

SPACES OF D.C. MAPPINGS ON ARBITRARY INTERVALS

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Dedicated to the memory of Jean-Jacques Moreau.

ABSTRACT. Let X be a Banach space. Using derivatives in the sense of vector distributions, we show that the space $DC([0, 1], X)$ of all d.c. mappings from $[0, 1]$ into X , in a natural norm, is isomorphic to the space $M_{bv}([0, 1], X)$ of all vector measures with bounded variation. The same is proved for the space $BDC_b((0, \infty), X)$ of all bounded d.c. mappings with a bounded control function. The result for the space $DC([0, 1], \mathbb{R})$ of all continuous d.c. functions was (essentially) proved by M. Zippin (2000) by a quite different method. The space $BDC_b((0, \infty), \mathbb{R})$ consists of all differences of two bounded convex functions. Internal characterizations of its members were given by O. Böhme (1985), but our characterization of its Banach structure is new.

1. INTRODUCTION

If C is a convex subset of a (real) normed linear space, then $f: C \rightarrow \mathbb{R}$ is called a *d.c. function* (DC function, or a delta-convex function) if it can be represented as a difference of two continuous convex functions on C . In [19], the notion of a d.c. function was extended to the notion of a *d.c. mapping between arbitrary Banach spaces* (see Definition 2.1 below) and a theory of such mappings was developed (see Introduction in [10] for a brief review).

For functions of several variables no satisfactory internal characterization of d.c. functions is known. However, internal characterizations of (special) d.c. functions on intervals in \mathbb{R} were given in [14], [3] and [21]. Our main results partly generalize the results of [3] and [21] to the case of d.c. mappings; some of them are new also in the case of d.c. functions.

Functions which are differences of two *Lipschitz* convex functions on $[a, b]$ are characterized in [14] (as functions with bounded convexity). A generalization of this result to the case of d.c. mappings is given in [20]. Internal characterizations of d.c. mappings F defined on *open intervals* are also well-known (see [20, Theorem B]). Here we present a new characterization (Theorem 3.1 below) which deals with the second distributional derivative D^2F and provides a generalization of the classical characterization of d.c. functions on open intervals (see [15, p. 54]).

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In [3], several internal characterizations of functions which are differences of two *bounded* convex functions on $(0, \infty)$ are given. We consider the corresponding set $BDC_b((0, \infty), X)$ of bounded d.c. mappings from $(0, \infty)$ to X having a bounded control function, and give internal characterizations of these mappings.

The space $K[0, 1]$ of differences of (possibly discontinuous) convex functions on $[0, 1]$ was considered in [21]. Using the theory of $L_1(\mu)$ preduals it is proved that $K[0, 1]$ equipped with a specific norm (which is equivalent to a natural norm on $K[0, 1]$) is isometric to the dual space $C(F)^*$, where $F = \{-1\} \cup [0, 1] \cup \{2\}$. We give an analogue of Zippin's result for d.c. mappings. Namely, we prove (see Theorem 5.11) that if X is a Banach space, then the space $DC([0, 1], X)$ of all d.c. mappings from $[0, 1]$ to X , equipped with a certain norm (which is equivalent to a natural norm on $DC([0, 1], X)$), is isometric to the space $M_{bv}([0, 1], X)$ of all vector measures with bounded variation. Consequently, the space $DC([0, 1], X^*)$ is isomorphic to the dual space $C([0, 1], X)^*$.

Moreover, we prove (see Theorem 5.14) that the space $BDC_b((0, \infty), X)$ (in a natural norm) is also isomorphic to the space $M_{bv}([0, 1], X)$. This result is new also in the case $X = \mathbb{R}$.

Note that our main idea leading to results on the structure of the Banach spaces $DC([0, 1], X)$ and $BDC_b((0, \infty), X)$ is based on considering the second distributional derivative of mappings belonging to these spaces and is quite different from the main idea of [21]. Our method is relatively easy in the case of d.c. functions ($X = \mathbb{R}$), but in the general case it needs a number of results on vector measures and vectors distributions, which are collected in Preliminaries.

2. PRELIMINARIES

2.1. Basic notation.

Throughout the paper, all normed linear and Banach spaces will be real spaces. By an integral of a real (resp. vector) function we mean the Lebesgue (resp. Bochner) integral. If it is not specified otherwise, the integral is taken with respect to the one-dimensional Lebesgue measure. We use the standard convention that $\int_a^b f := -\int_b^a f$ if $b < a$.

If $M \subset A \subset \mathbb{R}$ and $f: A \rightarrow X$ are given, then we define the *variation of f on M* as

$$V(f, M) := \sup \left\{ \sum_{i=1}^n \|f(x_{i-1}) - f(x_i)\| \right\},$$

where the supremum is taken over all $(x_i)_{i=0}^n \subset M$ such that $x_0 < x_1 < \dots < x_n$. (We set $V(f, M) := 0$, if M is empty or a singleton.) We say that f has *bounded variation on M* , provided $V(f, M) < \infty$.

For $a, b \in \mathbb{R}$, we denote $V_a^b f := V(f, [a, b])$ if $a < b$, $V_a^b f := -V_b^a f$ if $b < a$, and $V_a^b f := 0$ if $a = b$. Further, for $a \in \mathbb{R}$, set $V_a^\infty f := V(f, [a, \infty))$. Obviously, $V_a^\infty f = \lim_{x \rightarrow \infty} V_a^x f$.

For basic well-known properties of the variation of a vector function, see, e.g., [12] and [6]. In particular, we will use consequences of the additivity of variation

(see [6, (P3) on p. 263]):

$$(1) \quad V(f, M) = V(f, M \cap (-\infty, t]) + V(f, M \cap [t, \infty)), \quad \text{if } t \in M.$$

As usual, for f as above we denote $\|f\|_\infty = \sup_{x \in A} \|f(x)\|$.

In the present article, we use the definition of d.c. mappings from [19] also for mappings defined on a non-open convex set:

Definition 2.1. Let X, Y be normed linear spaces, $C \subset X$ a convex set, and $F: C \rightarrow Y$ a continuous mapping. We say that F is *d.c.* (or “delta-convex”, or DC) if there exists a continuous (necessarily convex) function $f: C \rightarrow \mathbb{R}$ such that $y^* \circ F + f$ is convex on C whenever $y^* \in Y^*$, $\|y^*\| \leq 1$. In this case we say that f controls F , or that f is a *control function* for (of) F .

Remark 2.2. Similarly as in [19], we have the following facts.

- (a) A real function F on C is a d.c. function iff it is d.c. in the sense of Definition 2.1. Indeed, if $F = g - h$ where g, h are continuous and convex then $f := g + h$ is a control function of F ; and if f controls F then $F = \frac{1}{2}(F + f) - \frac{1}{2}(f - F)$ is a d.c. function.
- (b) If $Y = \mathbb{R}^n$, and $F = (F_1, \dots, F_n)$, then F is d.c. iff all functions F_i are d.c.

Let us make the following agreement: *in the rest of this section, $I \subset \mathbb{R}$ will be an open interval, and X a Banach space.*

2.2. Vector measures.

There are two possibilities for defining X -valued measures on I : measures as σ -additive set functions ([7], [9]), and measures as continuous linear operators on $C_c(I)$, the space of continuous functions with compact support ([2], [5], [11]). We are going to follow the first possibility.

Let $\mathcal{B}(I)$ be the Borel σ -algebra on I , and $\mathcal{B}_c(I)$ the field of those $B \in \mathcal{B}(I)$ whose closure \overline{B} is a compact subset of I . We shall need the following two classes of X -valued measures on I .

By $M_{bv}(I, X)$ we mean the class of all X -valued vector measures on $\mathcal{B}(I)$ with bounded variation. So $\mu \in M_{bv}(I, X)$ means that $\mu: \mathcal{B}(I) \rightarrow X$ is σ -aditive and

$$|\mu|(B) := \sup \left\{ \sum_{i=1}^n \|\mu(B_i)\| \right\} < \infty \quad (B \in \mathcal{B}(I)),$$

where the supremum is taken over all finite Borel decompositions $\{B_1, \dots, B_n\}$ of B .

The definition of $M_{lbv}(I, X)$, the class of X -valued vector measures on $\mathcal{B}_c(I)$ with “locally bounded variation”, is obtained in the same way by everywhere writing $\mathcal{B}_c(I)$ instead of $\mathcal{B}(I)$.

Recall that, for $\mu \in M_{bv}(I, X)$ or $\mu \in M_{lbv}(I, X)$, the variation $|\mu|$ is a nonnegative finite measure on $\mathcal{B}(I)$ or $\mathcal{B}_c(I)$, respectively.

