NEW RESULTS CONCERNING MOSER-TYPE INEQUALITIES IN LORENTZ-SOBOLEV SPACES

ROBERT ČERNÝ

ABSTRACT. Let $n \in \mathbb{N}$, $n \geq 2$, $q \in (1, \infty]$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded set. We obtain sharp constants concerning the Moser-type inequalities corresponding to the Lorentz-Sobolev space $W_0^1 L^{n,q}(\Omega)$ equipped with the norm

$$||\nabla u||_{(n,q)} := \begin{cases} ||t^{\frac{1}{n}-\frac{1}{q}} |\nabla u|^{**}(t)||_{L^q((0,\infty))} & \text{for } q \in (1,\infty) \\ \sup_{t\in(0,\infty)} t^{\frac{1}{n}} |\nabla u|^{**}(t) & \text{for } q = \infty \ . \end{cases}$$

We also derive the Concentration-Compactness Principle in the case $q \in (1, \infty)$ with respect to the above norm.

1. Introduction

Throughout the paper $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open bounded set, ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, $\mathcal{L}_n$ is the n-dimensional Lebesgue measure and $|\Omega|$ stands for $\mathcal{L}_n(\Omega)$. We use the standard notation $q' = \frac{q}{q-1}$ (with the convention $\infty' = 1$ and $1' = \infty$).

The Moser-Trudinger inequality is a crucial tool when proving the existence and the regularity of nontrivial weak solutions to elliptic partial differential equations with critical growth (see for example the pioneering works [3] and [4] by Adimurthi).

Its original statement obtained in [14] for the space $W_0^{1,n}(\Omega)$ is the following

$$\sup_{||\nabla u||_{L^n(\Omega)}\leq 1} \int_\Omega \exp((K|u(x)|)^{n'})\, dx \begin{cases} \leq C(n, K, |\Omega|) & \text{when } K \leq n\omega_n^{\frac{1}{n}} \\ = \infty & \text{when } K > n\omega_n^{\frac{1}{n}} \ . \end{cases} \tag{1.1}$$

Further applications required several versions and generalizations of the Moser-Trudinger inequality such as a version for unbounded domains (see [1]), a version without boundary conditions (see [9]), a version for higher order Sobolev spaces (see [2]), the Concentration-Compactness Alternative (see [12] and [7]) and others. The most important part of the Concentration-Compactness Alternative is the following improvement of the Moser-Trudinger inequality. This result is used to prove the existence and the multiplicity of weak solutions to elliptic PDEs in the limiting situations when the Moser-Trudinger inequality is not powerful enough.

Theorem 1.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Let $u \in W_0^{1,n}(\Omega)$ be a non-trivial function and let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a sequence satisfying*

$$\int_\Omega |\nabla u_k(x)|^n\, dx \leq 1 \text{ for every } k \in \mathbb{N} \ , \quad u_k \rightharpoonup u \text{ in } W_0^{1,n}(\Omega) \quad \text{and} \quad u_k \to u \text{ a.e. in } \Omega \ .$$

Let us set

$$P = \begin{cases} (1 - \int_\Omega |\nabla u(x)|^n\, dx)^{-\frac{1}{n}} & \text{when } \int_\Omega |\nabla u(x)|^n\, dx < 1 \\ \infty & \text{when } \int_\Omega |\nabla u(x)|^n\, dx = 1 \ . \end{cases}$$

Then for every $p < P$ there is $C > 0$ such that

$$\int_\Omega \exp\Big(\Big(n\omega_n^{\frac{1}{n}} p\, |u_k(x)|\Big)^{n'}\Big)\, dx \leq C \ .$$

Moreover, such an upper bound for p is sharp.

Date: January 8, 2015.

2000 *Mathematics Subject Classification.* 46E35, 46E30, 26D10.

Key words and phrases. Sobolev spaces, Lorentz-Sobolev spaces, Moser-Trudinger inequality, Concentration-Compactness Principle, Sharp constants.

The standard proof of Moser-type inequalities uses the symmetrization (based on the Pólya-Szegö Principle). This approach is also suitable for generalizations concerning the Orlicz-Sobolev spaces embedded into exponential and multiple exponential Orlicz spaces (see [10] and [8]). For the Lorentz-Sobolev spaces, the proof of the corresponding Moser-type inequalities (see for example [5]) uses a suitable relation between u^* (the non-increasing rearrangement of u) and $|\nabla u|^*$. In particular, one has for every $u \in W_0^{1,1}(\Omega)$

$$u^*(t) \leq \frac{1}{n\omega_n^{\frac{1}{n}}} \left(t^{-\frac{1}{n'}} \int_0^t |\nabla u|^*(s)\, ds + \int_t^{|\Omega|} |\nabla u|^*(s) s^{-\frac{1}{n'}}\, ds \right) \qquad \text{for every } t \in (0, |\Omega|) \tag{1.2}$$

(see for example [16] or [5]). With such estimates, it is natural to have a statement of the Moser-type inequality with respect to the quantity

$$||f||_{n,q} := \begin{cases} ||t^{\frac{1}{n}-\frac{1}{q}} f^*(t)||_{L^q((0,|\Omega|))} & \text{for } q < \infty \\ \sup_{t\in(0,|\Omega|)} t^{\frac{1}{n}} f^*(t) & \text{for } q = \infty \ . \end{cases} \tag{1.3}$$

This quantity is generally not a norm (it is a norm for $q \leq n$, while for $q > n$ it is a quasi-norm, see [13, Theorem 1]). Let us also recall that even though the above quasi-norm is not a norm in general, it is equivalent to an actual norm

$$||f||_{(n,q)} := \begin{cases} ||t^{\frac{1}{n}-\frac{1}{q}} f^{**}(t)||_{L^q((0,\infty))} & \text{for } q < \infty \\ \sup_{t\in(0,\infty)} t^{\frac{1}{n}} f^{**}(t) & \text{for } q = \infty \ . \end{cases} \tag{1.4}$$

Here $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\, ds$ for every $t > 0$ and the equivalence follows from the estimates

$$||f||_{n,q} \leq ||f||_{(n,q)} \leq \frac{n}{n-1} ||f||_{n,q} \ . \tag{1.5}$$

See [11, Section 2] for the proof in the case $1 \leq q < \infty$, the proof for $q = \infty$ is an easy exercise.

The Moser-type inequality for Lorentz-Sobolev spaces $W_0L^{n,q}(\Omega)$ was obtained in [5] (the paper also contains an application to the proof of the regularity of the weak solutions to some PDEs) and it has the following form. If $q \in (1, \infty)$, then

$$\sup_{||\nabla u||_{n,q}\leq 1} \int_\Omega \exp((K|u(x)|)^{q'})\, dx \begin{cases} \leq C(n, K, q, |\Omega|) & \text{when } K \leq n\omega_n^{\frac{1}{n}} \\ = \infty & \text{when } K > n\omega_n^{\frac{1}{n}} \end{cases} \tag{1.6}$$

and if $q = \infty$, then

$$\sup_{||\nabla u||_{n,\infty}\leq 1} \int_\Omega \exp(K|u(x)|)\, dx \begin{cases} \leq C(n, K, |\Omega|) & \text{when } K < n\omega_n^{\frac{1}{n}} \\ = \infty & \text{when } K \geq n\omega_n^{\frac{1}{n}} \ . \end{cases} \tag{1.7}$$

Notice that, since $\infty' = 1$, the main difference between cases $q \in (1, \infty)$ and $q = \infty$ is the uniform boundedness of the integrals in the case $K = n\omega_n^{\frac{1}{n}}$ for $q \in (1, \infty)$.

There is no Moser-type inequality for $q = 1$, since $W_0L^{n,1}(\Omega)$ is embedded into $L^\infty(\Omega)$ (to see this, it is enough to combine the definition of the quasi-norm (1.3) with estimate (1.2)). Let us also recall that there is also the Trudinger-type embedding for $q \in (1, \infty)$ which states that for every $K \geq 0$ and every $u \in W_0L^{n,q}(\Omega)$ we have

$$\int_\Omega \exp((K|u(x)|)^{q'})\, dx < \infty \ . \tag{1.8}$$

The improvement of (1.6) of the same type as Theorem 1.1 was obtained in [6].

Theorem 1.2. *Let $n \in \mathbb{N}$, $n \geq 2$, $q \in (1, \infty)$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Let $u \in W_0L^{n,q}(\Omega)$ be a non-trivial function and let $\{u_k\} \subset W_0L^{n,q}(\Omega)$ be a sequence such that*

$$||\nabla u_k||_{n,q} \leq 1 \ , \qquad u_k \rightharpoonup u \text{ in } W_0L^{n,q}(\Omega) \qquad \text{and} \qquad u_k \to u \text{ a.e. in } \Omega \ .$$

Let us set

$$P := \begin{cases} \left(1 - ||\nabla u||_{n,q}^q\right)^{-\frac{1}{q}} & \text{when } ||\nabla u||_{n,q} < 1 \\ \infty & \text{when } ||\nabla u||_{n,q} = 1 \ . \end{cases}$$

If $q \in (1, n]$, then for every $p < P$ there is $C > 0$ such that

$$\int_\Omega \exp\Big(\Big(n\omega_n^{\frac{1}{n}} p|u_k(x)|\Big)^{q'}\Big)\,dx \le C \qquad \text{for every } k \in \mathbb{N}\ . \tag{1.9}$$

Moreover, the assumption $p < P$ is sharp.

If $q \in (n, \infty)$, then there is $\tilde{P} \in (1, P]$ such that (1.9) holds for every $p < \tilde{P}$, but we do not have $\tilde{P} = P$ in general.

Notice that in the case $q \in (1, n]$ (in this case the quantity (1.3) is a norm) the result is of the same type as Theorem 1.1. On the other hand, when $q \in (n, \infty)$, the fact that the quantity we work with is not weakly lower semicontinuous entails some loss of integrability.

There is no version of Theorem 1.2 for $q = \infty$, as can be seen considering the function $u(x) = \omega_n^{-\frac{1}{n}} \log(\frac{R}{|x|})$, $0 < |x| < R$, with some $R > 0$ (if we set $u_k := u$, then we have $u_k \to u$ in the norm, the norm is equal to 1 and the Moser functional corresponding to the number $n\omega_n^{\frac{1}{n}}$ is infinite).

The aim of this paper is to obtain versions of results (1.6), (1.7) and Theorem 1.2 with respect to the norm (1.4). Our results are the following. If $q \in (1, \infty)$, then

$$\sup_{||\nabla u||_{(n,q)} \le 1} \int_\Omega \exp((K|u(x)|)^{q'})\,dx \begin{cases} \le C(n, K, q, |\Omega|) & \text{when } K \le \frac{n^2}{n-1}\omega_n^{\frac{1}{n}} \\ = \infty & \text{when } K > \frac{n^2}{n-1}\omega_n^{\frac{1}{n}} \end{cases}, \tag{1.10}$$

if $q = \infty$, then

$$\sup_{||\nabla u||_{(n,\infty)} \le 1} \int_\Omega \exp(K|u(x)|)\,dx \begin{cases} \le C(n, K, |\Omega|) & \text{when } K < \frac{n^2}{n-1}\omega_n^{\frac{1}{n}} \\ = \infty & \text{when } K \ge \frac{n^2}{n-1}\omega_n^{\frac{1}{n}} \end{cases} \tag{1.11}$$

and we obtain a version of Theorem 1.2 containing a sharp result without the restriction $q \in (1, n]$.

