{1}{\log^{\frac{1}{2}}(\frac{1}{s})}\Big). \end{aligned}$$

Lemma 5.1. *We have* $L_s(u) \le (1+\mathrm{o}(s))\|\nabla u\|_{n,q}$.

Proof. Using (5.1), Hölder's inequality, (4.4), $q \le n$, the relation between $|\nabla m_s|$ and $|\nabla m_s|^*$ (compare (4.1) and (4.2)) and the Hardy-Littlewood inequality we obtain

$$\begin{aligned}
L_s(u) &\le \int_0^{\omega_n R^n} t^{\frac{q}{n}-1}\Big|g_s'\Big(\Big(\frac{t}{\omega_n}+s^nR^n\Big)^{\frac{1}{n}}\Big)\Big|^{q-1}\Big|h'\Big(\Big(\frac{t}{\omega_n}+s^nR^n\Big)^{\frac{1}{n}}\Big)\Big|\,dt\\
&= \int_0^{\omega_n R^n} \Big(t^{\frac{q}{n}-1}\Big)^{\frac{q-1}{q}}(|\nabla m_s|^*(t))^{q-1}\Big(t^{\frac{q}{n}-1}\Big)^{\frac{1}{q}}\Big|h'\Big(\Big(\frac{t}{\omega_n}+s^nR^n\Big)^{\frac{1}{n}}\Big)\Big|\,dt\\
&\le \|\nabla m_s\|_{n,q}^{q-1}\Big(\int_0^{\omega_n R^n} t^{\frac{q}{n}-1}\Big|h'\Big(\Big(\frac{t}{\omega_n}+s^nR^n\Big)^{\frac{1}{n}}\Big)\Big|^q\,dt\Big)^{\frac{1}{q}}\\
&\le \|\nabla m_s\|_{n,q}^{q-1}\Big(\int_0^{\omega_n R^n} t^{\frac{q}{n}-1}\big((|\nabla u|\chi_{B(R)\setminus B(sR)})^*(t)\big)^q\,dt\Big)^{\frac{1}{q}}\\
&= \|\nabla m_s\|_{n,q}^{q-1}\|\nabla u\|_{n,q} \le (1+\mathrm{o}(s))\|\nabla u\|_{n,q}.
\end{aligned}$$

□

Lemma 5.2. *For every fixed radial Lorentz-Sobolev function u we have $L_s(u)\to 0$ as $s \to 0$.*

Proof. Fix $\varepsilon > 0$. By the absolute continuity of the Lebesgue integral we can find $\tau \in (0,1)$ so small that

$$\int_0^{\tau} t^{\frac{q}{n}-1}(|\nabla u|^*(t))^q\,dt < \varepsilon^q. \tag{5.6}$$

Next, we have from (5.1) and (4.1)

$$\begin{aligned}
|L_s(u)| &\le \int_{\omega_n s^n \log^n(\frac{1}{s})R^n-\omega_n s^n R^n}^{\omega_n R^n-\omega_n s^n R^n} t^{\frac{q}{n}-1}\Big(\frac{n^{-\frac{1}{q}}\omega_n^{-\frac{1}{n}}\log^{-\frac{1}{q}}(\frac{1}{s})}{(\frac{t}{\omega_n}+s^nR^n)^{\frac{1}{n}}}\Big)^{q-1}\Big|h'\Big(\Big(\frac{t}{\omega_n}+s^nR^n\Big)^{\frac{1}{n}}\Big)\Big|\,dt\\
&\le C\log^{-\frac{q-1}{q}}\Big(\frac{1}{s}\Big)\int_{\omega_n s^n \log^n(\frac{1}{s})R^n-\omega_n s^n R^n}^{\omega_n R^n-\omega_n s^n R^n} t^{\frac{1}{n}-1}\Big|h'\Big(\Big(\frac{t}{\omega_n}+s^nR^n\Big)^{\frac{1}{n}}\Big)\Big|\,dt\\
&= C\log^{-\frac{q-1}{q}}\Big(\frac{1}{s}\Big)\Bigg(\int_{\omega_n s^n \log^n(\frac{1}{s})R^n-\omega_n s^n R^n}^{\tau}+\int_{\tau}^{\omega_n R^n-\omega_n s^n R^n}\Bigg)\\
&= C\log^{-\frac{q-1}{q}}\Big(\frac{1}{s}\Big)\Big(I_1+I_2\Big).
\end{aligned}$$