Observation 2.3. *Given $\mu \in M_{lbv}(I, X)$, then its variation $|\mu|$ admits a natural extension to a measure $|\mu|: \mathcal{B}(I) \rightarrow [0, \infty]$, given by*

$$|\mu|(B) := \sup \{ |\mu|(\tilde{B}) : \tilde{B} \in \mathcal{B}_c(I), \tilde{B} \subset B \}.$$

If, in addition, $|\mu|(I) < \infty$, then μ has a unique extension to a measure $\mu: \mathcal{B}(I) \rightarrow X$, and this extension belongs to $M_{bv}(I, X)$; see [7, p. 62, Theorem 1] (in which \mathcal{K} , \mathcal{C} , μ , \mathbf{m} are our $\mathcal{B}(I)$, $\mathcal{B}_c(I)$, $|\mu|$, μ , respectively). So, any measure $\mu \in M_{bv}(I, X)$ such that $|\mu|(I) < \infty$ can be considered as an element of $M_{bv}(I, X)$.

2.3. Multiplication of a vector measure by a function.

Let $\mu \in M_{bv}(I, X)$. If $\theta: I \rightarrow \mathbb{R}$ is a locally bounded and Borel measurable function, then it is possible to define the integrals $\int_B \theta d\mu \in X$, $B \in \mathcal{B}_c(I)$, as in [9, pp. 5-6]. Moreover, it is easy and well-known that

$$(2) \quad \left\| \int_B \theta d\mu \right\| \leq \int_B |\theta| d|\mu| \quad (B \in \mathcal{B}_c(I))$$

(e.g., this can be seen as a special case of [7, p. 142, 12] for $E = \mathbb{R}$, $F = X$).

Now, we can define $\theta\mu: \mathcal{B}_c(I) \rightarrow X$ by

$$(3) \quad (\theta\mu)(B) = \int_B \theta d\mu.$$

Remark 2.4. Let μ, θ be as above, $B \in \mathcal{B}_c(I)$, $x^* \in X^*$. By [9, p. 6],

$$(4) \quad x^* \left(\int_B \theta d\mu \right) = \int_B \theta d(x^* \circ \mu).$$

Consequently, $x^* \circ \theta\mu = \theta(x^* \circ \mu)$. This fact and the well-known scalar case easily imply that:

- (a) $\theta\mu$ is σ -additive on $\mathcal{B}_c(I)$;
- (b) the mapping $\theta \mapsto \theta\mu$ is linear;
- (c) $\theta_1(\theta_2\mu) = (\theta_1\theta_2)\mu$ if θ_1, θ_2 are locally bounded and Borel measurable on I .

We need also the following easy (probably known) fact for which we have not found any reference.

Lemma 2.5. *Let μ and θ be as above. Then $\theta\mu \in M_{bv}(I, X)$ and*

$$(5) \quad |\theta\mu|(B) = \int_B |\theta| d|\mu| \quad (B \in \mathcal{B}_c(I)).$$

Proof. By Remark 2.4, $\theta\mu$ is σ -additive. Let $\{B_1, \dots, B_n\}$ be a Borel decomposition of $B \in \mathcal{B}_c(I)$. By (2), $\sum_1^n \|(\theta\mu)(B_j)\| = \sum_1^n \left\| \int_{B_j} \theta d\mu \right\| \leq \sum_1^n \int_{B_j} |\theta| d|\mu| = \int_B |\theta| d|\mu|$. This shows that $\theta\mu \in M_{bv}(I, X)$ and the inequality “ \leq ” in (5) holds.

Now, fix $B \in \mathcal{B}_c(I)$ and $\varepsilon > 0$. Let $\{J_n\}$ be a countable disjoint covering of \mathbb{R} by intervals of length ε . Let $\{B_n\} \subset \mathcal{B}(I)$ be a countable disjoint covering of B , such that $B_n \subset \theta^{-1}(J_n)$ ($n \in \mathbb{N}$). For each n , fix $c_n \in J_n \setminus \{0\}$; moreover, let $\{B_{n,1}, \dots, B_{n,m_n}\}$ be a Borel decomposition of B_n such that

$$|\mu|(B_n) \leq \sum_{j=1}^{m_n} \|\mu(B_{n,j})\| + \varepsilon |c_n|^{-1} |\mu|(B_n).$$

Then

$$\begin{aligned}
\int_B |\theta| d|\mu| &= \sum_n \int_{B_n} |\theta| d|\mu| \leq \sum_n \left(\int_{B_n} |c_n| d|\mu| + \varepsilon |\mu|(B_n) \right) \\
&= \sum_n |c_n| |\mu|(B_n) + \varepsilon |\mu|(B) \leq \sum_n \sum_{j=1}^{m_n} |c_n| |\mu|(B_{n,j}) + 2\varepsilon |\mu|(B) \\
&= \sum_n \sum_{j=1}^{m_n} \left\| \int_{B_{n,j}} c_n d\mu \right\| + 2\varepsilon |\mu|(B) \\
&\leq \sum_n \sum_{j=1}^{m_n} \left\| \int_{B_{n,j}} \theta d\mu \right\| + 3\varepsilon |\mu|(B) \\
&= \sum_n \sum_{j=1}^{m_n} \|(\theta\mu)(B_{n,j})\| + 3\varepsilon |\mu|(B) \\
&\leq \sum_n \sum_{j=1}^{m_n} |\theta\mu|(B_{n,j}) + 3\varepsilon |\mu|(B) = |\theta\mu|(B) + 3\varepsilon |\mu|(B).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the inequality “ \geq ” in (5) follows, and we are done. \square

Now, in view of Observation 2.3, the above lemma immediately implies the following

Corollary 2.6. *Let μ and θ be as above. Then $|\theta\mu|(I) = \int_I |\theta| d|\mu|$. In particular, $\theta\mu \in M_{bv}(I, X)$ if and only if $\int_I |\theta| d|\mu| < \infty$.*

2.4. Banach spaces of vector measures with bounded variation.

Let X be a Banach space, $B \subset \mathbb{R}$ a nonempty Borel set. By $M_{bv}(B, X)$ we mean the vector space of all Borel X -valued vector measures with bounded variation on B . Thus a mapping $\mu: \mathcal{B}(B) \rightarrow X$ belongs to $M_{bv}(B, X)$ if and only if μ is σ -additive and $\|\mu\|_{M_{bv}} := |\mu|(B) < \infty$. It is well known that $\|\cdot\|_{M_{bv}}$ is a complete norm on $M_{bv}(B, X)$. Throughout the paper, the space $M_{bv}(B, X)$ will be always considered in this norm.

In the last section, we shall use the following fact.

Lemma 2.7. *Let X be a Banach space, $I \subset \mathbb{R}$ an open interval, and $A = \{a_1, \dots, a_n\} \subset \mathbb{R} \setminus I$ a finite set of cardinality $n \geq 0$. Then the Banach space $M_{bv}([0, 1], X)$ is isometric to any of the spaces*

$$(6) \quad M_{bv}(I \cup A, X) \quad \text{and} \quad M_{bv}(I, X) \oplus_1 \left(\bigoplus_{i=1}^n X \right)_1,$$

where the last space is the ℓ_1 -direct sum of $M_{bv}(I, X)$ and n copies of X .

Proof. It is clear that the formula

$$\Phi(\mu) := (\mu|_{\mathcal{B}(I)}, \mu(\{a_1\}), \dots, \mu(\{a_n\})), \quad \mu \in M_{bv}(I \cup A, X),$$

defines a linear isometry between the two spaces from (6).

Since I is homeomorphic to $(0, 1)$, the spaces $M_{bv}(I, X)$ and $M_{bv}((0, 1), X)$ are isometric. So we can suppose $I = (0, 1)$.

Let $\psi: (0, 1) \cup A \rightarrow [0, 1]$ be a bijection such that both ψ and ψ^{-1} are Borel measurable. To construct such a ψ , it suffices to fix an infinite countable set $N \subset (0, 1)$ and define ψ piecewise as: the identity on $(0, 1) \setminus N$, and a bijection of $N \cup A$ onto $N \cup \{0, 1\}$. Now, the mapping Ψ , defined by

$$\Psi(\mu)(B) := \mu(\psi^{-1}(B)), \quad B \in \mathcal{B}([0, 1]), \quad \text{for } \mu \in M_{bv}((0, 1) \cup A, X),$$

is clearly a linear isometry of $M_{bv}((0, 1) \cup A, X)$ onto $M_{bv}([0, 1], X)$. \square

We shall need also the following theorem which follows from a result by I. Singer [17] (see also [7, Corollary 2, p. 387]) and the observation that each $\mu \in M_{bv}([0, 1], X^*)$ is regular since the real nonnegative measure $|\mu|$ on $[0, 1]$ is regular. As usual, $C([0, 1], X)$ denotes the Banach space of all continuous X -valued functions on $[0, 1]$, in the supremum norm.