Theorem 1.3. *Let $n \in \mathbb{N}$, $n \ge 2$, $q \in (1, \infty)$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Let $u \in W_0 L^{n,q}(\Omega)$ be a non-trivial function and let $\{u_k\} \subset W_0 L^{n,q}(\Omega)$ be a sequence such that*

$$||\nabla u_k||_{(n,q)} \le 1\ , \qquad u_k \rightharpoonup u \text{ in } W_0 L^{n,q}(\Omega) \qquad \text{and} \qquad u_k \to u \text{ a.e. in } \Omega\ .$$

Let us set

$$P := \begin{cases} \Big(1 - ||\nabla u||^q_{(n,q)}\Big)^{-\frac{1}{q}} & \text{when } ||\nabla u||_{(n,q)} < 1 \\ \infty & \text{when } ||\nabla u||_{(n,q)} = 1\ . \end{cases} \tag{1.12}$$

Then for every $p < P$ there is $C > 0$ such that

$$\int_\Omega \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}} p|u_k(x)|\Big)^{q'}\Big)\,dx \le C \qquad \text{for every } k \in \mathbb{N}\ .$$

Moreover the constant P is sharp.

Finally, let us also give the full statement of the Concentration-Compactness Principle with respect to the norm (1.4).

Theorem 1.4. *Let $n \in \mathbb{N}$, $n \ge 2$, $q \in (1, \infty)$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Let $u \in W_0 L^{n,q}(\Omega)$ and $\{u_k\} \subset W_0 L^{n,q}(\Omega)$ be a sequence such that*

$$||\nabla u_k||_{(n,q)} \le 1\ , \qquad u_k \rightharpoonup u \text{ in } W_0 L^{n,q}(\Omega) \qquad \text{and} \qquad u_k \to u \text{ a.e. in } \Omega\ .$$

Let us set

$$A = \sup_{x \in \bar{\Omega}} \lim_{r \to 0_+} \limsup_{k \to \infty} \Big(\int_0^\infty \Big(t^{\frac{1}{n} - \frac{1}{q}} (|\nabla u_k| \chi_{B(x,r)})^{**}(t)\Big)^q\,dt\Big)^{\frac{1}{q}} \in [0, 1]\ .$$

(i) *If $u = 0$ and $A = 1$, then there is $x_0 \in \bar{\Omega}$ such that*

$$1 = A = \lim_{r \to 0_+} \limsup_{k \to \infty} \Big(\int_0^\infty \Big(t^{\frac{1}{n} - \frac{1}{q}} (|\nabla u_k| \chi_{B(x_0,r)})^{**}(t)\Big)^q\,dt\Big)^{\frac{1}{q}}\ . \tag{1.13}$$

The point x_0 is not unique in general. If x_0 is unique and

$$\int_\Omega \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}} |u_k(x)|\Big)^{q'}\Big)\,dx \to c + \mathcal{L}_n(\Omega) \tag{1.14}$$

for some $c \in [0,\infty)$, then

$$\exp\Bigl(\Bigl(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}|u_k(x)|\Bigr)^{q'}\Bigr) \overset{*}{\rightharpoonup} c\delta_{x_0} + \mathcal{L}_n|_\Omega \quad \text{in } \mathcal{M}(\bar{\Omega}) \ . \tag{1.15}$$

(ii) *If $u = 0$ and $A < 1$, then for every $p < A^{-1}$, there is a constant $C > 0$ such that*

$$\int_\Omega \exp\Bigl(\Bigl(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}p|u_k(x)|\Bigr)^{q'}\Bigr)\,dx \le C \qquad \text{for every } k \in \mathbb{N} \ . \tag{1.16}$$

Moreover, the assumption $p < A^{-1}$ is sharp.
(iii) *If $u \neq 0$ and P is defined by* (1.12)*, then for every $p < P$ there is $C > 0$ such that* (1.16) *holds. Moreover the constant P is sharp.*

Furthermore, in both cases (ii) and (iii) we have

$$\exp\Bigl(\Bigl(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}|u_k(x)|\Bigr)^{q'}\Bigr) \to \exp\Bigl(\Bigl(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}|u(x)|\Bigr)^{q'}\Bigr) \quad \text{in } L^1(\Omega) \ .$$

The paper is organized as follows. After Preliminaries we recall some properties of Moser functions corresponding to inequality (1.6). In the fourth section we derive (1.10) and (1.11) from (1.6) and (1.7), respectively.

Section 5 is devoted to the proof of Theorem 1.4(i). In Section 6 we prove Theorem 1.4(ii). The proof is divided into two parts. First, we prove the boundedness for $p < A^{-1}$, then we show that (1.16) does not hold even for $p = A^{-1}$ in general. Let us note that in Sections 5 and 6 we use the techniques from [6] (the paper dealing with the quasi-norm (1.3)) with minor changes only.

The proof of Theorem 1.4(iii) is given in Section 7 and it has two parts. First, we derive the boundedness for $p < P$, then we prove the optimality of P. At the end of Section 7, we prove the remaining statements of Theorem 1.4. Finally, Theorem 1.3 follows from Theorem 1.4(iii). The proofs in Section 7 again use the approach from [6], however we had to overcome several technical difficulties. In particular, in paper [6], the set Ω is decomposed into suitable subsets and on each such a set the contribution of $|\nabla u_k|$ to $||\nabla u_k||_{n,q}$ is studied separately. This is not possible in this paper because of the maximal function involved in the norm (1.4).

Section 8 contains some remarks concerning a version of the norm (1.4) in which we integrate over $(0, |\Omega|)$ only.

2. Preliminaries

Notation. The n-dimensional Lebesgue measure is denoted by $\mathcal{L}_n$ and $|\Omega|$ stands for $\mathcal{L}_n(\Omega)$. Further, $\mathcal{L}_n|_\Omega$ is the restriction of $\mathcal{L}_n$ to Ω, i.e. $\mathcal{L}_n|_\Omega(A) = \mathcal{L}_n(A \cap \Omega)$ for every measurable set $A \subset \mathbb{R}^n$. The characteristic function of A is denoted by χ_A. If u is a measurable function on Ω, then by $u = 0$ (or $u \neq 0$) we mean that u is equal (or not equal) to the zero function a.e. on Ω.

Next, $\bar{\Omega}$ denotes the closure of Ω. By $\mathcal{M}(\bar{\Omega})$ we denote the set of all Radon measures on $\bar{\Omega}$. We write that $\mu_j \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\bar{\Omega})$ if $\int_{\bar{\Omega}} \psi\, d\mu_j \to \int_{\bar{\Omega}} \psi\, d\mu$ for every test-function $\psi \in C(\bar{\Omega})$. It is well known that each sequence bounded in $L^1(\Omega)$ contains a subsequence converging weakly* in $\mathcal{M}(\bar{\Omega})$.

By $B(x,R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with the radius $R > 0$. If $x = 0$, we simply write $B(R)$.

By the symbol $W_0L^{n,q}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ with respect to the (quasi-)norm $||\nabla u||_{n,q}$ (or the equivalent norm $||\nabla u||_{(n,q)}$). We write that $u_k \rightharpoonup u$ in $W_0L^{n,q}(\Omega)$, $q \in (1,\infty)$, if

$$\int_\Omega \frac{\partial u_k}{\partial x_i} v\,dx \to \int_\Omega \frac{\partial u}{\partial x_i} v\,dx \qquad \text{for every } v \in L^{n',q'}(\Omega) \text{ and } i = 1,\dots,n \ .$$

By C we denote a generic positive constant which may depend on n, q, $|\Omega|$, p and K. This constant may vary from expression to expression as usual. Sometimes we say that for every $\varepsilon > 0$ something is true. Then the constants C in such a case may depend also on fixed $\varepsilon > 0$.

Non-increasing rearrangement. The non-increasing rearrangement f^* of a measurable function f on Ω is

$$f^*(t) = \sup\Big\{ s \geq 0 : \big|\{x \in \Omega : |f(x)| > s\}\big| > t \Big\} \qquad \text{for } t \in (0, \infty) \ .$$

Notice that for $t > |\Omega|$ we have $f^*(t) = 0$. Therefore it is not important whether we integrate over $(0, |\Omega|)$ or $(0, \infty)$ in the definition (1.3). Further, we define the maximal function of f^* by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds \qquad \text{for } t \in (0, \infty) \ .$$

This quantity does not vanish for $t > |\Omega|$. In the literature, the norm (1.4) is sometimes defined with the integration over $(0, \infty)$, sometimes over $(0, |\Omega|)$. The proof of inequality (1.5) given in [11] concerns the case of integration over $(0, \infty)$. It is obvious that it also holds in the case of integration over $(0, |\Omega|)$, since we obtain smaller integrals on one hand, on the other hand we always have $f^{**}(t) \geq f^*(t)$.

We are going to use the Hardy-Littlewood inequality for measurable functions

$$\int_\Omega |f(x) g(x)|\, dx \leq \int_0^{|\Omega|} f^*(t) g^*(t)\, dt \ .$$

Finally, let us derive a version of (1.2) for the quantity $|\nabla u|^{**}$. By the Fubini theorem we have

$$\begin{aligned}
\int_t^\infty s^{-\frac{n-1}{n}} |\nabla u|^{**}(s)\, ds &= \int_t^\infty s^{-2+\frac{1}{n}} \Big(\int_0^s |\nabla u|^*(r)\, dr \Big)\, ds \\
&= \int_t^\infty \int_0^s s^{-2+\frac{1}{n}} |\nabla u|^*(r)\, dr\, ds \\
&= \int_0^t \int_t^\infty s^{-2+\frac{1}{n}} |\nabla u|^*(r)\, ds\, dr + \int_t^\infty \int_r^\infty s^{-2+\frac{1}{n}} |\nabla u|^*(r)\, ds\, dr \\
&= -\frac{n}{n-1} \left(\Big[s^{-\frac{n-1}{n}} \Big]_t^\infty \int_0^t |\nabla u|^*(r)\, dr + \int_t^\infty \Big[s^{-\frac{n-1}{n}} \Big]_r^\infty |\nabla u|^*(r)\, dr \right) \\
&= \frac{n}{n-1} \left(t^{-\frac{n-1}{n}} \int_0^t |\nabla u|^*(r)\, dr + \int_t^\infty r^{-\frac{n-1}{n}} |\nabla u|^*(r)\, dr \right) .
\end{aligned}$$

Thus (1.2) reads

$$u^*(t) \leq \frac{n-1}{n^2 \omega_n^{\frac{1}{n}}} \int_t^{|\Omega|} |\nabla u|^{**}(s) s^{-\frac{1}{n'}}\, ds \qquad \text{for every } t \in (0, |\Omega|) \ . \tag{2.1}$$

3. Moser functions

The sharpness of our results is proved using sequences of suitably modified Moser functions. These are the functions that are often used to prove the second parts of (1.1), (1.6) and (1.7).