From (5.6), Hölder's inequality and from the fact that the non-increasing rearrangement of the function $|h'((\frac{\cdot}{\omega_n}+s^nR^n)^{\frac{1}{n}})|$ is $(|\nabla u|\chi_{B(R)\setminus B(sR)})^*$ we obtain for s sufficiently small

$$\begin{aligned}
I_1 &\le \int_{\omega_n s^n \log^n(\frac{1}{s})R^n-\omega_n s^n R^n}^{\tau} t^{\frac{1}{n}-1}|\nabla u|^*(t)\,dt\\
&\le \Bigg(\int_0^{\tau}(|\nabla u|^*(t)t^{\frac{1}{n}-\frac{1}{q}})^q\,dt\Bigg)^{\frac{1}{q}}\Bigg(\int_{\omega_n s^n \log^n(\frac{1}{s})R^n-\omega_n s^n R^n}^{\tau}(t^{\frac{1}{q}-1})^{q'}\,dt\Bigg)^{\frac{1}{q'}}\\
&\le \varepsilon\Bigg(\Big[\log(t)\Big]_{\omega_n s^n \log^n(\frac{1}{s})R^n-\omega_n s^n R^n}^{\tau}\Bigg)^{\frac{1}{q'}}\\
&\le C\varepsilon\log^{\frac{1}{q'}}\Big(\frac{1}{s}\Big).
\end{aligned}$$

For the second integral, we use the finiteness of the Dirichlet norm of u and Hölder's inequality to infer

$$\begin{aligned} I_2 &\le \int_\tau^{\omega_n R^n - \omega_n s^n R^n} t^{\frac{1}{n}-1} |\nabla u|^*(t)\,dt \\ &\le \Big(\int_0^{\omega_n R^n} (|\nabla u|^*(t) t^{\frac{1}{n}-\frac{1}{q}})^q\,dt\Big)^{\frac{1}{q}} \Big(\int_\tau^{\omega_n R^n} (t^{\frac{1}{q}-1})^{q'}\,dt\Big)^{\frac{1}{q'}} \\ &\le C\Big(\Big[\log(t)\Big]_\tau^{\omega_n R^n}\Big)^{\frac{1}{q'}} = C\log^{\frac{1}{q'}}\Big(\frac{1}{\tau}\Big). \end{aligned}$$

Therefore

$$|L_s(u)| \le C\log^{-\frac{q-1}{q}}\Big(\frac{1}{s}\Big)\Big(I_1+I_2\Big) \le C\varepsilon + C\log^{-\frac{q-1}{q}}\Big(\frac{1}{s}\Big)\log^{\frac{1}{q'}}\Big(\frac{1}{\tau}\Big)$$

and the result follows easily. □

Lemma 5.3. *If $u_k \rightharpoonup u$, $\|\nabla u_k\|_{L^{n,q}} \le 1$ and $\limsup_{k\to\infty} J_P(u_k) > J_P(u)$, then there is a subsequence $\{u_{k_m}\} \subset \{u_k\}$ and a sequence $\{s_m\} \subset (0,1)$ such that $s_m \to 0$ and*

$$\liminf_{m\to\infty} L_{s_m}(u_{k_m}) \ge (1-\theta)^{\frac{1}{q}}.$$

Proof. We proceed by contradiction. Suppose that there are $\delta > 0$, $k_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$L_s(u_k) \le (1-2\varepsilon)(1-\theta)^{\frac{1}{q}} \qquad \text{for every } s < 2\delta \text{ and every } k \ge k_0. \tag{5.7}$$

Furthermore, we can suppose that δ is so small that

$$(1-2\varepsilon)\Big(1+\frac{C}{\log^{\frac{1}{2}}(\frac{1}{\delta})}\Big) \le 1-\varepsilon, \tag{5.8}$$

where the constant C is a fixed number coming from (5.5).