Theorem 2.8 (Singer [17]). *Let X be a Banach space. Then the Banach space $M_{bv}([0, 1], X^*)$ is isometric to the dual space $C([0, 1], X)^*$.*

2.5. Vector distributions.

Let $\mathcal{D}(I)$ be the space of real-valued C^∞ -functions on I with compact support and, given a compact set $K \subset I$, let $\mathcal{D}_K(I) = \{\varphi \in \mathcal{D}(I) : \text{supp}(\varphi) \subset K\}$.

An X -valued distribution on I can be defined (see [15, p. 30, and Théorème III, p. 66], cf. also [16]) as a linear mapping $T: \mathcal{D}(I) \rightarrow X$ such that each of its restrictions

$$T|_{\mathcal{D}_K(I)}: \mathcal{D}_K(I) \rightarrow X \quad (K \subset I \text{ compact})$$

is a continuous linear operator if $\mathcal{D}_K(I)$ is equipped with the locally convex topology induced by the pseudonorms $N_p(\varphi) := \|\varphi^{(p)}\|_\infty$ ($p = 0, 1, 2, \dots$).

Any locally Bochner integrable vector function $G: I \rightarrow X$ induces the distribution $T_G(\varphi) = \int_I \varphi G dx$, since for each $\varphi \in \mathcal{D}_K(I)$ clearly $\|\int_I \varphi G dx\| \leq \|\varphi\|_\infty \cdot \int_K \|G(x)\| dx$. As usual, we say (slightly incorrectly) that T_G is a function.

More generally, let $\mu \in M_{bv}(I, X)$ and $\varphi \in \mathcal{D}(I)$. Fix $a, b \in I$ so that $\text{supp}(\varphi) \subset (a, b)$, and denote by $\tilde{\mu}$ the restriction of μ to $\mathcal{B}((a, b))$. Since $\tilde{\mu} \in M_{bv}((a, b), X)$, we can define the integral $T(\varphi) := \int_I \varphi d\tilde{\mu}$ as in [9, pp. 5-6]. Since clearly the integral $T(\varphi)$ does not depend on the choice of (a, b) , we can denote it by $\int_I \varphi d\mu$ or $T_\mu(\varphi)$. Then T_μ is an X -valued distribution, since $\|\int_I \varphi d\mu\| \leq \|\varphi\|_\infty \cdot |\mu|((a, b))$ (see [9, p. 6]). Moreover, if $\mu_1, \mu_2 \in M_{bv}(I, X)$ are distinct then $T_{\mu_1} \neq T_{\mu_2}$; indeed, this follows from the well-known real case (see [15]) by observing that

$$(7) \quad T_{x^* \circ \mu} = x^* \circ T_\mu \quad \text{for each } x^* \in X^* \text{ and } \mu \in M_{bv}(I, X),$$

which follows immediately from [9, p. 6]. So, as usual in the classical case, we can identify μ and T_μ , and we shall simply say that T_μ is a measure (from $M_{bv}(I, X)$).

2.6. Distributional derivatives.

Let T be an X -valued distribution on I . The *distributional derivative* of T is the X -valued distribution DT on I , defined by $DT(\varphi) = -T(\varphi')$, $\varphi \in \mathcal{D}(I)$. The *second distributional derivative* of T is $D^2T := D(DT)$. The following fact is well-known (for scalar distributions see [15, p. 51]).

Fact 2.9. *Let T be an X -valued distribution on I , such that $DT = 0$. Then T is (given by) a constant vector function.*

Proof. For each $x^* \in X^*$, we have $D(x^* \circ T) = x^* \circ DT = 0$, and hence $x^* \circ T = T_{g_{x^*}}$ where $g_{x^*}(x) \equiv c_{x^*} \in \mathbb{R}$. Thus

$$(8) \quad x^*(T(\varphi)) = c_{x^*} \int_I \varphi \quad (\varphi \in \mathcal{D}(I)).$$

Fix some $\varphi_0 \in \mathcal{D}(I)$ such that $\int_I \varphi_0 = 1$. Then (8) implies that $c_{x^*} = x^*(T(\varphi_0))$, $x^* \in X^*$, and hence, by (8) again, $T(\varphi) = T(\varphi_0) \int_I \varphi$, $\varphi \in \mathcal{D}(I)$. This means that $T = T_G$ for $G(x) \equiv T(\varphi_0)$. \square

Observation 2.10. *Let $G: I \rightarrow X$ be a locally Bochner integrable function, $z \in I$, and $F(x) = \int_z^x G$ ($x \in I$). Then $DF = G$.*

Proof. The assertion follows easily by the well-known case $X = \mathbb{R}$. Indeed, for each $x^* \in X^*$, we have $x^* \circ DF = D(x^* \circ F) = x^* \circ G$. \square

The following proposition, which generalizes the well-known real case, comes essentially from [7].

Proposition 2.11.

- (a) *If $G: I \rightarrow X$ is a right-continuous vector function with locally bounded variation, then there exists a unique $\mu = \mu_G \in M_{lbv}(I, X)$ such that*

$$\mu((x, y]) = G(y) - G(x), \quad x, y \in I, x < y.$$

Moreover, each $\mu \in M_{lbv}(I, X)$ is of the form μ_G for some G , and $\mu_{G_1} = \mu_{G_2}$ if and only if $G_1 - G_2$ is constant.

- (b) *If $\mu = \mu_G$, then*

$$|\mu|((x, y]) = V_x^y G, \quad x, y \in I, x < y.$$

- (c) *$\mu_G = DG$, that is, μ_G is the distributional derivative of G .*

Proof. The proof of (a),(b) for left-continuous (instead of right-continuous) vector functions follows from [7] (see Chap. III, §17, 2 and 3). Part (c) is well-known for $X = \mathbb{R}$ (see, e.g., [15, p. 53]). The general case reduces to the scalar case by using (7) and the definition of the distributional derivative; see also [2, p. 46]. \square

3. SECOND DISTRIBUTIONAL DERIVATIVES OF D.C. MAPPINGS ON OPEN INTERVALS

Theorem 3.1. *Let $I \subset \mathbb{R}$ be an open interval, X a Banach space, and $F: I \rightarrow X$ a continuous mapping. The following assertions are equivalent.*

- (i) *F is a d.c. mapping.*
- (ii) *The distributional derivative DF is (given by) a vector function of locally bounded variation.*
- (iii) *The second distributional derivative D^2F is a measure from $M_{lbv}(I, X)$.*

Moreover, the mapping $D^2: DC(I, X) \rightarrow M_{lbv}(I, X)$ is linear and onto, and its kernel $\text{Ker}(D^2)$ coincides with the set of all affine mappings of I into X .

Proof. Assume (i). By [19] or [20], F'_+ is a well-defined X -valued function on I . By [20, Theorem 3.1] and [20, Proposition 3.4, p. 329], F'_+ is right-continuous and has locally bounded variation. By [20, Theorem 3.1] we have $F(x) = F(z) + \int_z^x F'_+$, $x \in$

I , and so $DF = F'_+$ by Observation 2.10. Hence we can apply Proposition 2.11 to conclude that $D^2F = D(F'_+)$ is a measure from $M_{lbv}(I, X)$, that is, (iii) holds.

Assume (iii), i.e., $D^2F = \mu \in M_{lbv}(I, X)$. By Proposition 2.11, there exists a right-continuous vector function G of locally bounded variation such that $DG = \mu$. Since $D(G - DF) = 0$, Fact 2.9 implies that $G - DF$ is a constant function, and hence (ii) holds.

Assume (ii). Fix some $z \in I$ and define $\tilde{F}(x) = \int_z^x (DF)$. These Bochner integrals exist; indeed, the integrand is bounded, and continuous outside an at most countable set (see, e.g., [6, Theorem 4.1(b)]). Now, \tilde{F} is locally d.c. by [20, Theorem 3.1], and hence d.c. on I by [19, Theorem 1.20]. Since $D(F - \tilde{F}) = 0$ by Observation 2.10, we obtain that $F - \tilde{F}$ is a constant function, and hence (i) holds.

Now it is easy to see that D^2 is onto. Indeed, given $\mu \in M_{lbv}(I, X)$, then $\mu = DG$ for some right-continuous vector function G of locally bounded variation (Proposition 2.11). Then, as above, the corresponding vector function $\tilde{F}(x) := \int_z^x G$ is d.c., and $D^2\tilde{F} = DG = \mu$.