Fix $R > 0$. For every $s \in (0,1)$, we define the function $m_s \in W_0 L^{n,q}(B(R))$ by

$$m_s(x) = \begin{cases} n^{-\frac{1}{q}} \omega_n^{-\frac{1}{n}} \log^{\frac{q-1}{q}}(\frac{1}{s}) & 0 \leq |x| \leq sR \\ n^{-\frac{1}{q}} \omega_n^{-\frac{1}{n}} \log^{-\frac{1}{q}}(\frac{1}{s}) \log(\frac{R}{|x|}) & sR \leq |x| \leq R \ . \end{cases} \tag{3.1}$$

These functions satisfy

$$|\nabla m_s|(x) = \begin{cases} 0 & 0 \leq |x| < sR \\ n^{-\frac{1}{q}} \omega_n^{-\frac{1}{n}} \log^{-\frac{1}{q}}(\frac{1}{s}) \frac{1}{|x|} & sR < |x| < R \end{cases} \tag{3.2}$$

and thus

$$|\nabla m_s|^*(t) = \begin{cases} n^{-\frac{1}{q}} \omega_n^{-\frac{1}{n}} \log^{-\frac{1}{q}}(\frac{1}{s}) \frac{1}{(\frac{t}{\omega_n} + s^n R^n)^{\frac{1}{n}}} & 0 < t < \omega_n R^n - \omega_n s^n R^n \\ 0 & \omega_n R^n - \omega_n s^n R^n < t < \omega_n R^n \end{cases} \tag{3.3}$$

(indeed, the value of $|\nabla m_s|^*(t)$ corresponds to the value of $|\nabla m_s|$ on the sphere $\partial B(\varrho)$, where $\varrho > 0$ satisfies $t = |B(\varrho)| - |B(sR)| = \omega_n(\varrho^n - s^n R^n)$). It can be shown that (see [5] or [6, Section 4])

$$\|\nabla m_s\|_{n,q} \overset{s\to 0_+}{\to} 1 \ . \tag{3.4}$$

If we define the functions normalized with respect to the quasi-norm (1.3)

$$v_s = \frac{1}{||\nabla m_s||_{n,q}} m_s \ , \quad s \in (0,1) \ ,$$

then by [6, Lemma 4.1] we have for every $L > 0$

$$\int_{B(R)} \exp\Big(\Big(L + n\omega_n^{\frac{1}{n}} |v_s|\Big)^{q'}\Big)\, dx \overset{s\to 0_+}{\to} \infty \ . \tag{3.5}$$

Since we are going to work with the norm (1.4) it is convenient for us to define

$$w_s = \frac{1}{||\nabla m_s||_{(n,q)}} m_s, \quad s \in (0,1) \ . \tag{3.6}$$

Now, (1.5) and (3.5) imply that for every $L > 0$ we have

$$\int_{B(R)} \exp\Big(\Big(L + \frac{n^2}{n-1}\omega_n^{\frac{1}{n}} |w_s|\Big)^{q'}\Big)\, dx \overset{s\to 0_+}{\to} \infty \ . \tag{3.7}$$

In the proofs concerning the sharpness of our results, we use the sequence $\{w_{\frac{1}{k}}\}_{k\in\mathbb{N}}$ (we use $\frac{1}{k}$ in the place of the parameter s in the definition of m_s). It can be seen from (3.1), (3.6), (3.2), (3.4) and (1.5) that the sequence concentrates at the origin in the following way

$$\eta > 0 \quad \Longrightarrow \quad \sup_{\eta<|x|<R} |\nabla w_{\frac{1}{k}}(x)| \overset{k\to\infty}{\to} 0 \quad \text{and} \quad \sup_{\eta<|x|<R} |w_{\frac{1}{k}}(x)| \overset{k\to\infty}{\to} 0 \ . \tag{3.8}$$

We also have

$$w_{\frac{1}{k}} \to 0 \quad \text{on } B(R)\setminus\{0\} \qquad \text{and} \qquad w_{\frac{1}{k}} \rightharpoonup 0 \quad \text{in } W_0L^{n,q}(B(R)) \ .$$

4. Moser-type inequality

In this section we discuss the proofs of (1.10) and (1.11). For the reader familiar with Moser-type inequalities, the validity of (1.10) and (1.11) obviously follows from (1.6) and (1.7) via (1.5), (1.2) and (2.1).

Let us give some details. First, if $\{u_k\}$ is a Moser sequence for (1.6) (that is $\|\nabla u_k\|_{n,q} \le 1$ for every $k \in \mathbb{N}$ and the integrals tend to infinity as $k \to \infty$ whenever $K > n\omega_n^{\frac{1}{n}}$), then $\{\frac{n-1}{n}u_k\}$ is a Moser sequence for (1.10). Indeed, we have $\|\frac{n-1}{n}\nabla u_k\|_{(n,q)} \le 1$ for every $k \in \mathbb{N}$ by (1.5) and the borderline exponent is now $\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}$ instead of $n\omega_n^{\frac{1}{n}}$.

Concerning the boundedness, at the beginning of the standard proof of (1.6) in the case $K \le n\omega_n^{\frac{1}{n}}$ one applies (1.2) to obtain

$$\begin{aligned}
\int_\Omega \exp\Big(\Big(K|u(x)|\Big)^{q'}\Big)\, dx &= \int_0^{|\Omega|} \exp\Big(\Big(Ku^*(t)\Big)^{q'}\Big)\, dt \\
&\le \int_\Omega \exp\Bigg(\Bigg(K\frac{1}{n\omega_n^{\frac{1}{n}}}\Bigg(t^{-\frac{1}{n'}}\int_0^t |\nabla u|^*(s)\, ds + \int_t^{|\Omega|} |\nabla u|^*(s)s^{-\frac{1}{n'}}\, ds\Bigg)\Bigg)^{q'}\Bigg)\, dx \ .
\end{aligned}$$

Then some standard procedures follow. These procedures use Hölder's inequality, assumption $\|\nabla u\|_{n,q} \le 1$, Young inequality and several changes of variables. In our case, we use (2.1) to obtain

$$\begin{aligned}
\int_\Omega \exp\Big(\Big(K|u(x)|\Big)^{q'}\Big)\, dx &= \int_0^{|\Omega|} \exp\Big(\Big(Ku^*(t)\Big)^{q'}\Big)\, dt \\
&\le \int_\Omega \exp\Bigg(\Bigg(K\frac{n-1}{n^2\omega_n^{\frac{1}{n}}}\int_t^{|\Omega|} |\nabla u|^{**}(s)s^{-\frac{1}{n'}}\, ds\Bigg)^{q'}\Bigg)\, dx \ .
\end{aligned}$$

It is almost obvious that the procedures recalled above work again, since the difference between the borderline exponent in (1.6) and (1.10) is properly compensated by the the difference between the multiplicative constants in (1.2) and (2.1). Besides, one of the integrals in the exponent is missing now, this makes the proof a bit simpler.

5. Proof of Theorem 1.4(i)

Proof of Theorem 1.4(i). The proof is divided into several steps.

STEP 1. (proof of (1.13) and of the non-uniqueness of x_0)
By the definition of the number A, there are points $x_j \in \bar{\Omega}$, $j \in \mathbb{N}$, such that

$$\lim_{r\to 0_+} \limsup_{k\to\infty} \Big(\int_0^\infty \Big(t^{\frac{1}{n}-\frac{1}{q}} (|\nabla u_k| \chi_{B(x_j,r)})^{**}(t) \Big)^q \, dt \Big)^{\frac{1}{q}} \overset{j\to\infty}{\longrightarrow} 1 \ . \tag{5.1}$$

Since $\bar{\Omega}$ is compact, passing to a subsequence we can suppose that $\{x_j\}$ is a convergent sequence in $\bar{\Omega}$. Let x_0 be its limit. We claim that

$$\lim_{r\to 0_+} \limsup_{k\to\infty} \Big(\int_0^\infty \Big(t^{\frac{1}{n}-\frac{1}{q}} (|\nabla u_k| \chi_{B(x_0,r)})^{**}(t) \Big)^q \, dt \Big)^{\frac{1}{q}} = 1 \ . \tag{5.2}$$

Let us prove (5.2) by contradiction. If there is $\delta > 0$ such that

$$\lim_{r\to 0_+} \limsup_{k\to\infty} \Big(\int_0^\infty \Big(t^{\frac{1}{n}-\frac{1}{q}} (|\nabla u_k| \chi_{B(x_0,r)})^{**}(t) \Big)^q \, dt \Big)^{\frac{1}{q}} = 1 - 2\delta \ ,$$

then there is $r_0 > 0$ such that

$$\limsup_{k\to\infty} \Big(\int_0^\infty \Big(t^{\frac{1}{n}-\frac{1}{q}} (|\nabla u_k| \chi_{B(x_0,2r_0)})^{**}(t) \Big)^q \, dt \Big)^{\frac{1}{q}} \le 1 - \delta \ .$$

Next, as $x_j \to x_0$ as $j \to \infty$, we can find $j_0 \in \mathbb{N}$ so large that $x_j \in B(x_0, r_0)$ for every $j > j_0$. Therefore we also have

$$B(x_j, r) \subset B(x_0, 2r_0) \qquad \text{for every } r \in (0, r_0) \text{ and every } j > j_0 \ .$$

We infer

$$\lim_{r\to 0_+} \limsup_{k\to\infty} \Big(\int_0^\infty \Big(t^{\frac{1}{n}-\frac{1}{q}} (|\nabla u_k| \chi_{B(x_j,r)})^{**}(t) \Big)^q \, dt \Big)^{\frac{1}{q}} \overset{j\to\infty}{\le} 1 - \delta \qquad \text{for every } j > j_0 \ .$$

This contradicts (5.1) and thus (1.13) is proved.

The point x_0 is not unique in general, as can be seen considering a sequence $u_1, v_1, u_2, v_2, \dots$, where $\{u_k\}$ and $\{v_k\}$ are two Moser sequences centered at two different points.

STEP 2. (concentration of the modulars)
Suppose that $x_0 \in \bar{\Omega}$ is a unique point satisfying (1.13) and assumption (1.14) is satisfied. Our aim is to prove that for every fixed $\eta > 0$ we have

$$\int_{\Omega \setminus B(x_0,\eta)} \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k| \Big)^{q'} \Big) - 1 \Big) \, dx \to 0 \tag{5.3}$$

and

$$\int_{B(x_0,\eta)} \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k| \Big)^{q'} \Big) - 1 \Big) \, dx \to c \ . \tag{5.4}$$

Since (5.3) and assumption (1.14) obviously imply (5.4), it is enough to prove (5.3). Furthermore, as $\bar{\Omega} \setminus B(x_0, \eta)$ is compact, it is enough to prove that for every $x \neq x_0$ we can find the radius r_x such that

$$\int_{B(x,r_x)} \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k| \Big)^{q'} \Big) - 1 \Big) \, dx \to 0 \ . \tag{5.5}$$

Let us fix $x \neq x_0$ and let us prove (5.5). Since $x \neq x_0$ and x_0 is a unique point satisfying (1.13), we can find $\varepsilon > 0$, $r_0 > 0$ and $k_0 \in \mathbb{N}$ such that

$$|||\nabla u_k| \chi_{B(x,r_0)}||_{(n,q)} \le 1 - 2\varepsilon \qquad \text{for every } k > k_0 \ . \tag{5.6}$$

Let ψ be a smooth function such that $0 \le \psi \le 1$, $\psi \equiv 0$ on $\mathbb{R}^n \setminus B(x, r_0)$ and $\psi \equiv 1$ on $B(x, \frac{r_0}{2})$. Let us set $v_k = \psi u_k$. Hence v_k are $W_0 L^{n,q}$-functions vanishing outside $B(x, r_0)$. Our aim is to show that there is $k_1 > k_0$ such that

$$||\nabla v_k||_{(n,q)} \le 1 - \varepsilon \qquad \text{for every } k > k_1 \ . \tag{5.7}$$

We have $\nabla v_k = \psi \nabla u_k + u_k \nabla \psi$. Next, since $u_k \rightharpoonup 0$ in $W_0 L^{n,q}(\Omega)$, we also have $u_k \to 0$ in $L^{n,q}(\Omega)$. We infer that $u_k \nabla \psi \to 0$ in $L^{n,q}(\Omega)$, since $\max_{x \in \mathbb{R}^n} |\nabla \psi(x)|$ is finite. Now, (5.6) and the triangle inequality yield that we can find $k_1 > k_0$ such that for every $k > k_1$ we have

$$\begin{aligned} ||\nabla v_k||_{(n,q)} &= ||\psi \nabla u_k + u_k \nabla \psi||_{(n,q)} \le ||\psi \nabla u_k||_{(n,q)} + ||u_k \nabla \psi||_{(n,q)} \\ &\le (1 - 2\varepsilon) + \varepsilon = 1 - \varepsilon \end{aligned}$$

and (5.7) is proved. Hence for every $k > k_1$ we have by (1.10)

$$\begin{aligned} \int_{B(x, \frac{r_0}{2})} \exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} \frac{1}{1-\varepsilon} |u_k|\Big)^{q'}\Big)\, dx &= \int_{B(x, \frac{r_0}{2})} \exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} \frac{1}{1-\varepsilon} |v_k|\Big)^{q'}\Big)\, dx \\ &\le \int_{B(x, r_0)} \exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} \frac{1}{1-\varepsilon} |v_k|\Big)^{q'}\Big)\, dx \\ &\le \int_{B(x, r_0)} \exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} \frac{|v_k|}{||\nabla v_k||_{(n,q)}}\Big)^{q'}\Big)\, dx \le C \ . \end{aligned}$$

Therefore we can use $u_k \to 0$ a.e. and the Vitali convergence theorem for equiintegrable sequences of functions to obtain (5.5) with $r_x = \frac{r_0}{2}$. Thus, we also have (5.3) and (5.4).