Passing to a subsequence, we can also suppose that $u_k \to u$ a.e. in $B(R)$. By (1.5) and (1.8) we have

$$\begin{aligned} J_P(u_k) &= \int_{B(R)} \exp\Big(\Big(n\omega_n^{\frac{1}{n}} P|u_k|\Big)^{q'}\Big)\,dx \\ &= n\omega_n \int_0^R \exp\Big(\Big(n\omega_n^{\frac{1}{n}}(1-\theta)^{-\frac{1}{q}} h_k(y)\Big)^{q'}\Big) y^{n-1}\,dy \\ &= n\omega_n R^n \int_0^1 \exp\Big(\Big(n\omega_n^{\frac{1}{n}}(1-\theta)^{-\frac{1}{q}} h_k(Rz)\Big)^{q'}\Big) z^{n-1}\,dz \\ &= n\omega_n R^n\Big(\int_0^\delta + \int_\delta^1\Big). \end{aligned}$$

We are going to obtain a common majorant for the above integrands. First, from (5.5) and Lemma 5.1 we obtain

$$\exp\Big(\Big(n\omega_n^{\frac{1}{n}}(1-\theta)^{-\frac{1}{q}} h_k(Rz)\Big)^{q'}\Big) z^{n-1} \le C \qquad \text{for } z \in (\delta, 1).$$

Thus, we plainly have an integrable majorant on $(\delta, 1)$.

Next, the following computation based on (5.5), (5.7) and (5.8) gives us an integrable majorant on $(0,\delta)$ and it also proves the integrability of the majorant. We have

$$\begin{aligned}
&\int_0^\delta \exp\Big(\Big(n\omega_n^{\frac{1}{n}}(1-\theta)^{-\frac{1}{q}}h_k(Rz)\Big)^{q'}\Big)z^{n-1}\,dz\\
&\quad\leq \int_0^\delta \exp\Big(\Big(n\omega_n^{\frac{1}{n}}(1-\theta)^{-\frac{1}{q}}L_{\psi(z)}(u_k)n^{-\frac{1}{q}}\omega_n^{-\frac{1}{n}}\log^{\frac{q-1}{q}}\Big(\frac{1}{z}\Big)\Big(1+\frac{C}{\log^{\frac{1}{2}}(\frac{1}{z})}\Big)\Big)^{q'}\Big)z^{n-1}\,dz\\
&\quad= \int_0^\delta \exp\Big(n(1-\theta)^{-\frac{q'}{q}}L^{q'}_{\psi(z)}(u_k)\log\Big(\frac{1}{z}\Big)\Big(1+\frac{C}{\log^{\frac{1}{2}}(\frac{1}{z})}\Big)^{q'}\Big)z^{n-1}\,dz\\
&\quad\leq \int_0^\delta \exp\Big(n(1-2\varepsilon)^{q'}\log\Big(\frac{1}{z}\Big)\Big(1+\frac{C}{\log^{\frac{1}{2}}(\frac{1}{z})}\Big)^{q'}\Big)z^{n-1}\,dz\\
&\quad\leq \int_0^\delta \exp\Big(n(1-\varepsilon)^{q'}\log\Big(\frac{1}{z}\Big)\Big)z^{n-1}\,dz\\
&\quad\leq \int_0^\delta z^{n-1-n(1-\varepsilon)^{q'}}\,dz\leq C.
\end{aligned}$$

Hence we have an integrable majorant and thus we can use the Lebesgue Dominated Convergence Theorem to obtain $J_P(u_k)\to J_P(u)$, a contradiction. □

Lemma 5.4. *Let $\{s_k\}\subset(0,1)$, $s_k\to 0$ and let $\{u_k\}\subset W^1_0L^{n,q}(B(R))$ be radial functions satisfying $||\nabla u_k||_{n,q}\leq(1+\mathrm{o}(1))$. If $L_{s_k}(u_k)\to 1$, then*

$$u_k-m_{s_k}\to 0 \qquad \text{in } W^1_0L^{n,q}(B(R)).$$

Proof. The proof easily follows from the uniform convexity of the norm $||\cdot||_{n,q}$ applied to the gradients of the functions u_k and m_{s_k}. Let us give the details.

First, we infer from Lemma 5.1

$$||\nabla u_k||_{n,q}\to 1.$$

Now, since we have $L_{s_k}(m_{s_k})\to 1$ (see (5.1) and (4.5)) and $||\nabla m_{s_k}||_{n,q}\to 1$ (see (4.4)), we obtain from $L_{s_k}(u_k)\to 1$ and $||\nabla u_k||_{n,q}\to 1$

$$\begin{aligned}
&L_{s_k}\left(\frac{\frac{m_{s_k}}{||\nabla m_{s_k}||_{n,q}}+\frac{u_k}{||\nabla u_k||_{n,q}}}{2}\right)\\
&\quad=\frac{1}{2}\Big(L_{s_k}\Big(\frac{m_{s_k}}{||\nabla m_{s_k}||_{n,q}}\Big)+L_{s_k}\Big(\frac{u_k}{||\nabla u_k||_{n,q}}\Big)\Big)\to 1.
\end{aligned}$$