Finally, if $F \in DC(I, X)$ is such that $D^2F = 0$ then, by Fact 2.9, DF is a constant function, equal to some $c \in X$. Define $H(x) := xc$, $x \in I$, and notice that DH is the constant function c by Observation 2.10. Since $D(F - H) = 0$, another application of Fact 2.9 gives existence of $b \in X$ such that $F(x) = b + xc$ almost everywhere in I , and hence everywhere in I by continuity. This completes the proof. \square

Notice that the above proof gives the following

Corollary 3.2. *Let I, X, F be as in Theorem 3.1 and satisfy the conditions (i)–(iii) therein. Then F'_+ is a right-continuous vector function on I with locally bounded variation, and $DF = F'_+$.*

Corollary 3.3. *Let I, X be as in Theorem 3.1, and $z \in I$. Then the mapping $\Phi: DC(I, X) \rightarrow X \times X \times M_{lbv}(I, X)$, given by $\Phi(F) = (F(z), F'_+(z), D^2F)$, is a linear bijection.*

Proof. The assertion easily follows from the following reasoning, based on Theorem 3.1. Given $(p, q, \mu) \in X \times X \times M_{lbv}(I, X)$, there exists $\tilde{F} \in DC(I, X)$ such that $D^2\tilde{F} = \mu$. Adding to \tilde{F} a suitable X -valued affine function, we easily obtain the (clearly) unique $F \in DC(I, X)$ such that $\Phi(F) = (p, q, \mu)$. \square

4. CHARACTERIZATIONS OF D.C. MAPPINGS VIA ONE-SIDED DERIVATIVES

4.1. D.c. mappings on arbitrary intervals.

Lemma 4.1. *Let $I \subset \mathbb{R}$ be an open interval and $f: I \rightarrow \mathbb{R}$ a continuous function such that the unilateral derivatives $f'_\pm(x)$ exist at each $x \in I$. If $f_+(x) \leq f'_-(y)$ whenever $x < y$ belong to I , then f is convex.*

Proof. Suppose that f is not convex. There exist three points $a < b < c$ in I such that

$$\frac{f(c) - f(a)}{c - a} < \frac{f(b) - f(a)}{b - a}.$$

Consider the function $g(x) := f(x) - \frac{f(c)-f(a)}{c-a}(x-a)$. An easy calculation shows that $g(a) = g(c) < g(b)$. Applying Proposition 2 in [4, Chapter 1, §2] to the intervals $[a, b]$ and $[b, c]$, we obtain two points $x \in (a, b)$ and $y \in (b, c)$ such that $g'_+(x) > 0 > g'_-(y)$. But this implies that $f'_+(x) > f'_-(y)$, which contradicts the assumptions. \square

Theorem 4.2. *Let X be a Banach space, $I \subset X$ an arbitrary interval, $F: I \rightarrow X$ a continuous mapping, and $z \in \text{int } I$. Then the following assertions are equivalent.*

- (i) F is a d.c. mapping.
- (ii) $F'_+(t)$ exists for each $t \in \text{int } I$, and $f_z(x) := \int_z^x (V_z^t F'_+) dt$ is finite for each $x \in I$.

Moreover, if (i), (ii) are satisfied, then

$$(9) \quad F(x) = F(z) + \int_z^x F'_+$$

for each $x \in I$, and given a function $f: I \rightarrow \mathbb{R}$, we have the equivalence

$$(10) \quad f \text{ controls } F \Leftrightarrow f - f_z \text{ is continuous and convex on } I.$$

Proof. Let (i) hold. By [19] or [20], $F'_+(t)$ exists throughout $\text{int } I$. For each interval $[c, d] \subset \text{int } I$ containing z , F has clearly a Lipschitz control function on $[c, d]$. By [20, Theorem 3.1(c)], f_z controls F on $[c, d]$. Since every locally convex function on an open interval is convex, we easily get that f_z controls F on $\text{int } I$.

Now suppose that $s := \sup I \in I$. Let φ be a control function for F on I . By adding an affine function, we can suppose that $\varphi(z) = 0 (= f_z(z))$ and $\varphi'_+(z) = (f_z)'_+(z)$. For each $t \in (z, s)$, we can apply [20, Theorem 3.1(b,d)] on $[z, t]$ to get $(f_z)'_-(t) - (f_z)'_+(z) \leq \varphi'_-(t) - \varphi'_+(z)$, which implies $(f_z)'_-(t) \leq \varphi'_-(t)$. Therefore, for each $x \in (z, s)$, we have

$$f_z(x) = \int_z^x (f_z)'_- \leq \int_z^x \varphi'_- = \varphi(x).$$

Since f_z is nondecreasing on $[z, s]$, we obtain

$$f_z(s) = \lim_{x \rightarrow s^-} f_z(x) \leq \lim_{x \rightarrow s^-} \varphi(x) = \varphi(s) < \infty,$$

and hence $f_z(s)$ is finite.

If $i := \inf I \in I$, we proceed in a quite similar way. In this case, we consider a control function ψ for F such that $\psi(z) = 0$ and $\psi'_-(z) = (f_z)'_-(z)$. For any $t \in (i, z)$, we get $-(f_z)'_+(t) \leq -\psi'_+(t)$. Integration gives $f_z(x) \leq \psi(x)$ whenever $x \in (i, z)$. Since f_z is nonincreasing on $[i, z]$, we obtain that $f_z(i)$ is finite, which completes the proof of (ii).

Now suppose that (ii) holds. Then $V_z^t F'_+$ is clearly finite for each $t \in \text{int } I$, that is, F'_+ has locally bounded variation on $\text{int } I$. Using the obvious fact that $f_z - f_c$ is an affine function on I , we obtain by [20, Theorem 3.1] that f_z controls F on any compact interval $[c, d] \subset \text{int } I$, and therefore on $\text{int } I$. Since F and f_z are continuous on I , it follows easily that f_z controls F on I , and hence (i) holds.

The implication “ \Leftarrow ” in (10) is obvious. To prove “ \Rightarrow ”, suppose that f controls F and consider two arbitrary interior points $x < y$ of I . Since $f_z - f_x$ is affine on I ,

[20, Theorem 3.1(b,d)] gives $(f_z)'_-(y) - (f_z)'_+(x) \leq f'_-(y) - f'_+(x)$. Consequently,

$$(f - f_z)'_+(x) \leq (f - f_z)'_-(y).$$

By Lemma 4.1, the function $f - f_z$ is convex on $\text{int } I$ and hence, by continuity of f and f_z , also on I .

Finally, the formula $F(x) = F(z) + \int_z^x F'_+$ holds on $\text{int } I$ by [20, Theorem 3.1]. Our aim is to show that it holds on I . Assume that $s = \sup I \in I$. Consider an increasing sequence $\{x_n\}$ in (z, s) such that $x_n \rightarrow s$. Then the functions $G_n := F'_+ \cdot \chi_{[z, x_n]}$ converge to F'_+ pointwise on $[z, s)$ and are Bochner integrable on $[z, s]$. Moreover, since $\|G_n(t)\| \leq \|F'_+(z)\| + V_z^t F'_+$ for $t \in [z, s]$, the Vector Dominated Convergence Theorem (see [1, Theorem 11.45]) implies that so is also F'_+ . If $i = \inf I \in I$, an analogous reasoning leads to the conclusion that F'_+ is Bochner integrable on $[i, z]$ as well. Now, the desired formula holds on the whole I by continuity. \square

Remark 4.3. Condition (10) shows that f_z is in a sense a “minimal” (up to an affine function) control function of F . More precisely, (10) easily implies that a real function g on I has the property that, for each real function f on I ,

$$f \text{ controls } F \iff f - g \text{ is continuous and convex}$$

if and only if $g - f_z$ is an affine function on I .

4.2. D.C. mappings with a bounded control function.

Theorem 4.4 (Case $I = (a, b)$). *Let $a < z < b$ be real numbers, X a Banach space, and $F: (a, b) \rightarrow X$ a continuous mapping. Let f_z be defined as in Theorem 4.2. Then the following are equivalent.*

- (i) F is d.c. with a bounded control function.
- (ii) F'_+ exists on (a, b) , and f_z is continuously extendable to $[a, b]$.
- (iii) F'_+ exists on (a, b) , and the integral $\int_a^b (V_z^x F'_+) dx$ converges.

Moreover, if the above conditions are satisfied, then F is continuously extendable to $[a, b]$.

Proof. To prove (i) \Rightarrow (ii), suppose that f is a bounded control function for F . By (10), there exists a convex function g on (a, b) such that $f = f_z + g$. Since g is bounded from below, we have that f_z is bounded from above. So, since f_z is clearly nondecreasing on (z, b) and nonincreasing on (a, z) , we obtain (ii). Since f_z controls F , (ii) implies (i). Finally, since the function $v(x) := V_z^x F'_+$ is nonnegative on $[z, b)$, nonpositive on $[a, z)$, and $f_z(x) = \int_z^x v$, it is easy to see that (ii) is equivalent to (iii).