STEP 3. (convergence in measures)

Fix an arbitrary test function $\psi \in C(\bar{\Omega})$ and let $\varepsilon > 0$. There exists $\eta > 0$ such that

$$|\psi(x) - \psi(x_0)| < \frac{\varepsilon}{2 \max(c, 1)} \qquad \text{whenever } x \in \bar{\Omega} \text{ and } |x - x_0| < \eta \ . \tag{5.8}$$

We have that

$$\begin{aligned} &\Big| \int_{\bar{\Omega}} \psi \, d(c\delta_{x_0}) - \int_\Omega \psi \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k|\Big)^{q'}\Big) - 1\Big)\, dx \Big| \\ &= \Big| c\psi(x_0) - \int_\Omega \psi \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k|\Big)^{q'}\Big) - 1\Big)\, dx \Big| \\ &\le \int_{\Omega \setminus B(x_0, \eta)} |\psi| \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k|\Big)^{q'}\Big) - 1\Big)\, dx \\ &\qquad + \int_{B(x_0, \eta)} |\psi - \psi(x_0)| \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k|\Big)^{q'}\Big) - 1\Big)\, dx + \\ &\qquad + |\psi(x_0)| \cdot \Big| c - \int_{B(x_0, \eta)} \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k|\Big)^{q'}\Big) - 1\Big)\, dx \Big| \\ &= I_1 + I_2 + I_3 \ . \end{aligned}$$

By (5.3) and the fact that $\sup_\Omega |\psi| < \infty$, there exists $k_1 \in \mathbb{N}$ such that $I_1 < \varepsilon$ for $k > k_1$. Furthermore, on making use of (5.4) and (5.8), we obtain

$$\begin{aligned} I_2 &= \int_{B(x_0, \eta)} |\psi - \psi(x_0)| \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k|\Big)^{q'}\Big) - 1\Big)\, dx \\ &\le \frac{\varepsilon}{2 \max(c, 1)} \int_{B(x_0, \eta)} \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k|\Big)^{q'}\Big) - 1\Big)\, dx \to \frac{\varepsilon}{2} \frac{c}{\max(c, 1)} \ . \end{aligned}$$

Therefore we can find $k_2 > k_1$ such that $I_2 < \varepsilon$ for $k > k_2$. Finally, owing to (5.4), there exists $k_3 > k_2$ such that $I_3 < \varepsilon$ for $k > k_3$. Thus,

$$\lim_{k \to \infty} \int_{\bar{\Omega}} \psi \, d(c\delta_{x_0}) - \int_\Omega \psi \Big(\exp\Big(\Big(\frac{n^2}{n-1} \omega_n^{\frac{1}{n}} |u_k|\Big)^{q'}\Big) - 1\Big)\, dx = 0,$$

and (1.15) follows. $\square$

6. Proof of Theorem 1.4(ii)

Proof of Theorem 1.4(ii): boundedness. Fix $p < A^{-1}$. Since $\bar{\Omega}$ is compact and we consider functions vanishing outside Ω, it is enough to show that for every $x \in \bar{\Omega}$, there is a radius $r_x > 0$ such that

$$\int_{B(x,r_x)} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}} p|u_k|\Big)^{q'}\Big)\,dx \le C \qquad \text{for every } k \in \mathbb{N}\ . \tag{6.1}$$

Let us prove this property. Fix $x \in \bar{\Omega}$. Let $\varepsilon > 0$ be so small that

$$(1+\varepsilon)p < (1-\varepsilon)A^{-1}\ . \tag{6.2}$$

First, we observe from the definition of the number A that there are $k_0 \in \mathbb{N}$ and $r_0 > 0$ such that

$$|||\nabla u_k|\chi_{B(x,r_0)}||_{(n,q)} \le (1+\varepsilon)A \qquad \text{for every } k > k_0\ . \tag{6.3}$$

Let ψ be a smooth function such that $0 \le \psi \le 1$, $\psi \equiv 0$ on $\mathbb{R}^n \setminus B(x,r_0)$ and $\psi \equiv 1$ on $B(x,\frac{r_0}{2})$. Let us set $v_k = p\psi u_k$. Hence v_k are $W_0L^{n,q}$-functions vanishing outside $B(x,r_0)$. Our aim is to show that there is $k_1 > k_0$ such that

$$||\nabla v_k||_{(n,q)} \le 1 \qquad \text{for every } k > k_1\ . \tag{6.4}$$

We have $\nabla v_k = p\psi\nabla u_k + pu_k\nabla\psi$. Next, since $u_k \rightharpoonup 0$ in $W_0L^{n,q}(\Omega)$, we also have $u_k \to 0$ in $L^{n,q}(\Omega)$. We infer that $pu_k\nabla\psi \to 0$ in $L^{n,q}(\Omega)$, since $\max_{x\in\mathbb{R}^n}|\nabla\psi(x)|$ is finite. Now, (6.2), (6.3) and the triangle inequality yield that we can find $k_1 > k_0$ such that for every $k > k_1$ we have

$$\begin{aligned}||\nabla v_k||_{(n,q)} &= ||p\psi\nabla u_k + pu_k\nabla\psi||_{(n,q)} \le ||p\psi\nabla u_k||_{(n,q)} + ||pu_k\nabla\psi||_{(n,q)}\\ &\le pA(1+\varepsilon) + \varepsilon \le (1-\varepsilon) + \varepsilon = 1\end{aligned}$$

and (6.4) is proved. Hence for every $k > k_1$ we have by (1.10)

$$\begin{aligned}\int_{B(x,\frac{r_0}{2})} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}} p|u_k|\Big)^{q'}\Big)\,dx &= \int_{B(x,\frac{r_0}{2})} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}} |v_k|\Big)^{q'}\Big)\,dx\\ &\le \int_{B(x,r_0)} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}} |v_k|\Big)^{q'}\Big)\,dx \le C\ .\end{aligned}$$

For the remaining finite number of indexes we use the Trudinger-type embedding (1.8). Thus, we see that we can set $r_x = \frac{r_0}{2}$ to obtain (6.1) and we are done. ☐

Proof of Theorem 1.4(ii): sharpness. Let us define a function $u \in W_0L^{n,q}(B(3R))$ by

$$u(x) = \begin{cases} L & \text{for } 0 \le |x| \le 2R\\ L(3-|x|) & \text{for } 2R \le |x| \le 3R\ ,\end{cases}$$

where $L > 0$ is a small number specified below. Next, for every $k \in \mathbb{N}$ we set

$$u_k = u + Aw_{\frac{1}{k}} \in W_0L^{n,q}(B(3R))\ ,$$

where the functions $w_{\frac{1}{k}}$ come from (3.6). We plainly have $u_k \to u$ a.e. in $B(3R)$ and thus it is easy to prove that $u_k \rightharpoonup u$ in $W_0L^{n,q}(B(3R))$.

Next, since $||\nabla Aw_{\frac{1}{k}}||_{(n,q)} = A < 1$, the constant L could have been chosen so small that

$$||\nabla u_k||_{(n,q)} \le ||\nabla Aw_{\frac{1}{k}}||_{(n,q)} + ||\nabla u||_{(n,q)} \le A + CL \le 1\ .$$

Furthermore, by (3.6) and (3.8), we obtain

$$\begin{aligned}&\sup_{x\in\bar{B}(3R)} \lim_{r\to 0_+} \limsup_{k\to\infty} \Big(\int_0^\infty \Big(t^{\frac{1}{n}-\frac{1}{q}}(|\nabla u_k|\chi_{B(x,r)})^{**}(t)\Big)^q\,dt\Big)^{\frac{1}{q}}\\ &= \lim_{r\to 0_+} \limsup_{k\to\infty} \Big(\int_0^\infty \Big(t^{\frac{1}{n}-\frac{1}{q}}(|\nabla u_k|\chi_{B(r)})^{**}(t)\Big)^q\,dt\Big)^{\frac{1}{q}}\\ &= A\lim_{r\to 0_+} \limsup_{k\to\infty} \Big(\int_0^\infty \Big(t^{\frac{1}{n}-\frac{1}{q}}(|\nabla w_{\frac{1}{k}}|\chi_{B(r)})^{**}(t)\Big)^q\,dt\Big)^{\frac{1}{q}} = A\ .\end{aligned}$$

Finally, by (3.7), we have for every $p \geq A^{-1}$

$$\int_{B(3R)} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}} p|u_k|\Big)^{q'}\Big)\,dx \geq \int_{B(R)} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}} A^{-1}|u_k|\Big)^{q'}\Big)\,dx$$
$$= \int_{B(R)} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}} A^{-1}L + \frac{n^2}{n-1}\omega_n^{\frac{1}{n}}|w_{\frac{1}{k}}|\Big)^{q'}\Big) \overset{k\to\infty}{\to} \infty\ .$$

Thus, we are done. □

7. Proof of Theorem 1.4(iii)

We start with some preliminary work. First, if v is a $W_0L^{n,q}$-function with $\operatorname{supp} v \subset \Omega$, we can use (2.1) and Hölder's inequality to obtain for every $s \in (0, |\operatorname{supp} v|)$

$$\begin{aligned} v^*(s) &\leq \frac{n-1}{n^2\omega_n^{\frac{1}{n}}} \int_s^{|\operatorname{supp} v|} |\nabla v|^{**}(r) r^{-\frac{n-1}{n}}\,dr \\ &\leq \frac{n-1}{n^2\omega_n^{\frac{1}{n}}} \Big(\int_s^{|\operatorname{supp} v|} \Big(r^{\frac{1}{n}-\frac{1}{q}}|\nabla v|^{**}(r)\Big)^q dr\Big)^{\frac{1}{q}} \Big(\int_s^{|\operatorname{supp} v|} \Big(r^{\frac{1}{q}-1}\Big)^{\frac{q}{q-1}} dr\Big)^{\frac{1}{q'}} \\ &\leq \frac{n-1}{n^2\omega_n^{\frac{1}{n}}} \Big(\int_s^{|\operatorname{supp} v|} \Big(r^{\frac{1}{n}-\frac{1}{q}}|\nabla v|^{**}(r)\Big)^q dr\Big)^{\frac{1}{q}} \Big(\int_s^{|\Omega|} r^{-1}\,dr\Big)^{\frac{1}{q'}} \\ &= \frac{n-1}{n^2\omega_n^{\frac{1}{n}}} \Big(\int_s^{|\operatorname{supp} v|} \Big(r^{\frac{1}{n}-\frac{1}{q}}|\nabla v|^{**}(r)\Big)^q dr\Big)^{\frac{1}{q}} \log^{\frac{1}{q'}}\Big(\frac{|\Omega|}{s}\Big)\ . \end{aligned} \tag{7.1}$$

Second, let us introduce the truncation operators T^L and T_L acting on any function $v\colon \Omega \mapsto \mathbb{R}$. They are defined by

$$T^L(v) = \min\{|v|, L\}\operatorname{sign}(v) \qquad \text{and} \qquad T_L(v) = v - T^L(v)\ .$$

Our aim is to prove the following result.

