EXAMPLES OF SINGULAR INTEGRAL OPERATORS WITH ROUGH KERNELS

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ABSTRACT. For each $1 < p < 2$ we construct a simple example of singular integral operator which is bounded in the range (p, p') and unbounded outside $[p, p']$. This is done by proving that certain theorem in the article of D. Fan and Y. Pan [2] is sharp.

1. INTRODUCTION

An open question related to the classical Calderon-Zygmund operator with kernel

$$K(y) = \frac{\Omega(y/|y|)}{|y|^n}$$

is to provide an example of function Ω in $L^1(S^{n-1})$ such that the operator is bounded on L^p only in certain prescribed range (p, p'). In the previous works [3], [4] we provided examples of such kernels so that the operator was bounded only on L^2, or bounded on all L^p, $1 < p < \infty$ but not of the weak type $1-1$. However, we were not able to solve the problem in general.

Here, we provide a simple example of singular integral operator which is bounded in the range (p, p') and unbounded outside $[p, p']$ for the slightly more general kernel of the type

$$K(y) = b(|y|)\frac{\Omega(y/|y|)}{|y|^n} \tag{1}$$

where $y \in \mathbb{R}^n$, Ω is a complex-valued integrable function on the sphere $\mathbf{S}^{n-1}$ with mean value zero with respect to the surface measure and integrable function b is defined on $\mathbb{R}^+$. In the article [2] D. Fan and Y. Pan study in great detail the boundedness of integral singular operators with such kernels. They study operators in the form

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \mathcal{P}(y))K(y)dy, \tag{2}$$

where $\mathcal{P}$ is a polynomial. The L^p boundes of the operator T is proved in the cases $\Omega \in L^q$, $q > 1$ and $\Omega \in H^1$ in terms if integrability of the function b. In particular,

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they define the class Δ_γ of functions on $\mathbb{R}^+$ which satisfy

$$\sup_{R>0} \frac{1}{R} \int_0^R |b|^\gamma(t)dt < \infty. \tag{3}$$

The following two theorems are proved in [2]:

Theorem 1. *Let Ω be a complex-valued integrable function on the sphere $\mathbf{S}^1$ with mean value zero such that $\Omega \in L^q$, $q > 1$. Let b be a function in the class Δ_γ, $\gamma > 1$. Suppose that $\mathcal{P}(-y) = -\mathcal{P}(y)$. Then the operator T is bounded on L^p for $1 < p < \infty$.*

Theorem 2. *Let Ω be a complex-valued integrable function on the sphere $\mathbf{S}^1$ with mean value zero such that $\Omega \in H^1$. Let b be a function in the class Δ_γ, $\gamma > 1$. Suppose that $|1/2 - 1/p| < \min\{1/2, 1/\gamma'\}$. Then the operator T is bounded on L^p for $1 < p < \infty$.*

We provide examples which show that both theorems are sharp in terms of γ. First we show that in the Theorem 1 it is not possible to take $\gamma = 1$. The situation is simple, because we construct an example of an operator which is not bounded on L^2. This is equivalent to showing that the Fourier transform of the kernel is not bounded. Therefore, all we need to do is to compute the Fourier transform of the kernel at one point. Showing that the Theorem 2 is sharp is of more interest, because it provides range of very simple examples of singular integral operators which are bounded precisely in prescribed range (p, p') for any $1 < p < 2$. All the examples we provide are in the simplest case $n = 2$ and $\mathcal{P}(y) = y$.

2. Example related to Theorem 1

In this section we show that the Theorem 1 is sharp in terms of γ, that is the Theorem no longer holds if we take $\gamma = 1$. The example can be quickly sumarized as follows: Suppose $n = 2$ and $\mathcal{P}(x) = x$. If $\gamma = 1$, then we may essentially take