Now suppose that (i)-(iii) hold. The vector function F'_+ , being the pointwise limit of the functions $H_n(x) := n(F(x + \frac{1}{n}) - F(x))$ (where we set $F(x) := 0$ for $x \geq b$), is (strongly) measurable on (a, b) ; see [1, §11]. Since clearly

$$\|F'_+(y) - F'_+(x)\| \leq v(y) - v(x) \quad \text{for } a < x < y < b,$$

we obtain $\|F'_+(x)\| \leq \|F'_+(z)\| + |v(z)| + |v(x)|$ for $x \in (a, b)$. So the integrals $\int_z^a F'_+$ and $\int_z^b F'_+$ converge. Now, (9) easily implies that F is continuously extendable to $[a, b]$. This completes the proof. \square

Theorem 4.5 (Case $I = (a, \infty)$). *Let $a < z$ be real numbers, X a Banach space, and $F: (a, \infty) \rightarrow X$ a continuous mapping. Then the following are equivalent.*

- (i) F is d.c. with a bounded control function.
- (ii) $F'_+(x)$ exists for each $x \in (a, \infty)$, and the integral $\int_a^\infty (V_t^\infty F'_+) dt$ converges.
- (iii) $F'_+(x)$ exists for each $x \in (a, \infty)$, and the function f_z from Theorem 4.2 is bounded on (a, z) and has an asymptote at ∞ .

Moreover, if the above conditions are satisfied, then the following limits exist in X :

$$F_a := \lim_{x \rightarrow a^+} F(x), \quad F'_\infty := \lim_{x \rightarrow \infty} F'_+(x), \quad q := \lim_{x \rightarrow \infty} (F(x) - xF'_\infty).$$

In particular, F has an asymptote at ∞ .

Proof. To prove (i) \Rightarrow (ii), suppose that f is a bounded control function for F . Since f is convex, it is easy to see that f is nonincreasing and has a finite limit at ∞ . By (10), there exists a convex function g on (a, ∞) such that $f = f_z + g$. Set $v(x) := V_z^x F'_+$. We know that $f_z(x) = \int_z^x v$, and there exists a nondecreasing function w on (a, ∞) such that $g(x) = g(z) + \int_z^x w$ (see [14, §12, Theorem A]). We claim that $L := \lim_{x \rightarrow \infty} v(x) = V_z^\infty F'_+ \in \mathbb{R}$. Indeed, if $L = \infty$, choose $\tilde{z} > z$ such that $v(\tilde{z}) > 1 - w(z)$. Then, for each $x > \tilde{z}$, we have $f_z(x) \geq (1 - w(z))(x - \tilde{z})$ and $g(x) \geq g(z) + w(z)(x - z)$. This easily implies that $\lim_{x \rightarrow \infty} f(x) = \infty$, which is a contradiction. Similarly, since $f_z \geq 0$ on (z, ∞) , we easily see that $\tilde{L} := \lim_{x \rightarrow \infty} w(x) \in \mathbb{R}$. Since clearly

$$\varphi(x) := \int_z^x (L - v(t)) dt = o(x), \quad \psi(x) := \int_z^x (\tilde{L} - w(t)) dt = o(x), \quad \text{as } x \rightarrow \infty,$$

and the function

$$f(x) = L(x - z) - \varphi(x) + g(z) + \tilde{L}(x - z) - \psi(x)$$

is bounded on (a, ∞) , we obtain $L + \tilde{L} = 0$, and hence $\varphi(x) + \psi(x) = g(z) - f(x)$. Since φ and ψ are nonnegative and nondecreasing on $[z, \infty)$, we obtain that $\lim_{x \rightarrow \infty} \varphi(x) \in \mathbb{R}$, that is, the integral $\int_z^\infty (L - v(x)) dx = \int_z^\infty (V_x^\infty F'_+) dx$ converges.

Applying Theorem 4.4 to f on (a, b) , where $b = z + 1$, we obtain that $\int_z^a v(x) dx$ converges. Consequently $\int_z^a (L - v(x)) dx = \int_z^a (V_x^\infty F'_+) dx$ converges and so (ii) follows.

To prove (ii) \Rightarrow (iii), suppose that (ii) holds. Then clearly $L := V_z^\infty F'_+ \in \mathbb{R}$ and $\int_z^a v = \int_z^a (L - V_x^\infty F'_+) dx$ converges, and therefore f_z is bounded on $(a, z]$. Further, $\int_z^\infty (L - v(x)) dx$ converges, i.e.,

$$\lim_{x \rightarrow \infty} \left(L(x - z) - \int_z^x v \right) = \lim_{x \rightarrow \infty} (Lx - Lz - f_z(x)) \in \mathbb{R}.$$

Consequently, f_z has an asymptote at ∞ , and (iii) is proved.

If (iii) holds, then F is controlled by f_z by Theorem 4.2. Moreover, if $y(x) = kx + r$ is an asymptote of f_z at ∞ , then $f(x) := f_z(x) - kx$ is a bounded control function of F . So, the equivalence of (i)-(iii) is proved.

Now suppose that (i)-(iii) hold. Since $\lim_{x \rightarrow \infty} v(x) = L \in \mathbb{R}$ and clearly

$$(11) \quad \|F'_+(y) - F'_+(x)\| \leq v(y) - v(x) \quad \text{for } a < x < y < \infty,$$

the completeness of X implies that the limit F'_∞ exists in X . The vector function F'_+ , being the pointwise limit of the functions $H_n(x) := n(F(x + \frac{1}{n}) - F(x))$ ($n \in \mathbb{N}$), is (strongly) measurable on (z, ∞) ; see [1, §11]. Consequently, since (11) implies that $\|F'_\infty - F'_+(x)\| \leq L - v(x)$ for $x > z$, we obtain convergence of the integral $\int_z^\infty (F'_\infty - F'_+(x)) dx =: p \in X$. By (9), we obtain

$$\lim_{x \rightarrow \infty} [(x - z)F'_\infty - (F(x) - F(z))] = p,$$

which implies that the limit q exists in X .

Applying Theorem 4.4 to f on (a, b) , where $b = a + 1$, we obtain that the limit F_a exists in X , which completes the proof. \square

Remark 4.6. Theorem 4.5 immediately implies that a function $F: (0, \infty) \rightarrow \mathbb{R}$ is a difference of two bounded convex functions ($F \in \mathcal{D}$ in the notation of [3]) if and only if F is bounded continuous and the condition (ii) of Theorem 4.5 holds. This characterization is very similar to a characterization of O. Böhme [3, Section 2] ($\mathcal{D} = \mathcal{K}_0$ in the notation therein), which also follows from Theorem 4.5.

Corollary 4.7. *Let X be a Banach space, $a \in \mathbb{R}$, and $F: (a, \infty) \rightarrow X$ a d.c. mapping with a bounded control function. Then the function*

$$\hat{f}(x) := \int_x^\infty (V_t^\infty F'_+) dt, \quad x \in (a, \infty)$$

is a bounded control function of F such that

$$(12) \quad \hat{f} \text{ is nonincreasing, } \lim_{x \rightarrow \infty} \hat{f}(x) = 0 \text{ and } \hat{f}(a+) = \int_a^\infty (V_t^\infty F'_+) dt \in \mathbb{R}.$$

Further,

- (a) *a function $f: (a, \infty) \rightarrow \mathbb{R}$ is a bounded control function of F if and only if $f - \hat{f}$ is a bounded convex function on (a, ∞) ;*
- (b) *moreover, if f is a bounded control function for F , then*

$$\|\hat{f}\|_\infty = \hat{f}(a+) \leq 2\|f\|_\infty.$$

Proof. \hat{f} is clearly nonincreasing and, by Theorem 4.5(ii), $\hat{f}(a+) \in \mathbb{R}$. Thus (12) follows.

Fix some $z \in (a, \infty)$. Let v and L be as in the proof of Theorem 4.5, and f_z as in Theorem 4.2. Then $\hat{f}(x) = \int_x^\infty (V_z^\infty F'_+ - V_z^t F'_+) dt = \int_x^\infty (L - v(t)) dt$. We know that $c := \int_z^\infty (L - v) \in \mathbb{R}$ (see the proof of Theorem 4.5). Noticing that

$$(13) \quad \hat{f}(x) = c - \int_z^x (L - v) = c - L(x - z) + f_z(x), \quad x > a,$$

we have that $\hat{f} - f_z$ is an affine function and thus Remark 4.3 implies that \hat{f} is a (bounded) control function for F and (a) holds.

Finally, if f is a bounded control function of F , then the convex function $f - \hat{f}$ is bounded and hence nonincreasing. Consequently, $f(a+) - \hat{f}(a+) \geq \lim_{x \rightarrow \infty} (f - \hat{f})(x) = \lim_{x \rightarrow \infty} f(x)$, which implies that $0 \leq \hat{f}(a+) \leq f(a+) - \lim_{x \rightarrow \infty} f(x) \leq 2\|f\|_\infty$. This completes the proof. \square

It is easy to see that Theorem 4.5 implies the following.