Combining this result with Lemma 5.1 and the triangle inequality we obtain

$$\left|\left|\frac{\frac{\nabla m_{s_k}}{||\nabla m_{s_k}||_{n,q}}+\frac{\nabla u_k}{||\nabla u_k||_{n,q}}}{2}\right|\right|_{n,q}\to 1.$$

Therefore the uniform convexity of the the norm $||\cdot||_{n,q}$ implies

$$\Big|\Big|\frac{\nabla m_{s_k}}{||\nabla m_{s_k}||_{n,q}}-\frac{\nabla u_k}{||\nabla u_k||_{n,q}}\Big|\Big|_{n,q}\to 0.$$

Finally, since $||\nabla m_{s_k}||_{n,q}\to 1$ and $||\nabla u_k||_{n,q}\to 1$, we have

$$\begin{aligned}
&||\nabla m_{s_k}-\nabla u_k||_{n,q}\\
&\quad\leq\Big|\Big|\nabla m_{s_k}-\frac{\nabla m_{s_k}}{||\nabla m_{s_k}||_{n,q}}\Big|\Big|_{n,q}+\Big|\Big|\frac{\nabla m_{s_k}}{||\nabla m_{s_k}||_{n,q}}-\frac{\nabla u_k}{||\nabla u_k||_{n,q}}\Big|\Big|_{n,q}+\Big|\Big|\frac{\nabla u_k}{||\nabla u_k||_{n,q}}-\nabla u_k\Big|\Big|_{n,q}\\
&\quad\to 0.
\end{aligned}$$

Thus, we are done. □

6. Proof of Theorem 1.5

Proof of Theorem 1.5. The strategy of the proof was taken from the proof of Theorem 1.3 given in [3]. However, there are still some technical difficulties that we have to overcome. These difficulties occur in the case when the limit function u is nontrivial and they are caused by the fact that in the Lebesgue spaces we have for two functions with disjoint support

$$||f+g||_p = \left(||f||_p^p + ||g||_p^p\right)^{\frac{1}{p}},$$

while a corresponding formula does not hold in Lorentz spaces in general. However, it was observed in [4] that if $\{f_k\}$ is a concentrating sequence and $\{g_k\}$ is a sequence with a nice behavior, then we have

$$||f_k+g_k||_{n,q} - \left(||f_k||_{n,q}^q + ||g_k||_{n,q}^q\right)^{\frac{1}{q}} \overset{k\to\infty}{\to} 0.$$

We use this principle in the proofs of inequalities (6.6) and (6.7) below. The proof is divided into five parts. The procedure from [3] has three steps. Moreover, inequalities (6.6) and (6.7), which belong to Step 2, are proved separately at the end of the proof of Theorem 1.5.

Assume that $\theta \in [0,1)$ and $\limsup_{k\to\infty} J_P(u_k) > J_P(u)$. Passing to a subsequence we can suppose that the limit exists and $\lim_{k\to\infty} J_P(u_k) > J_P(u)$. Passing to a subsequence again we can also suppose that $u_k \to u$ in $L^{n,q}(\Omega)$ and $u_k \to u$ a.e. in Ω.

We define truncation operators T^L and T_L acting on any function $v \in W_0^1 L^{n,q}(\mathrm{B(R)})$ by

$$T^L(v) = \min\{|v|, L\}\operatorname{sign}(v) \qquad \text{and} \qquad T_L(v) = v - T^L(v).$$

Notice that the weak convergence $u_k \rightharpoonup u$ implies $T^L(u_k) \rightharpoonup T^L(u)$ and $T_L(u_k) \rightharpoonup T_L(u)$ (indeed, $T^L(u_k)$ is bounded, hence it has a weakly convergent subsequence and the convergence a.e. implies that the weak limit has to be $T^L(u)$, similarly for $T_L(u_k)$). We often use the following simple observation. Since $q \le n$, we have that $t \mapsto t^{\frac{q}{n}-1}$ is non-increasing on $(0,\infty)$ and thus

(6.1)
$$\int_0^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla v|^*(t))^q\,dt \le \int_0^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla T_L(v)|^*(t))^q\,dt + \int_0^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla T^L(v)|^*(t))^q\,dt.$$