Proposition 7.1. *If $\varepsilon > 0$ and the assumptions of Theorem 1.4(iii) are satisfied, then there is $L > 0$ and $k_0 \in \mathbb{N}$ such that for every $k > k_0$*

$$\int_0^{|\operatorname{supp} T_L(u_k)|} t^{\frac{q}{n}-1}(|\nabla T_L(u_k)|^{**}(t))^q\,dt \leq 1 - (1-\varepsilon)\int_0^\infty t^{\frac{q}{n}-1}(|\nabla u|^{**}(t))^q\,dt\ .$$

We decompose the proof of Proposition 7.1 into several lemmata. Fix $\varepsilon \in (0, \frac{1}{2})$. We start with the following constructions.

First, by the absolute continuity of the Lebesgue integral, we can find $\eta_2 \in (0, |\operatorname{supp}|\nabla u|^*|)$ so small that

$$\int_0^{\eta_2} t^{\frac{q}{n}-1}(|\nabla u|^{**}(t))^q\,dt \leq \varepsilon \int_0^\infty t^{\frac{q}{n}-1}(|\nabla u|^{**}(t))^q\,dt\ . \tag{7.2}$$

Next, since $L^{n,q}$ is continuously embedded into L^1, we can also use the absolute continuity of the Lebesgue integral to obtain $\eta_1 \in (0, \eta_2)$ so small that

$$\int_0^{\eta_1} |\nabla u|^*(t)\,dt \leq \varepsilon \int_0^{\eta_2} |\nabla u|^*(t)\,dt\ . \tag{7.3}$$

Further, let $\tau > 0$ be so small that

$$\int_0^{|\Omega|} \tau\,dt \leq \varepsilon \int_0^{\eta_2} |\nabla u|^*(t)\,dt\ . \tag{7.4}$$

To simplify our notation, let us define $\sigma \in (0, |\Omega|]$ by

$$\sigma = \sup\{t \in (0, |\Omega|)\colon |\nabla u|^*(t) > \tau\}\ . \tag{7.5}$$

Next, we chose the truncation level $L > 0$.

Lemma 7.2. *If $L > 0$ is large enough, then*

$$|\{x \in \Omega : |u_k(x)| > L\}| < \eta_1 \qquad \text{for every } k \in \mathbb{N} \tag{7.6}$$

and

$$\int_0^\infty t^{\frac{q}{n}-1}(|\nabla T^L(u)|^{**}(t))^q\,dt \geq (1-\varepsilon)\int_0^\infty t^{\frac{q}{n}-1}(|\nabla u|^{**}(t))^q\,dt\ . \tag{7.7}$$

Proof. Estimate (7.6) easily follows from the assumption $\|\nabla u_k\|_{(n,q)} \leq 1$, $k \in \mathbb{N}$, the continuous embedding of $W_0L^{n,q}$ into $L^{n,q}$ and from the Chebyshev inequality.

When proving (7.7), it is enough to realize that $u = T^L(u) + T_L(u)$ and thus it is enough to show that we can make $||\nabla T_L(u)||_{(n,q)}$ arbitrarily small via a suitable choice of very large L. But this is easy, since we can use $\Omega = \bigcup_{m\in\mathbb{N}}(\Omega \cap \{|u| < m\})$ and the absolute continuity of the Lebesgue integral of $(t^{\frac{1}{n}-\frac{1}{q}}|\nabla u|^*(t))^q \in L^1((0,\infty))$ to show that the quasi-norm $||\nabla T_L(u)||_{n,q}$ can be made as small as we wish. Hence, by (1.5), the norm $||\nabla T_L(u)||_{(n,q)}$ is also as small as we wish. □

Now, when the constant L is chosen, we decompose the interval $[\eta_1, \sigma]$ into finite number of subintervals choosing $m \in \mathbb{N}$ and $\{a_j\}_{j=0}^m$ such that $\eta_1 = a_0 < a_1 < a_2 < \cdots < a_m = \sigma$ and

$$|\nabla T^L(u)|^* \in [(1-\varepsilon)|\nabla T^L(u)|^*(a_j), |\nabla T^L(u)|^*(a_j)] \qquad \text{on } [a_j, a_{j+1}) \tag{7.8}$$

(the non-increasing rearrangement is continuous from the right hand side, see [11, Section 1]). We can also suppose that the points a_j are chosen in such a way that for each number $j = 1, \dots, m-1$ such that $a_0 \neq a_j \neq a_m$ we have

$$\Big|\Big\{x \in \Omega\colon |\nabla T^L(u)|(x) = |\nabla T^L(u)|^*(a_j)\Big\}\Big| = 0\ . \tag{7.9}$$

Let us define the sets G_j, $j = 0, \dots, m-1$, to be the sets whose points are sent to each interval (a_j, a_{j+1}) when computing $|\nabla T^L(u)|^*$. In particular, if (7.9) holds for every $j = 0, \dots, m$, then we simply set

$$G_j = \Big\{x \in \Omega\colon |\nabla T^L(u)|(x) \in \Big(|\nabla T^L(u)|^*(a_{j+1}), |\nabla T^L(u)|^*(a_j)\Big)\Big\}\ , \qquad j = 0, \dots, m-1$$

and we have

$$|G_j| = a_{j+1} - a_j \qquad \text{for every } j = 0, \dots, m-1\ . \tag{7.10}$$

Otherwise, it is still not difficult to see that the sets G_j can be chosen to be measurable, disjoint, satisfying (7.10),

$$\Big\{|\nabla T^L(u)| \in \Big(|\nabla T^L(u)|^*(a_{j+1}), |\nabla T^L(u)|^*(a_j)\Big)\Big\} \subset G_j$$

and

$$\Big\{|\nabla T^L(u)| \in \Big[|\nabla T^L(u)|^*(a_{j+1}), |\nabla T^L(u)|^*(a_j)\Big]\Big\} \supset G_j\ .$$

Finally, we define the following modification of $|\nabla T^L(u)|^{**}$

$$|\nabla T^L(u)|^\diamond(t) = \frac{1}{t}\int_0^t |\nabla T^L(u)|^*(s)\chi_{(\eta_1,\sigma)}(s)\,ds \qquad \text{for } t > 0\ . \tag{7.11}$$

Notice that with the above definitions the following identities hold

$$\begin{aligned}\int_0^\infty t^{\frac{q}{n}-1}(|\nabla T^L(u)|^\diamond(t))^q\,dt &= \int_0^\infty t^{\frac{q}{n}-1-q}\Big(\int_0^t |\nabla T^L(u)|^*(s)\chi_{(\eta_1,\sigma)}(s)\,ds\Big)^q\,dt \\ &= \int_{\eta_1}^\infty t^{\frac{q}{n}-1-q}\Big(\int_{\eta_1}^t |\nabla T^L(u)|^*(s)\chi_{(\eta_1,\sigma)}(s)\,ds\Big)^q\,dt \\ &= \int_{\eta_1}^\infty t^{\frac{q}{n}-1-q}\Big(\int_0^{t-\eta_1} (|\nabla T^L(u)|\chi_{\bigcup_{j=0}^{m-1}G_j})^*(s)\,ds\Big)^q\,dt\ .\end{aligned} \tag{7.12}$$

Lemma 7.3. *We have for every $k \in \mathbb{N}$*

$$\int_0^{\eta_1} t^{\frac{q}{n}-1}(|\nabla T_L(u_k)|^{**}(t))^q\,dt + \int_{\eta_1}^{\infty} t^{\frac{q}{n}-1-q}\Big(\int_0^{t-\eta_1} |\nabla T^L(u_k)|^*(s)\,ds\Big)^q\,dt \le \int_0^{\infty} t^{\frac{q}{n}-1}(|\nabla u_k|^{**}(t))^q\,dt\ .$$

Proof. The estimate follows from the inequalities

$$\int_0^t |\nabla T_L(u_k)|^*(s)\,ds \le \int_0^t |\nabla u_k|^*(s)\,ds \qquad \text{for every } t \in (0,\eta_1)$$

and

$$\int_0^{t-\eta_1} |\nabla T^L(u_k)|^*(s)\,ds \le \int_0^t |\nabla u_k|^*(s)\,ds \qquad \text{for every } t \in (\eta_1,\infty)\ .$$

Both inequalities are obvious. □

Lemma 7.4. *There is $k_0 \in \mathbb{N}$ such that for every $k > k_0$ and every $t > 0$ we have*

$$\int_0^t (|\nabla T^L(u_k)|\chi_{\bigcup_{j=0}^{m-1} G_j})^*(s)\,ds \ge (1-\varepsilon)^2 \int_0^t (|\nabla T^L(u)|\chi_{\bigcup_{j=0}^{m-1} G_j})^*(s)\,ds\ . \tag{7.13}$$

Proof. First, let us show that for every $j = 0,\dots,m-1$ there is $\tilde{k}_j \in \mathbb{N}$ such that for every $k > \tilde{k}_j$ we have

$$\int_0^t (|\nabla T^L(u_k)|\chi_{G_j})^*(s)\,ds \ge (1-\varepsilon)^2 \int_0^t (|\nabla T^L(u)|\chi_{G_j})^*(s)\,ds \qquad \text{for every } t > 0\ . \tag{7.14}$$

The assumption $u_k \rightharpoonup u$ in $W_0L^{n,q}(\Omega)$ can be used to obtain that $T^L(u_k) \rightharpoonup T^L(u)$ in $W_0L^{n,q}(\Omega)$ (indeed, $\{T^L(u_k)\}$ is a bounded sequence, hence it has a weakly convergent subsequence and the convergence a.e. implies that the weak limit has to be $T^L(u)$). This implies $\nabla T^L(u_k) \rightharpoonup \nabla T^L(u)$ in $L^{n,q}(\Omega)$. Hence $\nabla T^L(u_k) \rightharpoonup \nabla T^L(u)$ in $L^{n,q}(G_j)$ for each j (indeed, the dual space to $L^{n,q}(\Omega)$ is $L^{n',q'}(\Omega)$ where the simple functions are dense, hence it is enough to consider only characteristic functions as the test-functions and thus each such a test-function for the weak convergence in $L^{n,q}(G_j)$ is also a test-function for the weak convergence in $L^{n,q}(\Omega)$). Furthermore, we also have $\nabla T^L(u_k) \rightharpoonup \nabla T^L(u)$ in $L^1(G_j)$. Hence the weak lower semicontinuity of the L^1-norm yields that there is $\tilde{k}_j$ such that

$$\int_{G_j} |\nabla T^L(u_k)|\,dx \ge (1-\varepsilon)\int_{G_j} |\nabla T^L(u)|\,dx \qquad \text{for every } j \text{ and every } k > \tilde{k}_j\ .$$