Corollary 4.8. *Let $F: (a, \infty) \rightarrow X$ be d.c. with a bounded control function. Then*

- (i) *F is bounded if and only if $F'_\infty := \lim_{x \rightarrow \infty} F'_+(x) = 0$.*
- (ii) *There exists a unique $A = F'_\infty \in X$ such that the function $G(x) = F(x) - xA$, $x \in (a, \infty)$ is a bounded d.c. function with a bounded control function.*

5. SPACES OF D.C. MAPPINGS

5.1. The space $BDC_b(C, X)$.

Spaces of d.c. functions were considered in several articles, see, e.g., [18] and [21]. If C is a convex subset of a normed linear space V , we will denote by $\widetilde{DC}(C)$ the set of all real d.c. functions on C which can be written as difference of two continuous convex bounded functions on C . Then the natural norm on $\widetilde{DC}(C)$ is defined as

$$\|F\|_{dc} := \inf\{\|g\|_\infty + \|h\|_\infty : g, h \text{ continuous convex, } F = g - h\}.$$

We shall see below that this norm is complete. If in the definition of $\|F\|_{dc}$ we consider only $g \geq 0$, $h \geq 0$, then we obtain the definition of the norm $\|F\|_{dc+}$. Considering the functions $\tilde{g} := g + c$ and $\tilde{h} := h + c$ where $c = \max\{\|g\|_\infty, \|h\|_\infty\}$, we easily see that

$$\|F\|_{dc} \leq \|F\|_{dc+} \leq 3\|F\|_{dc}.$$

Spaces of d.c. mappings (defined on an *open* convex sets) were studied in [8], where the natural norm $\|F\|_{\mathcal{D}}$ (see below) was used.

Definition 5.1. Let C be a convex subset of a normed linear space V , and X a Banach space. We shall denote:

- (a) by $DC(C, X)$ the vector space of all d.c. mappings $F: C \rightarrow X$;
- (b) by $DC_b(C, X)$ the vector space of all d.c. mappings $F: C \rightarrow X$ that admit a bounded control function.
- (c) by $BDC_b(C, X)$ the vector space of all bounded d.c. mappings $F: C \rightarrow X$ that admit a bounded control function.

We define

$$\begin{aligned} |F| &:= \inf\{\|f\|_\infty : f \text{ controls } F\} && \text{for } F \in DC_b(C, X), \\ \|F\|_{\mathcal{D}} &:= \|F\|_\infty + |F| && \text{for } F \in BDC_b(C, X). \end{aligned}$$

It is elementary to see that $\|\cdot\|_{\mathcal{D}}$ is a norm on $BDC_b(C, X)$. It is proved in [8] that this norm is complete. Let us present a short proof of this fact.

Proposition 5.2. *The space $BDC_b(C, X)$ is a Banach space in the norm $\|\cdot\|_{\mathcal{D}}$.*

Proof. Consider a sequence $\{F_n\}$ in $BDC_b(C, X)$ such that $\sum_{n=1}^{\infty} \|F_n\|_{\mathcal{D}} < \infty$. We have to show (see, e.g., [13, Lemma 1.22]) that the series $\sum_{n=1}^{\infty} F_n$ converges in $(BDC_b(C, X), \|\cdot\|_{\mathcal{D}})$. Since $\sum_{n=1}^{\infty} \|F_n\|_\infty < \infty$, the mapping $G := \sum_{n=1}^{\infty} F_n : C \rightarrow X$ is well-defined, bounded and continuous. For each $n \in \mathbb{N}$, let f_n be a control function of F_n such that $\|f_n\|_\infty < |F_n| + 2^{-n}$. Since $\sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$, the function $g := \sum_{n=1}^{\infty} f_n$ is bounded and continuous. Moreover, g controls G since, for each

$x^* \in S_{X^*}$, $x^* \circ G + g = \sum_{n=1}^{\infty} (x^* \circ F_n + f_n)$ is clearly convex and continuous. Finally, since $\sum_{n=N}^{\infty} f_n$ controls $\sum_{n=N}^{\infty} F_n$ ($N \in \mathbb{N}$), we have

$$\left\| \sum_{n=N}^{\infty} F_n \right\|_{\mathcal{D}} \leq \sum_{n=N}^{\infty} \|F_n\|_{\infty} + \sum_{n=N}^{\infty} \|f_n\|_{\infty} \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

which implies that the series $\sum_{n=1}^{\infty} F_n$ converges to G in the norm $\|\cdot\|_{\mathcal{D}}$. \square

Agreement 5.3. If not specified otherwise, by $BDC_b(C, X)$ we always mean the Banach space $(BDC_b(C, X), \|\cdot\|_{\mathcal{D}})$.

The argument of Remark 2.2(a) easily shows that $\widetilde{DC}(C) = BDC_b(C, \mathbb{R})$ and

$$\|F\|_{dc} \leq \|F\|_{\mathcal{D}} \leq 2\|F\|_{dc} \quad \text{for } F \in \widetilde{DC}(C).$$

Consequently, $\widetilde{DC}(C)$ is a Banach space in the norm $\|\cdot\|_{dc}$.

Remark 5.4. If $-\infty < a < b < \infty$, then $BDC_b((a, b), X) = DC_b((a, b), X)$ by Theorem 4.4.

5.2. The space $DC([0, 1], X)$.

Lemma 5.5. *Let $I \subset \mathbb{R}$ be an open interval, $z \in I$, and X a Banach space. Given $F \in DC(I, X)$, let f_z be as in Theorem 4.2, and let f be an arbitrary control function of F . Then*

$$(14) \quad (f_z)'_+(x) = V_z^x F'_+, \quad x \in I;$$

$$(15) \quad f_z(z) = (f_z)'_+(z) = 0 = \min f_z(I);$$

$$(16) \quad (f_z)'_+(x) \leq [f'_+(x) - f'_+(z)] \operatorname{sgn}(x - z), \quad x \in I;$$

$$(17) \quad f_z(x) \leq f(x) - f(z) - f'_+(z)(x - z), \quad x \in I.$$

Proof. Since F'_+ is right-continuous by [20, Theorem 3.1] and [20, Proposition 3.4(ii), p. 329], the function $t \mapsto V_z^t F'_+$ is right-continuous as well (see, e.g., [6, Lemma 5.2(b)]). This implies (14) in a standard way. Now, (15) is easy. Recall that $f - f_z$ is convex by (10). Since $(f - f_z)'_+(u) \leq (f - f_z)'_+(v)$ whenever $u \leq v$ belong to I , we obtain (16). Finally, (17) follows from (15) and the inequality $(f - f_z)(x) \geq (f - f_z)(z) + (f - f_z)'_+(z)(x - z)$, $x \in I$. \square

Lemma 5.6. *Let I and z be as above. Let f be a bounded convex function on I . Then*

$$|f'_+(z)| \leq \frac{2}{\delta} \|f\|_{\infty}$$

whenever $\delta > 0$ is such that $(z - \delta, z + \delta) \subset I$.

Proof. For any $d \in (0, \delta)$, we have $f'_+(z) \leq \frac{f(z+d) - f(z)}{d} \leq \frac{2\|f\|_{\infty}}{d}$ and $f'_+(z) \geq \frac{f(z) - f(z-d)}{d} \geq -\frac{2\|f\|_{\infty}}{d}$. Thus $|f'_+(z)| \leq \frac{2\|f\|_{\infty}}{d}$. Passing to limit as $d \nearrow \delta$ concludes the proof. \square

Remark 5.7. Obviously, $DC([0, 1], X) = BDC_b([0, 1], X)$. Moreover, we can clearly canonically identify the space $DC([0, 1], X)$ with $BDC_b((0, 1), X)$. Indeed, the mapping $G \mapsto G|_{(0,1)}$, $G \in DC([0, 1], X)$, is an isometry of $(DC([0, 1], X), \|\cdot\|_{\mathcal{D}})$ onto $(BDC_b((0, 1), X), \|\cdot\|_{\mathcal{D}})$. This follows immediately from the facts that each

$F \in BDC_b((0, 1), X)$ is continuously extendable to $[0, 1]$ by Theorem 4.4, and the same clearly holds for each bounded control function of F .

Theorem 5.8 (The space $DC([0, 1], X)$). *Let X be a Banach space, $z \in (0, 1)$, and let f_z be as in Theorem 4.2. Then each of the three norms*

$$\begin{aligned} \|F\|_{DC} &:= \|F(z)\| + \|F'_+(z)\| + |F|, \\ \|F\|_{DC} &:= \|F(z)\| + \|F'_+(z)\| + \|f_z\|_\infty, \\ \|F\|_\bullet &:= \|F(z)\| + \|F'_+(z)\| + \int_0^1 |V_z^t F'_+| dt \end{aligned}$$

on $DC([0, 1], X)$ is equivalent to the norm $\|\cdot\|_{\mathcal{D}}$.

Proof. It is easy to see that $\|\cdot\|_{DC}$, $\|\cdot\|_{DC}$ and $\|\cdot\|_\bullet$ are positively homogeneous and subadditive.