STEP 1.
Using Lemma 5.3 we find a sequence $\{s_k\} \subset (0,1)$, $s_k \to 0$, such that (passing to a subsequence of $\{u_k\}$ if necessary)

$$\liminf_{k\to\infty} L_{s_k}(u_k) \ge (1-\theta)^{\frac{1}{q}}. \tag{6.2}$$

Next, inequality (6.2) and Lemma 5.2 imply

$$\liminf_{k\to\infty} L_{s_k}(u_k - u) \ge (1-\theta)^{\frac{1}{q}}. \tag{6.3}$$

STEP 2.
In this step we prove

$$\limsup_{k\to\infty} ||\nabla(u_k-u)||_{n,q} \le (1-\theta)^{\frac{1}{q}}. \tag{6.4}$$

If $\theta = 0$, the proof trivially follows from the assumption $||\nabla u_k||_{n,q} \le 1$, $k \in \mathbb{N}$. Thus, let us suppose that $\theta \in (0,1)$ in the rest of this step (and also in the proofs of inequalities (6.6) and (6.7)).

Fix $\varepsilon > 0$. We also fix $L > 0$ so large that

$$\int_0^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla T_L(u)|^*(t))^q\,dt = \tau, \tag{6.5}$$

where $\tau \in (0, \frac{1}{8}\min\{\theta, 1-\theta\})$ is a small number specified below.

We have

$$\begin{aligned} ||\nabla(u_k-u)||_{n,q} &\le ||\nabla(T^L(u_k) - T^L(u))||_{n,q} + ||\nabla T_L(u_k)||_{n,q} + ||\nabla T_L(u)||_{n,q} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

If τ is small enough, then (6.5) implies that $I_3 < \varepsilon$.

Next, we claim that for k large enough the following inequality holds

$$\int_0^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla T_L(u_k)|^*(t))^q\,dt \le 1-\theta+3\tau. \tag{6.6}$$

Let us postpone the proof of (6.6). From (6.6) we observe that if τ is small enough, we also have $I_2 < (1-\theta)^{\frac{1}{q}} + \varepsilon$.

Let us proceed to the proof that $I_1 < \varepsilon$. The proof is based on Lemma 3.6 (recall that we have $T^L(u_k) \rightharpoonup T^L(u)$). Since the norm is homogeneous, since we have (by (1.8), (6.1) and (6.5))

$$\theta^{\frac{1}{q}} = ||\nabla u||_{n,q} \ge ||\nabla T^L(u)||_{n,q} \ge \Big(||\nabla u||_{n,q}^q - ||\nabla T_L(u)||_{n,q}^q\Big)^{\frac{1}{q}} = (\theta-\tau)^{\frac{1}{q}}$$

and since τ is as small as we wish, it remains to prove (so that Lemma 3.6 implies $||\nabla(T^L(u_k) - T^L(u))||_{n,q} < \varepsilon$)

$$||\nabla T^L(u_k)||_{n,q} \le (\theta+\zeta)^{\frac{1}{q}}, \tag{6.7}$$

where ζ is a small number depending on ε. Let us postpone the proof of (6.7).

Thus, when (6.6) and (6.7) are proved, we will have $I_1 + I_2 + I_3 \le \varepsilon + (1-\theta)^{\frac{1}{q}} + \varepsilon + \varepsilon$, which concludes the proof of (6.4).

STEP 3.
Our aim is to prove

$$(1-\theta)^{-\frac{1}{q}}(u_k-u) - m_{s_k} \overset{k\to\infty}{\to} 0 \qquad \text{in } W_0^1L^{n,q}(B(R)). \tag{6.8}$$

Combining (6.3) and (6.4) with Lemma 5.1 we obtain

$$L_{s_k}((1-\theta)^{-\frac{1}{q}}(u_k-u)) \overset{k\to\infty}{\to} 1 \qquad \text{and} \qquad ||(1-\theta)^{-\frac{1}{q}}\nabla(u_k-u)||_{n,q} \overset{k\to\infty}{\to} 1.$$

Now, Lemma 5.4 concludes the proof of (6.8).

It remains to prove inequalities (6.6) and (6.7) to complete the proof of Theorem 1.5.

PROOF OF (6.6).
The proof is based on the method used in [4, Proof of Theorem 1.3(iii): Step 2]. We omit several detailed computations, let us just recall the main ideas for the convenience of the reader.