Thus

$$\int_0^{|G_j|} (|\nabla T^L(u_k)|\chi_{G_j})^*(s)\,ds = \int_{G_j} |\nabla T^L(u_k)|\,dx \ge (1-\varepsilon)\int_{G_j} |\nabla T^L(u)|\,dx = (1-\varepsilon)\int_0^{|G_j|} (|\nabla T^L(u)|\chi_{G_j})^*(s)\,ds\ .$$

On one hand, this proves (7.14) for every $t \ge |G_j|$. On the other hand, using the definition of G_j and (7.8) we obtain from above

$$\int_0^{|G_j|} (|\nabla T^L(u_k)|\chi_{G_j})^*(s)\,ds \ge (1-\varepsilon)\int_0^{|G_j|} (|\nabla T^L(u)|\chi_{G_j})^*(s)\,ds \ge (1-\varepsilon)^2 \int_0^{|G_j|} |\nabla T^L(u)|^*(a_j)\,ds\ .$$

Now, since $(|\nabla T^L(u_k)|\chi_{G_j})^*$ is non-increasing on $(0,|G_j|)$ and $|\nabla T^L(u)|^*(a_j) \ge (|\nabla T^L(u)|\chi_{G_j})^*(s)$ for every $s \in (0,|G_j|)$, we easily infer that for every $t \in (0,|G_j|)$

$$\int_0^t (|\nabla T^L(u_k)|\chi_{G_j})^*(s)\,ds \ge (1-\varepsilon)^2 \int_0^t |\nabla T^L(u)|^*(a_j)\,ds \ge (1-\varepsilon)^2 \int_0^t (|\nabla T^L(u)|\chi_{G_j})^*(s)\,ds\ .$$

This completes the proof of (7.14).

Next, the proof of (7.13) follows from (7.14) by induction. Indeed, if $t \in (0, a_m - a_0]$ (it is enough to consider this case only, since in the case $t > a_m - a_0 = \sum_{j=0}^{m-1} |G_j|$, the integrals are the same as if $t = a_m - a_0$) and if $\ell \in \mathbb{N} \cup \{0\}$, $\ell < m-1$, is such that

$$\sum_{j=0}^{\ell} |G_j| < t \leq |\sum_{j=0}^{\ell+1} |G_j| \ ,$$

then we have by (7.14) for every $k > k_0 := \max\{\tilde{k}_0, \dots, \tilde{k}_{m-1}\}$

$$\begin{aligned}
&(1-\varepsilon)^2 \int_0^t (|\nabla T^L(u)|\chi_{\bigcup_{j=0}^{m-1} G_j})^*(s)\, ds \\
&= (1-\varepsilon)^2 \Bigg(\int_0^{|G_1|} (|\nabla T^L(u)|\chi_{G_1})^*(s)\, ds + \int_0^{|G_2|} (|\nabla T^L(u)|\chi_{G_2})^*(s)\, ds \\
&\qquad + \dots + \int_0^{t-\sum_{j=0}^{\ell}|G_j|} (|\nabla T^L(u)|\chi_{G_{\ell+1}})^*(s)\, ds \Bigg) \\
&\leq \int_0^{|G_1|} (|\nabla T^L(u_k)|\chi_{G_1})^*(s)\, ds + \int_0^{|G_2|} (|\nabla T^L(u_k)|\chi_{G_2})^*(s)\, ds \\
&\qquad + \dots + \int_0^{t-\sum_{j=0}^{\ell}|G_j|} (|\nabla T^L(u_k)|\chi_{G_{\ell+1}})^*(s)\, ds \\
&= \int_0^{|G_1|} (|\nabla T^L(u_k)|\chi_{G_1})^*(s)\, ds + \int_{|G_1|}^{|G_1|+|G_2|} (|\nabla T^L(u_k)|\chi_{G_2})^*(s - |G_1|)\, ds \\
&\qquad + \dots + \int_{\sum_{j=0}^{\ell}|G_j|}^{t} (|\nabla T^L(u_k)|\chi_{G_{\ell+1}})^*\Big(s - \sum_{j=0}^{\ell}|G_j|\Big)\, ds \\
&\leq \int_0^t (|\nabla T^L(u_k)|\chi_{\bigcup_{j=0}^{m-1} G_j})^*(s)\, ds
\end{aligned}$$

and thus we are done. □

Lemma 7.5. *We have*

$$\int_0^\infty t^{\frac{q}{n}-1} (|\nabla T^L(u)|^\diamond(t))^q\, dt \geq (1 - C\varepsilon) \int_0^\infty t^{\frac{q}{n}-1} (|\nabla T^L(u)|^{**}(t))^q\, dt \ ,$$

where $C > 0$ depends on q only.

Proof. We have

$$\int_0^\infty t^{\frac{q}{n}-1}\Big((|\nabla T^L(u)|^{**}(t))^q - (|\nabla T^L(u)|^\diamond(t))^q \Big)\, dt = \int_0^{\eta_2} + \int_{\eta_2}^{\sigma} + \int_\sigma^\infty = I_1 + I_2 + I_3 \ . \tag{7.15}$$

Let us estimate each summand on the rightmost side. First, by (7.2), (7.7) and $\varepsilon < \frac{1}{2}$ we have

$$\begin{aligned}
I_1 &\leq \int_0^{\eta_2} t^{\frac{q}{n}-1} (|\nabla T^L(u)|^{**}(t))^q\, dt \\
&\leq \int_0^{\eta_2} t^{\frac{q}{n}-1} (|\nabla u|^{**}(t))^q\, dt \\
&\leq \varepsilon \int_0^\infty t^{\frac{q}{n}-1} (|\nabla u|^{**}(t))^q\, dt \\
&\leq 2\varepsilon \int_0^\infty t^{\frac{q}{n}-1} (|\nabla T^L(u)|^{**}(t))^q\, dt \ .
\end{aligned} \tag{7.16}$$

Next, using (7.11) and the inequality $(x+y)^q \le x^q + q2^{q-1}(y^q + x^{q-1}y)$ valid for every $x, y \ge 0$ and $q \ge 1$ we arrive at

$$\begin{aligned}
I_2 &= \int_{\eta_2}^{\sigma} t^{\frac{q}{n}-1-q}\Big(\Big(\int_0^t |\nabla T^L(u)|^*(s)\,ds\Big)^q - \Big(\int_{\eta_1}^t |\nabla T^L(u)|^*(s)\,ds\Big)^q\Big)\,dt \\
&\le q2^{q-1}\int_{\eta_2}^{\sigma} t^{\frac{q}{n}-1-q}\Big(\int_0^{\eta_1} |\nabla T^L(u)|^*(s)\,ds\Big)^q\,dt \\
&\qquad + q2^{q-1}\int_{\eta_2}^{\sigma} t^{\frac{q}{n}-1-q}\Big(\int_{\eta_1}^t |\nabla T^L(u)|^*(s)\,ds\Big)^{q-1}\Big(\int_0^{\eta_1} |\nabla T^L(u)|^*(s)\,ds\Big)\,dt \\
&= q2^{q-1}(J_1 + J_2)\ .
\end{aligned}$$

Next, we use (7.3), (7.7) and $\varepsilon < \frac{1}{2}$ to obtain

$$\begin{aligned}
J_1 &= \int_{\eta_2}^{\sigma} t^{\frac{q}{n}-1-q}\Big(\int_0^{\eta_1} |\nabla T^L(u)|^*(s)\,ds\Big)^q\,dt \\
&\le \int_{\eta_2}^{\sigma} t^{\frac{q}{n}-1-q}\Big(\int_0^{\eta_1} |\nabla u|^*(s)\,ds\Big)^q\,dt \\
&\le \varepsilon^q\int_{\eta_2}^{\sigma} t^{\frac{q}{n}-1-q}\Big(\int_0^{t} |\nabla u|^*(s)\,ds\Big)^q\,dt \\
&= \varepsilon^q\int_{\eta_2}^{\sigma} t^{\frac{q}{n}-1}(|\nabla u|^{**}(t))^q\,dt \\
&\le \varepsilon^q\int_{0}^{\infty} t^{\frac{q}{n}-1}(|\nabla u|^{**}(t))^q\,dt \\
&\le 2\varepsilon^q\int_{0}^{\infty} t^{\frac{q}{n}-1}(|\nabla T^L(u)|^{**}(t))^q\,dt\ .
\end{aligned} \tag{7.17}$$

Furthermore, the Young inequality yields

$$\begin{aligned}
J_2 &= \int_{\eta_2}^{\sigma} t^{\frac{q}{n}-1-q}\Big(\varepsilon^{\frac{1}{q}}\int_{\eta_1}^t |\nabla T^L(u)|^*(s)\,ds\Big)^{q-1}\Big(\frac{1}{\varepsilon^{\frac{q-1}{q}}}\int_0^{\eta_1} |\nabla T^L(u)|^*(s)\,ds\Big)\,dt \\
&\le \frac{q-1}{q}\varepsilon\int_{\eta_2}^{\sigma} t^{\frac{q}{n}-1-q}\Big(\int_{\eta_1}^t |\nabla T^L(u)|^*(s)\,ds\Big)^q\,dt + \frac{1}{q}\int_{\eta_2}^{\sigma} t^{\frac{q}{n}-1-q}\Big(\frac{1}{\varepsilon^{\frac{q-1}{q}}}\int_0^{\eta_1} |\nabla T^L(u)|^*(s)\,ds\Big)^q\,dt\ .
\end{aligned}$$

Now, the first integral has a suitable appearance while the second integral can be estimated in the same way as in (7.17) and thus

$$J_2 \le \Big(\frac{q-1}{q}\varepsilon + \frac{1}{q}\varepsilon^{1-q}2\varepsilon^q\Big)\int_0^\infty t^{\frac{q}{n}-1}(|\nabla T^L(u)|^{**}(t))^q\,dt\ .$$

Hence

$$I_2 \le q2^{q-1}(J_1+J_2) \le q2^{q-1}\Big(2\varepsilon^q + \frac{q-1}{q}\varepsilon + \frac{2}{q}\varepsilon\Big)\int_0^\infty t^{\frac{q}{n}-1}(|\nabla T^L(u)|^{**}(t))^q\,dt\ . \tag{7.18}$$

It remains to estimate I_3. Using (7.11) and the inequality $(x+y)^q \le x^q + q2^{q-1}(y^q + x^{q-1}y)$ again we arrive at

$$\begin{aligned}
I_3 &= \int_{\sigma}^{\infty} t^{\frac{q}{n}-1-q}\Big(\Big(\int_0^t |\nabla T^L(u)|^*(s)\,ds\Big)^q - \Big(\int_{\eta_1}^\sigma |\nabla T^L(u)|^*(s)\,ds\Big)^q\Big)\,dt \\
&\le q2^{q-1}\int_{\sigma}^{\infty} t^{\frac{q}{n}-1-q}\Big(\int_0^{\eta_1} |\nabla T^L(u)|^*(s)\,ds + \int_\sigma^t |\nabla T^L(u)|^*(s)\,ds\Big)^q\,dt \\
&\quad + q2^{q-1}\int_{\sigma}^{\infty} t^{\frac{q}{n}-1-q}\Big(\int_{\eta_1}^t |\nabla T^L(u)|^*(s)\,ds\Big)^{q-1}\Big(\int_0^{\eta_1} |\nabla T^L(u)|^*(s)\,ds + \int_\sigma^t |\nabla T^L(u)|^*(s)\,ds\Big)\,dt\ .
\end{aligned}$$