Let $F \in DC([0, 1], X)$, and let f be a control function of F on $[0, 1]$. Clearly, $\|F\|_{DC} \leq \|F\|_{DC}$. Using (15), (17) and Lemma 5.6 (and denoting $c = \frac{2}{\delta}$ for some $\delta > 0$ such that $(z - \delta, z + \delta) \subset (0, 1)$), we obtain

$$\begin{aligned} \|f_z\|_\infty &= \max\{f_z(0), f_z(1)\} \\ &\leq \max\{f(0) - f(z) + f'_+(z)z, f(1) - f(z) - f'_+(z)(1 - z)\} \\ &\leq 2\|f\|_\infty + |f'_+(z)| \leq (2 + c)\|f\|_\infty, \end{aligned}$$

and also

$$\begin{aligned} \|F'_+(z)\| &= \sup_{\|x^*\|=1} \{(x^* \circ F + f)'_+(z) - f'_+(z)\} \\ &\leq \sup_{\|x^*\|=1} \{c\|x^* \circ F + f\|_\infty + c\|f\|_\infty\} \leq c\|F\|_\infty + 2c\|f\|_\infty. \end{aligned}$$

It follows easily that $\|F\|_{DC} \leq A\|F\|_{\mathcal{D}}$ for some constant $A > 0$ (non depending on F). Further, using Theorem 4.2, for $x \in [0, 1]$ we have

$$\begin{aligned} \|F(x)\| &= \|F(z) + \int_z^x F'_+(t) dt\| \\ &\leq \|F(z)\| + \|F'_+(z)\| |x - z| + \left| \int_z^x \|F'_+(t) - F'_+(z)\| dt \right| \\ &\leq \|F(z)\| + \|F'_+(z)\| + |f_z(x)| \\ &\leq \|F(z)\| + \|F'_+(z)\| + (2 + c)\|f\|_\infty. \end{aligned}$$

Hence $\|F\|_\infty \leq (2 + c)\|F\|_{DC}$, which implies that $\|F\|_{\mathcal{D}} \leq (3 + c)\|F\|_{DC}$.

Recall that $\|f_z\|_\infty = \max\{f_z(0), f_z(1)\}$. Moreover,

$$\int_0^1 |V_z^t F'_+| dt = \int_0^z |V_z^t F'_+| dt + \int_z^1 (V_z^t F'_+) dt = f_z(0) + f_z(1).$$

Now, the obvious inequalities $\max\{\alpha, \beta\} \leq \alpha + \beta \leq 2 \max\{\alpha, \beta\}$ ($\alpha, \beta \geq 0$) give that $\|\cdot\|_\bullet$ is equivalent to $\|\cdot\|_{DC}$. \square

Lemma 5.9. *Let X be a Banach space, and $F: (0, 1) \rightarrow X$ a d.c. mapping. Denote $\mu := D^2F$ and $\theta(x) := \min\{x, 1 - x\}$, $x \in (0, 1)$. Then*

$$(18) \quad |\theta\mu|((0, 1)) = \int_0^1 \theta d|\mu| = \int_0^1 |V_{1/2}^t F'_+| dt.$$

Proof. For simplicity denote $I = (0, 1)$ and $z = \frac{1}{2}$. First observe that $\mu \in M_{lbv}(I, X)$ by Theorem 3.1 and F'_+ has locally bounded variation on $(0, 1)$ (see Corollary 3.2).

Using Proposition 2.11 with $G := F'_+$, we obtain $|\mu|((x, y]) = V_x^y F'_+$ whenever $x < y$ belong to I . Consider the product measure space $((0, 1)^2, \eta) := ((0, 1), |\mu|) \times ((0, 1), \lambda)$ where λ is the Lebesgue measure. Using the Fubini theorem for computation of $\eta(M)$, where $M = \{(x, t) \in (0, z] \times (0, 1) : t < x\}$, we obtain

$$\begin{aligned} \int_0^z |V_z^t F'_+| dt &= \int_0^z (V_t^z F'_+) dt = \int_0^z |\mu|((t, z]) dt \\ &= \eta(M) = \int_{(0, z]} x d|\mu|(x) = \int_{(0, z]} \theta d|\mu|. \end{aligned}$$

Similarly, if $M^* = \{(x, t) \in (z, 1) \times (0, 1) : t \geq x\}$ then we obtain

$$\begin{aligned} \int_z^1 |V_z^t F'_+| dt &= \int_z^1 (V_z^t F'_+) dt = \int_z^1 |\mu|((z, t]) dt \\ &= \eta(M^*) = \int_{(z, 1)} (1-x) d|\mu|(x) = \int_{(z, 1)} \theta d|\mu|. \end{aligned}$$

Summing up the last two formulas and using Corollary 2.6, we obtain the equalities (18). \square

Theorem 5.10. *Let X be a Banach space, and $F: [0, 1] \rightarrow X$ a continuous mapping. Denoting $\theta(x) := \min\{x, 1-x\}$, $x \in (0, 1)$, the following assertions are equivalent.*

- (i) F is d.c.
- (ii) $D^2(F|_{(0,1)}) = \mu \in M_{lbv}((0, 1), X)$, and $\theta\mu \in M_{bv}((0, 1), X)$.

Moreover, if (i), (ii) are satisfied then

$$(19) \quad |\theta\mu|((0, 1)) = \int_0^1 \theta d|\mu| = \int_0^1 |V_{1/2}^t F'_+| dt < \infty.$$

Proof. If (i) or (ii) holds then $F|_{(0,1)}$ is d.c. and so, denoting $\mu := D^2(F|_{(0,1)})$, we obtain by Lemma 5.9

$$V := |\theta\mu|((0, 1)) = \int_0^1 \theta d|\mu| = \int_0^1 |V_{1/2}^t F'_+| dt.$$

If (i) holds, then $V < \infty$ by Theorem 4.4 and so (ii) holds. If (ii) holds then clearly $V < \infty$ and so $F|_{(0,1)} \in DC_b((0, 1), X)$ by Theorem 4.4 and consequently also $F|_{(0,1)} \in BDC_b((0, 1), X)$ (see Remark 5.4). Thus (i) holds by Remark 5.7. \square

Theorem 5.11. *Let X be a Banach space, and $z = \frac{1}{2}$. Then the Banach space $DC([0, 1], X)$ with the norm*

$$\|F\|_{\bullet} = \|F(z)\| + \|F'_+(z)\| + \int_0^1 |V_z^t F'_+| dt$$

(which is equivalent to the norm $\|\cdot\|_{\mathcal{D}}$ by Theorem 5.8) is isometric to $M_{bv}([0, 1], X)$. In particular, $DC([0, 1], X^*)$ in the norm $\|\cdot\|_{\mathcal{D}}$ is isomorphic to the dual space $C([0, 1], X)^*$.

Proof. Consider the ℓ_1 -sum $E := X \oplus_1 X \oplus_1 M_{bv}((0, 1), X)$. If $F \in DC([0, 1], X)$, then we obtain by Theorem 5.10 that $D^2(F|_{(0,1)}) = \mu \in M_{bv}((0, 1), X)$ and that $\theta\mu \in M_{bv}((0, 1), X)$, where $\theta(x) = \min\{x, 1 - x\}$. Thus we can consider the linear mapping

$$\Psi: DC([0, 1], X) \rightarrow E, \quad \Psi(F) := (F(z), F'_+(z), \theta\mu).$$

By Corollary 3.3, Ψ is one-to-one; let us show that it is also onto. So consider an arbitrary $(y_0, y_1, \nu) \in E$. By Lemma 2.5, $\mu := (1/\theta)\nu \in M_{bv}((0, 1), X)$. So by Corollary 3.3 there exists $F \in DC((0, 1), X)$ such that $F(z) = y_0$, $F'_+(z) = y_1$, $D^2F = \mu$. By Remark 2.4(c) we have $\theta\mu = \nu$. Since $\int_0^1 |V_z^t F'_+| dt = \|\theta\mu\|_{M_{bv}} = \|\nu\|_{M_{bv}}$ by Lemma 5.9, the integral $\int_0^1 (V_z^t F'_+) dt$ converges. So $F \in DC_b(I, X)$ by Theorem 4.4, and consequently also $F \in BDC_b(I, X)$ (see Remark 5.4). Thus there exists $\tilde{F} \in DC([0, 1], X)$ with $\tilde{F}|_{(0,1)} = F$ by Remark 5.7. Clearly $\Psi(\tilde{F}) = (y_0, y_1, \nu)$.

Finally, by Theorem 5.10, $\|\Psi(\tilde{F})\|_E = \|F(z)\| + \|F'_+(z)\| + \|\theta\mu\|_{M_{bv}} = \|F(z)\| + \|F'_+(z)\| + \int_0^1 |V_z^t F'_+| dt = \|\tilde{F}\|_\bullet$. This shows that Ψ is an isometry. The rest of the proof follows from Lemma 2.7 and Theorem 2.8. \square

5.3. The spaces $BDC_b((0, \infty), X)$ and $DC_b((0, \infty), X)$.

Recall that if X is a Banach space then also the space $BDC_b((0, \infty), X)$ in the norm $\|\cdot\|_{\mathcal{D}}$ is a Banach space.