First, by (6.1) and (6.5) we have $||\nabla T^L(u)||_{n,q}^q \ge \theta - \tau$. Thus, we can use the absolute continuity of the Lebesgue integral to obtain $\sigma \in (0, |B(R)|)$ so small that

$$\int_\sigma^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla T^L(u)|^*(t))^q\,dt \ge \theta - 2\tau. \tag{6.9}$$

Next, we decompose the interval $[\sigma, |B(R)|]$ into very short subintervals $[a_{j-1}, a_j]$, $j = 1, \dots, m$, so that the function $t \mapsto t^{\frac{q}{n}-1}$ is very close to a constant on each subinterval. Furthermore, let G_j, $j = 1, \dots, m$, be disjoint measurable subsets of $B(R)$ satisfying $|G_j| = a_j - a_{j-1}$ and chosen so that the values of $|\nabla T^L(u)|$ at the points of G_j correspond to the values of $|\nabla T^L(u)|^*$ at the points of $[a_{j-1}, a_j]$. Let G be the union of these sets. Now, we use (6.9), the weak lower semicontinuity of the L^q-norm (since $q \le n$, we have that $L^{n,q}$ is embedded into L^q) on each set G_j, we use the fact that $t \mapsto t^{\frac{q}{n}-1}$ is almost constant on each $[a_{j-1}, a_j]$ and the Hardy-Littlewood inequality to obtain

$$\int_\sigma^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla(T^L(u_k)|_G)|^*(t-\sigma))^q\,dt \ge \theta - 3\tau. \tag{6.10}$$

Finally, by the Chebyshev inequality we can see that if L is large enough, then $|\operatorname{supp} T_L(u_k)| < \sigma$ for every $k \in \mathbb{N}$. This property, Hardy-Littlewood inequality and the fact that $t \mapsto t^{\frac{q}{n}-1}$ is non-increasing imply

(6.11)
$$\int_0^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla u_k|^*(t))^q\,dt \ge \int_0^\sigma t^{\frac{q}{n}-1}(|\nabla(T_L(u_k))|^*(t))^q\,dt + \int_\sigma^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla(T^L(u_k))|^*(t-\sigma))^q\,dt.$$

Now, (6.6) follows from (6.10), (6.11) and the assumption $||\nabla u_k||_{n,q} \le 1$ for every $k \in \mathbb{N}$.

PROOF OF (6.7).
We restrict ourselves to the case $q < n$ (we do not need to care about $q = n$, since the Sobolev case of Theorem 1.5 is contained already in Theorem 1.3).

Fix $\zeta > 0$. By the uniform monotonicity of the Lorentz norm we can find $\gamma > 0$ so small that

$$0 \le g \le f,\ ||f||_{n,q} = 1, ||g||_{n,q} > \Big(\frac{\zeta}{2}\Big)^{\frac{1}{q}} \quad \Longrightarrow \quad ||f - g||_{n,q} < (1 - 3\gamma)^{\frac{1}{q}}. \tag{6.12}$$

We can also suppose that

$$\gamma \le \frac{\zeta}{4}. \tag{6.13}$$

Next, since we have (6.2), $T_L(u_k) \ge u_k - L$ and since an additive constant is irrelevant for the behavior of L_s with s very small (observe (5.3) and (5.4)), we obtain for k large enough

$$L_{s_k}(T_L(u_k)) \ge (1 - \theta - \tau)^{\frac{1}{q}}.$$

Thus, by Lemma 5.1, we have for k large enough

$$||\nabla T_L(u_k)||_{n,q} \ge (1 - \theta - 2\tau)^{\frac{1}{q}}. \tag{6.14}$$

Let us recall that $\tau > 0$ is a very small number and this number can be made as small as we wish.

Next, let $\xi > 0$ be so small that

$$\int_0^{|B(R)|} t^{\frac{q}{n}-1} \xi^q \, dt \le \gamma$$

and, for every $k \in \mathbb{N}$, let us set

$$\varrho_k := |\{|\nabla T_L(u_k)| > \xi\}|.$$

From the choice of ξ and (6.14) we infer

$$\int_0^{\varrho_k} t^{\frac{q}{n}-1} (|\nabla T_L(u_k)|^*(t))^q \, dt \ge 1 - \theta - 2\tau - \gamma. \tag{6.15}$$