Now, we have by (7.3) (recall $t \ge \sigma \ge \eta_2$)

$$\int_0^{\eta_1} |\nabla T^L(u)|^*(s)\,ds \le \int_0^{\eta_1} |\nabla u|^*(s)\,ds \le \varepsilon\int_0^t |\nabla u|^*(s)\,ds \tag{7.19}$$

and by (7.4) and (7.5)

$$\int_\sigma^t |\nabla T^L(u)|^*(s)\,ds \le \int_\sigma^t \tau\,ds \le \varepsilon \int_0^{\eta_2} |\nabla u|^*(s)\,ds \le \varepsilon \int_0^t |\nabla u|^*(s)\,ds\ . \tag{7.20}$$

Notice that in (7.19) and (7.20) we have estimated the left hand sides by the same quantity. Hence a careful inspection of the procedure leading to the estimate of I_2 shows, that we can use the same approach to estimate I_3. In particular, we obtain the following version of (7.18)

$$I_3 \le q2^{q-1}\Big(22^q\varepsilon^q + \frac{q-1}{q}2^q\varepsilon + \frac{1}{q}\varepsilon^{1-q}22^q\varepsilon^q\Big)\int_0^\infty t^{\frac{q}{n}-1}(|\nabla T^L(u)|^{**}(t))^q\,dt\ . \tag{7.21}$$

Finally, the proof follows from (7.15), (7.16), (7.18) and (7.21). □

Proof of Proposition 7.1. It is enough to establish the following chain of inequalities

$$\begin{aligned}
\int_0^{|\operatorname{supp} T_L(u_k)|} & t^{\frac{q}{n}-1}(|\nabla T_L(u_k)|^{**}(t))^q\,dt \\
&\le \int_0^{\eta_1} t^{\frac{q}{n}-1}(|\nabla T_L(u_k)|^{**}(t))^q\,dt \\
&\le \int_0^\infty t^{\frac{q}{n}-1}(|\nabla u_k|^{**}(t))^q\,dt - \int_{\eta_1}^\infty t^{\frac{q}{n}-1-q}\Big(\int_0^{t-\eta_1}|\nabla T^L(u_k)|^*(s)\,ds\Big)^q\,dt \\
&\le 1 - \int_{\eta_1}^\infty t^{\frac{q}{n}-1-q}\Big(\int_0^{t-\eta_1}(|\nabla T^L(u_k)|\chi_{\bigcup_{j=0}^{m-1}G_j})^*(s)\,ds\Big)^q\,dt \\
&\le 1-(1-\varepsilon)^2\int_{\eta_1}^\infty t^{\frac{q}{n}-1-q}\Big(\int_0^{t-\eta_1}(|\nabla T^L(u)|\chi_{\bigcup_{j=0}^{m-1}G_j})^*(s)\,ds\Big)^q\,dt \\
&= 1-(1-\varepsilon)^2\int_0^\infty t^{\frac{q}{n}-1}(|\nabla T^L(u)|^{\diamond}(t))^q\,dt \\
&\le 1-(1-\varepsilon)^2(1-C\varepsilon)\int_0^\infty t^{\frac{q}{n}-1}(|\nabla T^L(u)|^{**}(t))^q\,dt \\
&\le 1-(1-\varepsilon)^3(1-C\varepsilon)\int_0^\infty t^{\frac{q}{n}-1}(|\nabla u|^{**}(t))^q\,dt\ .
\end{aligned} \tag{7.22}$$

Let us prove (7.22). The first inequality follows from (7.6). The second inequality follows from Lemma 7.3. In the third inequality we have used the assumption $||\nabla u_k||_{(n,q)} \le 1$ and a trivial estimate. The fourth inequality follows from Lemma 7.4. The fifth relation is an equality following from (7.12). The sixth is an inequality which follows from Lemma 7.5 and the last inequality follows from (7.7).

Hence we have proven (7.22). Since the constant C depending on q only is irrelevant, we are done. □

Proof of Theorem 1.4(iii): boundedness. Suppose that $p < P$. We fix $p_1, p_2 \in (p, P)$ such that $p_1 < p_2$. Next, we fix $\varepsilon > 0$ so small that

$$\frac{1}{p_2} \ge \Big(1-(1-\varepsilon)||\nabla u||^q_{(n,q)}\Big)^{\frac{1}{q}}\ . \tag{7.23}$$

The existence of such ε is obvious if $||\nabla u||_{(n,q)} = 1$, otherwise we use the fact that

$$\frac{1}{p_2} > \frac{1}{P} = \Big(1-||\nabla u||^q_{(n,q)}\Big)^{\frac{1}{q}}\ .$$

From the definition of $T_L(u_k)$ we observe that $|u_k(x)| \le |T_L(u_k)(x)| + L$ for every $x \in \Omega$ and thus

$$u_k^*(s) \le T_L(u_k^*)(s) + L = (T_L(u_k))^*(s) + L \qquad \text{for every } s \in (0, |\Omega|)\ .$$

Thus, we can use (7.1), Proposition 7.1, (7.23) and $p_1 < p_2$ to obtain $s_0 \in (0, |\Omega|)$ such that we have for every $s \in (0, s_0]$ and every $k > k_0$

$$\begin{aligned} u_k^*(s) &\leq L + (T_L(u_k))^*(s) \\ &\leq L + \frac{n-1}{n^2\omega_n^{\frac{1}{n}}}\Big(\int_0^{|\operatorname{supp} T_L(u_k)|} r^{\frac{q}{n}-1}\Big(|\nabla T_L(u_k)|^{**}(r)\Big)^q\,dr\Big)^{\frac{1}{q}} \log^{\frac{1}{q'}}\Big(\frac{|\Omega|}{s}\Big) \\ &\leq L + \frac{n-1}{n^2\omega_n^{\frac{1}{n}}}\Big(1-(1-\varepsilon)\|\nabla u\|_{(n,q)}^q\Big)^{\frac{1}{q}} \log^{\frac{1}{q'}}\Big(\frac{|\Omega|}{s}\Big) \\ &\leq L + \frac{n-1}{n^2\omega_n^{\frac{1}{n}}}\frac{1}{p_2}\log^{\frac{1}{q'}}\Big(\frac{|\Omega|}{s}\Big) \\ &\leq \frac{n-1}{n^2\omega_n^{\frac{1}{n}}}\frac{1}{p_1}\log^{\frac{1}{q'}}\Big(\frac{|\Omega|}{s}\Big)\ . \end{aligned}$$

Thus, we obtain for every every $k > k_0$

$$\int_0^{|\Omega|} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}p|u_k^*|\Big)^{q'}\Big)\,ds \leq \int_0^{s_0}\Big(\frac{|\Omega|}{s}\Big)^{(\frac{p}{p_1})^{q'}}\,ds + \int_{s_0}^{|\Omega|}\Big(\frac{|\Omega|}{s_0}\Big)^{(\frac{p}{p_1})^{q'}}\,ds \leq C\ .$$

For the remaining finite number of indexes we can use (1.8). Thus, we are done. □

Proof of Theorem 1.4(iii): sharpness of P. Fix arbitrary $\theta \in (0,1)$. We define a function $u \in W_0L^{n,q}(B(3R))$ by

$$u(x) = \begin{cases} L & \text{for } 0 \leq |x| \leq 2R \\ L(3-|x|) & \text{for } 2R \leq |x| \leq 3R\ , \end{cases}$$

where $L > 0$ is such that

$$\theta^q = ||\nabla u||_{(n,q)}^q = \int_0^{|B(3R)\setminus B(2R)|} t^{\frac{q}{n}-1}L^q\,dt + \int_{|B(3R)\setminus B(2R)|}^{\infty} t^{\frac{q}{n}-1-q}L^q|B(3R)\setminus B(2R)|^q\,dt\ .$$

We have

$$P = \Big(1 - ||\nabla u||_{(n,q)}^q\Big)^{-\frac{1}{q}} = (1-\theta^q)^{-\frac{1}{q}}. \tag{7.24}$$

We are going to show that for any $p > P$, we can construct a sequence $\{u_k\} \subset W_0L^{n,q}(B(3R))$ such that

$$u_k \rightharpoonup u\ ,\quad \|\nabla u_k\|_{(n,q)} \leq 1 \text{ for every } k \text{ large enough}\ ,\quad u_k \to u \text{ a.e. in } (B(3R)) \tag{7.25}$$

and

$$\int_{B(3R)} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}p|u_k|\Big)^{q'}\Big)\,dx \overset{k\to\infty}{\to} \infty\ . \tag{7.26}$$

To show this, fix $p > P$. We set $u_k = u + \frac{1}{p}w_{\frac{1}{k}} \in W_0L^{n,q}(B(3R))$, $k \in \mathbb{N}$, where the functions $w_{\frac{1}{k}} \in W_0L^{n,q}(B(R))$ come from (3.6).

Let us prove (7.25) and (7.26). Concerning (7.25), the weak convergence and the convergence a.e. are obvious. Let us prove the estimate of the norm. Since $\frac{1}{p^q} + \theta^q < 1$ (see (7.24)), we can find $\varepsilon > 0$ so small that

$$\frac{1}{p^q} + (1+\varepsilon)^q\theta^q + 2\varepsilon \leq 1\ . \tag{7.27}$$

We want to prove that if $k \in \mathbb{N}$ is large enough, then

$$I := \int_0^\infty t^{\frac{q}{n}-1}(|\nabla u_k|^{**}(t))^q\,dt \leq 1\ .$$

Fix $k \in \mathbb{N}$. We decompose the integral I setting

$$I = I_1 + I_2 + I_3 = \int_0^{a_1} + \int_{a_1}^{a_2} + \int_{a_2}^{\infty}\ ,$$

where the numbers a_1, a_2 are chosen in the following way

$$a_1 = \sup\{t > 0\colon |\nabla u_k|^*(t) > L\} \ ,$$
$$a_2 = \sup\{t > 0\colon |\nabla u_k|^*(t) \geq L\} \ .$$

Let us also define the numbers $J_1, J_2, J_3, J_4 > 0$ by

$$\begin{aligned} \frac{1}{p^q} = \Big\|\frac{1}{p}\nabla w_{\frac{1}{k}}\Big\|^q_{(n,q)} &= \int_0^\infty t^{\frac{q}{n}-1}\Big(\frac{1}{p}|\nabla w_{\frac{1}{k}}|^{**}(t)\Big)^q\,dt \\ &= \int_0^{a_1} + \int_{a_1}^\infty = J_1 + J_2 \end{aligned} \tag{7.28}$$

and (it is easy to see that $a_2 - a_1 = |B(3R)\setminus B(2R)|$)

$$\theta^q = ||\nabla u||^q_{(n,q)} = \int_0^\infty t^{\frac{q}{n}-1}(|\nabla u|^{**}(t))^q\,dt = \int_0^{a_2-a_1} + \int_{a_2-a_1}^\infty = J_3 + J_4 \ . \tag{7.29}$$

We plainly have

$$I_1 = J_1 \ . \tag{7.30}$$

Let us estimate I_2. Notice, that by (3.3) there are $0 < C_1 < C_2$ (independent of k) such that

$$C_1 \log^{-\frac{n}{q}}(k) \leq a_1 \leq C_2 \log^{-\frac{n}{q}}(k) \ . \tag{7.31}$$