Theorem 5.12 (The space $BDC_b((0, \infty), X)$). *Let X be a Banach space, $J = (0, \infty)$, and $z \in J$. Then the formulas*

$$\begin{aligned} \|F\|_{BDC_b} &= \|F(z)\| + |F|, \\ \|F\|_{BDC_b} &= \|F(z)\| + \int_0^\infty (V_t^\infty F'_+) dt \end{aligned}$$

define two norms on $BDC_b(J, X)$ that are equivalent to the norm $\|\cdot\|_{\mathcal{D}}$.

Proof. First, notice that Theorem 4.5 implies the equivalence

$$F \in BDC_b(J, X) \Leftrightarrow F \in DC_b(J, X) \text{ and } F'_\infty := \lim_{x \rightarrow \infty} F'_+(x) = 0.$$

The function \hat{f} from Corollary 4.7, controls F , and $|F| \leq \|\hat{f}\|_\infty = \hat{f}(0+) \leq 2|F|$. Recall that $\hat{f}(0+) = \int_0^\infty (V_t^\infty F'_+) dt$. Hence the seminorms $\|\cdot\|_{BDC_b}$ and $\|\cdot\|_{BDC_b}$ are equivalent, in the sense that each of them is majorized by a multiple of the other one. We also have $\|F\|_{BDC_b} \leq \|F\|_{\mathcal{D}}$.

Now, for $x \in J$ and $F \in BDC_b(J, X)$, the equality (9) and corollaries 4.7 and 4.8 imply

$$\begin{aligned} \|F(x)\| &\leq \|F(z)\| + \left\| \int_z^x F'_+ \right\| \leq \|F(z)\| + \left| \int_z^x \|F'_+(t) - F'_\infty\| dt \right| \\ &\leq \|F(z)\| + \left| \int_z^x (V_t^\infty F'_+) dt \right| = \|F(z)\| + |\hat{f}(z) - \hat{f}(x)| \\ &\leq \|F(z)\| + \hat{f}(0+) \leq \|F(z)\| + 2|F|. \end{aligned}$$

Consequently, $\|F\|_{\mathcal{D}} \leq 3\|F\|_{BDC_b}$, and the proof is complete. \square

Theorem 5.13. *Let X be a Banach space, $J = (0, \infty)$, and $F: J \rightarrow X$ a continuous mapping. Denoting $\hat{\theta}(x) := x$, $x \in J$, the following assertions are equivalent.*

- (i) F is d.c. with a bounded control function.

(ii) $\mu := D^2F \in M_{lbv}(J, X)$, and $\hat{\theta}\mu \in M_{bv}(J, X)$.

Moreover, if (i), (ii) are satisfied then

$$(20) \quad |\hat{\theta}\mu|(J) = \int_J \hat{\theta} d|\mu| = \int_0^\infty (V_t^\infty F'_+) dt.$$

Proof. Suppose that (i) or (ii) holds. Then we know that $\mu := D^2F \in M_{lbv}(J, X)$ by Theorem 3.1, and F'_+ exists on J by Theorem 3.1 and Theorem 4.2. So, to finish the proof, by Theorem 4.5 it is sufficient to prove (20) under the assumption that (i) or (ii) holds.

Applying Corollary 3.2 and Proposition 2.11 for $G := F'_+$, we obtain $V_t^x F'_+ = |\mu|((t, x])$ for $0 < t < x < \infty$ and so $V_t^\infty F'_+ = |\mu|((t, \infty))$ for each $t \in J$. Consider the product measure space $((0, \infty)^2, \eta) := ((0, \infty), |\mu|) \times ((0, \infty), \lambda)$, where λ is the Lebesgue measure. Using the Fubini theorem for computation of $\eta(M)$, where $M := \{(x, t) \in (0, \infty)^2 : t < x\}$, we obtain

$$\begin{aligned} \int_0^\infty V_t^\infty F'_+ d\lambda(t) &= \int_0^\infty |\mu|((t, \infty)) d\lambda(t) = \eta(M) \\ &= \int_0^\infty x d|\mu|(x) = \int_J \hat{\theta} d|\mu| = |\hat{\theta}\mu|(J), \end{aligned}$$

where the last equality comes from Corollary 2.6. \square

Theorem 5.14. *Let X be a Banach space, and $z > 0$. Then the Banach space $BDC_b((0, \infty), X)$ in the norm*

$$\|F\|_{BDC_b} = \|F(z)\| + \int_0^\infty (V_t^\infty F'_+) dt$$

(which is equivalent to the norm $\|\cdot\|_{\mathcal{D}}$ by Theorem 5.12) is isometric to $M_{bv}([0, 1], X)$. In particular, $BDC_b((0, \infty), X^)$ in the norm $\|\cdot\|_{\mathcal{D}}$ is isomorphic to the dual space $C([0, 1], X)^*$.*

Proof. Consider the mapping $\Omega(F) := (F(z), \hat{\theta}D^2F)$, $F \in BDC_b((0, \infty))$ where $\hat{\theta}(x) := x$, $x \in (0, \infty)$. By Theorem 5.13, Ω is a mapping into the ℓ_1 -sum $E := X \oplus_1 M_{bv}((0, \infty), X)$, and it is clearly linear. To prove that Ω is onto E , consider an arbitrary $(p, \nu) \in E$. Set $\mu := (1/\hat{\theta})\nu$. By Lemma 2.5, $\mu \in M_{lbv}((0, \infty), X)$ and so by Corollary 3.3 there exists $\tilde{F} \in DC((0, \infty), X)$ such that $\tilde{F}(z) = p$ and $D^2\tilde{F} = \mu$. Since $\hat{\theta}\mu = \nu$ by Remark 2.4(c), Theorem 5.13 implies that $\tilde{F} \in DC_b((0, \infty), X)$. By Theorem 4.5, $(\tilde{F})'_\infty := \lim_{x \rightarrow \infty} (\tilde{F})'_+(x) \in X$. Setting $F(x) := \tilde{F}(x) - (x - z)(\tilde{F})'_\infty$ for $x \in (0, \infty)$, we have $F'_\infty := \lim_{x \rightarrow \infty} F'_+(x) = 0$ and so $F \in BDC_b((0, \infty), X)$ by Corollary 4.8. Obviously, $\Omega(F) = (p, \nu)$. Since (20) gives $\|F\|_{BDC_b} = \|p\| + \|\nu\|$, the mapping Ω is an isometry of $BDC_b((0, \infty), X)$ onto E .

The rest of the proof follows from Lemma 2.7 and Theorem 2.8. \square

It seems that in general there is no natural norm on the vector space $DC_b(C, X)$ (see Definition 5.1). However, in the case $C = (0, \infty)$ such natural norm exists.

Theorem 5.15 (The space $DC_b((0, \infty), X)$). *Let X be a Banach space, $J = (0, \infty)$, and $z \in J$. For $F \in DC_b(J, X)$, let $F'_\infty = \lim_{x \rightarrow \infty} F'_+(x)$ (see Theorem 4.5). Then the formula*

$$\|F\|_{DC_b} = \|F(z)\| + \|F'_\infty\| + \int_0^\infty (V_t^\infty F'_+) dt$$

defines a complete norm on $DC_b(J, X)$ which induces on $BDC_b(J, X)$ a norm equivalent to $\|\cdot\|_D$. The space $DC_b(J, X)$ equipped with the above norm is isometric to the space $M_{bv}([0, 1], X)$. In particular, $DC_b(J, X^)$ is isometric to the dual space $C([0, 1], X)^*$.*

Proof. Using Remark 4.8 we easily obtain that the mapping

$$\Phi: (BDC_b(J, X), \|\cdot\|_{BDC_b}) \oplus_1 X \rightarrow DC_b(J, X)$$

defined by

$$\Phi(G, A)(x) = G(x) + xA, \quad x \in (0, \infty)$$

is an algebraic isomorphism. The norm induced on $DC_b(J, X)$ by Φ is clearly just $\|\cdot\|_{DC_b}$, and so it is complete. By Theorem 5.14, $DC_b(J, X)$ with this norm is isometric to $M_{bv}([0, 1], X) \oplus_1 X$, which is isometric to $M_{bv}([0, 1], X)$ by Lemma 2.7. The last part follows from Theorem 2.8. \square

Remark 5.16. It can be shown that the norm $\|\cdot\|_{DC_b}$ is equivalent to the norm $\|F\|_{DC_b} = \|F(z)\| + \|F'_+(z)\| + \int_0^\infty (V_t^\infty F'_+) dt$.

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