Next, we claim that passing to a subsequence we obtain $\varrho_k \to 0$ as $k \to \infty$. Let us prove this claim by contradiction. We suppose that there are $\varrho_0 > 0$ and $k_0 \in \mathbb{N}$ such that $\varrho_k > \varrho_0$ for every $k > k_0$. This implies for every $k > k_0$

$$\int_{\frac{\varrho_0}{2}}^{|B(R)|} t^{\frac{q}{n}-1} (|\nabla T_L(u_k)|^*(t))^q \, dt \ge \int_{\frac{\varrho_0}{2}}^{\varrho_0} t^{\frac{q}{n}-1} \xi^q \, dt = C$$

and thus, by (6.6),

$$\int_0^{\frac{\varrho_0}{2}} t^{\frac{q}{n}-1} (|\nabla T_L(u_k)|^*(t))^q \, dt \le 1 - \theta + 3\tau - C.$$

This means that if τ is sufficiently small, then there is $\beta \in (0, \frac{1}{2})$ such that for every $k > k_0$

$$\int_0^{\frac{\varrho_0}{2}} t^{\frac{q}{n}-1} (|\nabla T_L(u_k)|^*(t))^q \, dt \le (1 - \theta)(1 - 2\beta). \tag{6.16}$$

Now, we follow the computation in (2.3), where we decompose the integral over (t, Ω) into integral over $(t, \frac{\varrho_0}{2})$ and integral over $(\frac{\varrho_0}{2}, |\Omega|)$, and we apply estimate (6.16) after Hölder's inequality when estimating the integral over $(t, \frac{\varrho_0}{2})$. We obtain for t small enough

$$\begin{aligned} u_k^*(t) &\le L + (T_L(u_k))^*(t) \le L + C + \frac{((1-\theta)(1-2\beta))^{\frac{1}{q}}}{n\omega_n^{\frac{1}{n}}} \log^{\frac{1}{q'}}\Big(\frac{|\Omega|}{t}\Big) + C \\ &= C + \frac{((1-\theta)(1-2\beta))^{\frac{1}{q}}}{n\omega_n^{\frac{1}{n}}} \log^{\frac{1}{q'}}\Big(\frac{|\Omega|}{t}\Big). \end{aligned}$$

Therefore we have an integrable majorant of the integrand of $J_P(u_k)$. Indeed, for suitably small $t_0 > 0$ we have

$$\begin{aligned}
J_P(u_k) &= \int_{B(R)} \exp\Big(\Big(n\omega_n^{\frac{1}{n}} P|u_k|\Big)^{q'}\Big)\,dx \\
&= \int_0^{|B(R)|} \exp\Big(\Big(n\omega_n^{\frac{1}{n}} (1-\theta)^{-\frac{1}{q}} u_k^*(t)\Big)^{q'}\Big)\,dt \\
&= \int_0^{|B(R)|} \exp\Big(\Big(C + (1-2\beta)^{\frac{1}{q}} \log^{\frac{1}{q'}}\Big(\frac{|\Omega|}{t}\Big)\Big)^{q'}\Big)\,dt \\
&\leq \int_0^{t_0} \exp\Big(\Big((1-\beta)^{\frac{1}{q}} \log^{\frac{1}{q'}}\Big(\frac{|\Omega|}{t}\Big)\Big)^{q'}\Big)\,dt + \int_{t_0}^{|B(R)|} \exp(C)\,dt \\
&= C\int_0^{t_0} t^{-(\beta-1)^{\frac{q'}{q}}}\,dt + C.
\end{aligned}$$

Thus $J_P(u_k) \to J_P(u)$ by the Lebesgue Dominated Convergence Theorem. This is a contradiction and thus we can pass to a subsequence to obtain $\varrho_k \to 0$ as $k \to \infty$.

Next, let us fix $D > 1$ so large that

$$\Big(\frac{D}{D+1}\Big)^{\frac{q}{n}-1} \leq 1 + \tau. \tag{6.17}$$

Now, we use (6.10), (6.15) and the Hardy-Littlewood inequality to obtain

$$1 \geq ||\nabla u_k||_{n,q}^q \geq \int_0^{\varrho_k} t^{\frac{q}{n}-1}(|\nabla T_L(u_k)|^*(t))^q\,dt + \int_\sigma^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t-\sigma))^q\,dt \geq 1 - 5\tau - \gamma. \tag{6.18}$$

Next, a trivial estimate

$$\int_0^{D\varrho_k} t^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t))^q\,dt \leq \int_0^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla u_k|^*(t))^q\,dt \leq 1$$

implies for k large enough (notice that $\frac{q}{n} - 1 < 0$ in our case $q < n$ and $D\varrho_k$ is much smaller than σ for k large)

$$\int_\sigma^{\sigma + D\varrho_k} t^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t-\sigma))^q\,dt < \gamma.$$