Thus, using (3.3) again

$$\begin{aligned} \int_0^{a_1} \frac{1}{p}|\nabla w_{\frac{1}{k}}|^*(s)\,ds &\leq C\log^{-\frac{1}{q}}(k)\int_0^{a_1} s^{-\frac{1}{n}}\,ds \leq C\log^{-\frac{1}{q}}(k)a_1^{1-\frac{1}{n}} \\ &\leq C\log^{-\frac{1}{q}}(k)\log^{-\frac{n}{q}(1-\frac{1}{n})}(k) = C\log^{-\frac{n}{q}}(k) \ . \end{aligned}$$

Now

$$\begin{aligned} I_2 &= \int_{a_1}^{a_2} t^{\frac{q}{n}-1}\Big(\frac{1}{p}|\nabla u_k|^{**}(t)\Big)^q\,dt \\ &= \int_{a_1}^{a_2} t^{\frac{q}{n}-1-q}\Big(\int_0^{a_1}\frac{1}{p}|\nabla w_{\frac{1}{k}}|^*(s)\,ds + \int_{a_1}^t L\,ds\Big)^q\,dt \\ &\leq \int_{a_1}^{a_2} t^{\frac{q}{n}-1-q}\Big(C\log^{-\frac{n}{q}}(k) + Lt\Big)^q\,dt \ . \end{aligned}$$

We apply the estimate $(x+y)^q \leq x^q + q2^{q-1}(x^{q-1}y + y^q)$ valid for every $x, y \geq 0$ and $q \geq 1$ together with (7.31) to obtain for k large enough

$$\begin{aligned} I_2 &\leq \int_{a_1}^{a_2} t^{\frac{q}{n}-1}L^q\,dt + C\int_{a_1}^{a_2} t^{\frac{q}{n}-2}L^{q-1}\log^{-\frac{n}{q}}(k)\,dt + C\int_{a_1}^{a_2} t^{\frac{q}{n}-1-q}\log^{-n}(k)\,dt \\ &\leq \Big(J_3 + \int_{a_1}^{a_2}(t^{\frac{q}{n}-1} - (t-a_1)^{\frac{q}{n}-1})L^q\,dt\Big) + C\log^{-\frac{n}{q}}(k)\Big|\big[t^{\frac{q}{n}-1}\big]_{a_1}^{a_2}\Big| + C\log^{-n}(k)\Big|\big[t^{\frac{q}{n}-q}\big]_{a_1}^{a_2}\Big| \\ &\leq (J_3 + \varepsilon) + C\Big(\log^{-\frac{n}{q}}(k) + \log^{-\frac{n}{q}}(k)a_1^{\frac{q}{n}-1}\Big) + C\Big(\log^{-n}(k) + \log^{-n}(k)a_1^{\frac{q}{n}-q}\Big) \\ &\leq J_3 + \varepsilon + C\Big(\log^{-\frac{n}{q}}(k) + \log^{-\frac{n}{q}}(k)\log^{-\frac{n}{q}(\frac{q}{n}-1)}(k)\Big) + C\Big(\log^{-n}(k) + \log^{-n}(k)\log^{-\frac{n}{q}(\frac{q}{n}-q)}(k)\Big) \\ &\leq J_3 + 2\varepsilon \ . \end{aligned} \tag{7.32}$$

Now, let us estimate I_3. Using (3.3) we obtain for k large enough

$$\int_0^{|\operatorname{supp}\nabla w_{\frac{1}{k}}|}\frac{1}{p}|\nabla w_{\frac{1}{k}}|^*(s)\,ds \leq C\log^{-\frac{1}{q}}(k)\int_0^{\omega_n} s^{-\frac{1}{n}}\,ds \leq C\log^{-\frac{1}{q}}(k) \ .$$

Hence we have for k large enough

$$
\begin{aligned}
I_3 &= \int_{a_2}^{\infty} t^{\frac{q}{n}-1}(|\nabla u_k|^{**}(t))^q\,dt \\
&\le \int_{a_2}^{\infty} t^{\frac{q}{n}-1-q}\Big(\int_0^{|\operatorname{supp}\nabla w_{\frac{1}{k}}|}\frac{1}{p}|\nabla w_{\frac{1}{k}}|^*(s)\,ds+\int_0^{|\operatorname{supp}\nabla u|}|\nabla u|^*(s)\,ds\Big)^q\,dt \\
&\le \int_{a_2}^{\infty} t^{\frac{q}{n}-1-q}\Big(C\log^{-\frac{1}{q}}(k)+\int_0^{|B(3R)\setminus B(2R)|}|\nabla u|^*(s)\,ds\Big)^q\,dt \\
&\le \int_{a_2}^{\infty} t^{\frac{q}{n}-1-q}\Big((1+\varepsilon)\int_0^{a_2-a_1}|\nabla u|^*(s)\,ds\Big)^q\,dt \\
&\le (1+\varepsilon)^q J_4\ .
\end{aligned}
\tag{7.33}
$$

Here we used the fact that $a_1 \to 0$ as $k \to \infty$. Now, from (7.30), (7.32), (7.33), (7.28), (7.29) and (7.27) we infer

$$
I = I_1 + I_2 + I_3 \le J_1 + J_3 + 2\varepsilon + (1+\varepsilon)^q J_4 \le \frac{1}{p^q} + (1+\varepsilon)^q\theta^q + 2\varepsilon \le 1\ .
$$

Finally, we use (3.7) to obtain

$$
\begin{aligned}
&\int_{B(3R)} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}p|u_k|\Big)^{q'}\Big)\,dx \\
&\qquad \ge \int_{B(R)} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}p\Big(L+\frac{1}{p}w_{\frac{1}{k}}\Big)\Big)^{q'}\Big)\,dx \\
&\qquad \ge \int_{B(R)} \exp\Big(\Big(\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}pL+\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}w_{\frac{1}{k}}\Big)^{q'}\Big)\,dx \overset{k\to\infty}{\to} \infty\ .
\end{aligned}
$$

This is (7.26) and we are done. □

Proof of Theorem 1.4. The property $A \in [0,1]$ follows from the assumption $||\nabla u_k||_{(n,q)} \le 1$, $k \in \mathbb{N}$. Statements (i), (ii) and (iii) have already been proved. The convergence of $\exp((\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}|u_k|)^{q'})$ in $L^1(\Omega)$ in the cases (ii) and (iii) follows from the Vitali convergence theorem for equiintegrable sequences of functions and the uniform boundedness of $||\exp((\frac{n^2}{n-1}\omega_n^{\frac{1}{n}}p|u_k|)^{q'})||_{L^1(\Omega)}$ with $p > 1$. □

Proof of Theorem 1.3. This theorem is actually Theorem 1.4(iii). □

8. Concluding remarks

It is a natural question to ask what happens to our results if we replace the norm (1.4) by a bit smaller norm

$$
||u|| := \begin{cases} ||t^{\frac{1}{n}-\frac{1}{q}}u^{**}(t)||_{L^q((0,|\Omega|))} & \text{for } q < \infty \\ \sup_{t\in(0,|\Omega|)} t^{\frac{1}{n}}u^{**}(t) & \text{for } q = \infty\ . \end{cases}
$$

It can be seen that all our results have the same statements as when considering the norm (1.4) up to integrating over $(0, |\Omega|)$ in the definition of the quantity A now. Some details are given below.

Results (1.10) **and** (1.11)**.** These results have the same statement as before. Indeed, on one hand, when proving the boundedness, in our key estimate (2.1) we always integrate over the interval $(t, |\Omega|)$ only. On the other hand, the old Moser functions are still admissible when proving the sharpness of the borderline exponent, since we work with a smaller norm.

Theorem 1.4(i). In the definition of the quantity A we integrate over $(0, |\Omega|)$ now. The rest of the statement and the proof are the same as before up to using the new norm, the new version of A and the new version of (1.10).

Theorem 1.4(ii). Again, it is enough to use the new norm, the new version of A and the new version of (1.10) in the proof of the boundedness. In the proof of the sharpness, the functions from (3.6) have to be normalized with respect to the new norm. Since the new norm is smaller than the original one, we plainly have (3.7) also for the new functions. Thus, the construction follows the same lines as before.

Theorem 1.4(iii) (and Theorem 1.3). The statement is the same as before and the proof requires minor changes only. This is not quite obvious since changing the norm influences the size of P. Nevertheless, the reason why integrating over $(|\Omega|, \infty)$ is irrelevant rests upon the fact that the borderline exponent in Moser-type inequalities is determined by the behavior of concentrating sequences (these sequences behave in the same way as in (3.8) and thus the integrals over $(|\Omega|, \infty)$ tend to zero).

Acknowledgement. The author was supported by the ERC CZ grant LL1203 of the Czech Ministry of Education. The author would like to thank Luboš Pick for fruitful discussions.

References

[1] Adachi S., Tanaka K., Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents, *Proc. Amer. Math. Soc.* **128** no. 7 (1999), 2051-2057.

[2] R. D.Adams, A sharp inequality of J. Moser for higher order derivatives, *Ann. of Math.* **128** (1988), 385-398.

[3] Adimurthi, Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian, *Ann. Sc. Norm. Sup. Pisa* **17** (1990), 393-413.

[4] Adimurthi, Positive solutions of the semilinear Dirichlet problem with Critical growth in the unit disc in $\mathbb{R}^2$, *Proc. Indian Acad. Sci.* **99** (1989), 49-73.

[5] Alvino A., Ferone V., Trombetti Q., Moser-Type Inequalities in Lorentz Spaces, *Potential Anal.* **5** (1996), 273-299.

[6] R. Černý, *Concentration-Compactness Principle for Moser-type Inequalities in Lorentz-Sobolev spaces*, to appear in Potential Anal. (preprint is available at www.karlin.mff.cuni.cz/kma-preprints/).

[7] R. Černý, A. Cianchi and S. Hencl, Concentration-compactness principle for Moser-Trudinger inequalities: new results and proofs, *Ann. Mat. Pura Appl.* **192** no. 2 (2013), 225–243.

[8] Černý R., Mašková S., A sharp form of an embedding into multiple exponential spaces, *Czechoslovak Math. J.* **60** no. 3 (2010), 751-782.

[9] Cianchi A., Moser-Trudinger inequalities without boundary conditions and isoperimetric problems, *Indiana Univ. Math. J.* **54** (2005), 669-705.

[10] Hencl S., A sharp form of an embedding into exponential and double exponential spaces, *J. Funct. Anal.* **204,** no. 1 (2003), 196–227.

[11] Hunt R. A., On $L(p,q)$ spaces, *Enseignement Math.* **12,** no. 2 (1966), 249–276.

[12] Lions P. L., The concentration-compactness principle in the calculus of variations. The limit case. I., *Rev. Mat. Iberoamericana* **1,** no. 1 (1985), 145-201.

[13] Lorentz G. G., On the theory of spaces Λ, *Pacific J. Math.* **1** (1951), 411-429.

[14] Moser J., A sharp form of an inequality by N. Trudinger, *Indiana Univ. Math. J.* **20** (1971), 1077–1092.

[15] Talenti G., Inequalities in rearrangement invariant function spaces, *Nonlinear Analysis, Function Spaces and Applications* **5** (1994), 177-230.

[16] Talenti G., An inequality between u^* and ∇u, *General Inequalities, 6 (Oberwolfach, 1990), Internat. Ser. Numer. Math., Birkhuser, Basel* **103** (1992), 175-182.

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic

E-mail address: rcerny@karlin.mff.cuni.cz