Hence, if $\tau < \frac{\gamma}{6}$, we obtain from the above inequality and (6.18)

$$1 \geq \int_0^{\varrho_k} t^{\frac{q}{n}-1}(|\nabla T_L(u_k)|^*(t))^q\,dt + \int_{\sigma + D\varrho_k}^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t-\sigma))^q\,dt \geq 1 - 3\gamma. \tag{6.19}$$

Therefore we can use the uniform monotonicity (6.12) to obtain

$$|||\nabla T^L(u_k)|^*\chi_{(0,D\varrho_k)}||_{n,q} < \Big(\frac{\zeta}{2}\Big)^{\frac{1}{q}} \tag{6.20}$$

(we applied (6.12) to $f = \frac{|\nabla u_k|}{||\nabla u_k||_{n,q}}$ and $g = \frac{|\nabla T^L(u_k)|\chi_G}{||\nabla u_k||_{n,q}} = \frac{|\nabla u_k|\chi_G}{||\nabla u_k||_{n,q}}$, where the set G is chosen so that the values of $|\nabla T^L(u_k)|$ on G correspond to the values of $|\nabla T^L(u_k)|^*$ on $(0, D\varrho_k)$. With this setting we have $f - g = \frac{|\nabla u_k|\chi_{B(R)\setminus G}}{||\nabla u_k||_{n,q}}$ and (6.19) implies $|||\nabla u_k|\chi_{B(R)\setminus G}\|_{n,q}^q \geq 1 - 3\gamma$. The normalization by $\frac{1}{||\nabla u_k||_{n,q}}$ is harmless since $1 - 3\gamma \leq ||\nabla u_k||_{n,q}^q \leq 1$).

We also have by the Hardy-Littlewood inequality

$$\begin{aligned}
1 \geq ||\nabla u_k||_{n,q}^q \geq \int_0^{\varrho_k} t^{\frac{q}{n}-1}(|\nabla T_L(u_k)|^*(t))^q\,dt &+ \int_{\varrho_k}^{(D+1)\varrho_k} t^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t-\varrho_k))^q\,dt \\
&+ \int_{(D+1)\varrho_k}^{\varrho_k + |\operatorname{supp}\nabla T^L(u_k)|} t^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t-\varrho_k))^q\,dt \\
&+ \int_{\varrho_k + |\operatorname{supp}\nabla T^L(u_k)|}^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla T_L(u_k)|^*(t - |\operatorname{supp}\nabla T^L(u_k)|))^q\,dt
\end{aligned}$$

and thus, by (6.15), for the third summand on the right hand side we obtain

$$\int_{(D+1)\varrho_k}^{\varrho_k+|\operatorname{supp}\nabla T^L(u_k)|} t^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t-\varrho_k))^q\,dt \le \theta+2\tau+\gamma. \tag{6.21}$$

Finally, we infer from (6.20), (6.17), (6.21), (6.13), $\frac{q}{n}-1<0$ and $\tau<\frac{\gamma}{6}$

$$\begin{aligned}
||\nabla T^L(u_k)||^q_{n,q} &= \int_0^{D\varrho_k} t^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t))^q\,dt + \int_{D\varrho_k}^{|B(R)|} t^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t))^q\,dt \\
&\le \frac{\zeta}{2} + \int_{(D+1)\varrho_k}^{|B(R)|+\varrho_k} (t-\varrho_k)^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t-\varrho_k))^q\,dt \\
&\le \frac{\zeta}{2} + \Big(\frac{D}{D+1}\Big)^{\frac{q}{n}-1} \int_{(D+1)\varrho_k}^{|B(R)|+\varrho_k} t^{\frac{q}{n}-1}(|\nabla T^L(u_k)|^*(t-\varrho_k))^q\,dt \\
&\le \frac{\zeta}{2} + (1+\tau)(\theta+2\tau+\gamma) \le \frac{\zeta}{2}+\theta+2\tau+\gamma+\tau(1+2+1) \le \theta+\frac{\zeta}{2}+2\gamma \le \theta+\zeta.
\end{aligned}$$

This is (6.7) which completes the proof of Theorem 1.5. □

Acknowledgement. The author was supported by the ERC CZ grant LL1203 of the Czech Ministry of Education.

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Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic

E-mail address: rcerny@karlin.mff.cuni.cz