$$b = \sum_{j \in I \subset \mathbb{N}} 2^j \delta_{2^j},$$

where δ_x is the Dirac mass supported at the point x and I is a finite index set which we specify later. If we denote the normalized surface measure of the sphere S^1 by σ, we get $\hat{\sigma}(\xi) \to 0$ as $|\xi| \to \infty$. Let us now write $y = (y_1, y_2)$. We observe that if we write $K_j(y) = \sigma(y/2^j)\sin(2\pi y_1)/2^j$, then K_j may be written in the form of (1) with $b_j = 2^j \delta_{2^j}/2\pi$ and $\Omega_j(y) = \sin(2\pi 2^j y_1)$. Clearly we get $\hat{K}_j((1,0)) \approx 1$. Now we take $\Omega = \sum_{j \in I} \Omega_j$, $b = \sum b_j$ and K as in (1). Clearly $\|\Omega\|_q \approx |I|^{1/2}$ for any $1 < q < \infty$. On the other hand, we get $\sum_{j \in I} \hat{K}_j((1,0)) \approx |I|$, while $\hat{K}((1,0)) = \sum_{j \in I} \hat{K}_j((1,0)) + E$, where the term E may be controlled by taking set I with large distances between its elements. Therefore it is not possible to control the L^∞ norm of $\hat{K}$ by the L^q norm of Ω for any $1 < q < \infty$.

Let us make this construction more quantitative, in particular, let us replace the class Δ_γ by a class of logaritmic integrability to see what results this example gives.

We define $\Delta_{1,\beta}$ as a class of functions b such that

$$\|b\|_{\Delta_{1,\beta}} = \sup_{R>0} \int_1^2 |f|(Rt) \log^\beta(2 + |f|(Rt))dt \leq \infty.$$

Clearly $\Delta_{1+\epsilon} \subset \Delta_{1,\beta} \subset \Delta_1$ for $\epsilon, \beta > 0$.

Theorem 3. *For any $0 \leq \beta < 1/2$, any constant $C > 0$ and any $1 \leq q < \infty$ there is a function $\|b\|_{\Delta_{1,\beta}} \leq 1$ and a function Ω integrable on the sphere $\mathbf{S}^1$ with mean value zero such that $\|\Omega\|_q \leq 1$ and $\|T\|_p > C$ for any $1 < p < \infty$.*

Proof. Following the above considerations, we take $\Omega_j(y) = \sin(2\pi 2^j y_1)$ and $\Omega = \sum_{j \in I} \Omega_j$. We take $b_j = 2^j \chi_{(2^j, 2^j+1/100)}$ and $b = \sum_{j\in I} b_j$. We define $K_{j,k}(y) = b_j(|y|)\frac{\Omega_k(y/|y|)}{|y|^2}$ and define the corresponding operator $T_{j,k}$. If we choose the test function $f(y) = \sin(2\pi y_1)$, we see that $|T_{j,k}f| \lesssim 2^{|j-k|}$, with the constant independent of k and j. This claim follows by a routine integral estimate. On the other hand for $y_1 \in (-1/100, 1/100) + 2\pi m$, $m \in \mathbb{Z}$ we get $T_{j,j}f \gtrsim 1$, as the phases correspond. We have $T = \sum_{j,k\in I} T_{j,k}$. Therefore, if we take set I to be an aritmetic progression with large enough step, we see that for any ball $|B| > 100$ we get $\|(Tf)\chi_B\|_p \approx |I||B|^{1/p}$. Let us take $s = \sup I$. We see that $(Tf)\chi_{B(0,2^s)} = T(f\chi_{B(0,2^{s+4})})\chi_{B(0,2^s)}$. $\|f\chi_{B(0,2^{s+4})}\|_p \approx 2^{s/p}$ and therefore $\|T\|_p \gtrapprox |I|$. The function Ω has mean 0 from symmetry. To estimate $\|\Omega\|_q$, we first observe that we have $\| \sum_{j\in I}(\sin 2\pi 2^j \cdot)\|_r \approx |I|^{1/2}$ for any $1 \leq r < \infty$, as it is a lacunary trigonometric series. Moreover, we need to handle the weight which comes from the curvature of the circle. We use Hölder:

$$\int_0^1 |\Omega|^q((t, \sqrt{1-t^2}))\frac{\sqrt{1+t^2}}{\sqrt{1-t^2}}dt \lesssim \left(\int_0^1 |\Omega|^{4q}((t, \sqrt{1-t^2}))dt\right)^{1/4} \approx |I|^{q/2}.$$

So, if we normalize Ω we get $\|T\|_p \approx |I|^{1/2}$. We see that $\|b\|_{\Delta_{1,\beta}} \approx |I|^\beta$ and that finishes the proof. □

The best known result in the logarithmic context is in the paper of Sato [5]. It proves the boundedness of the operator T under the condition that $b \in \Delta_{1,1+\epsilon}$, using an extrapolation method. Such result is natural, because since the work of Duoandikoetxea and Rubio de Francia [1] the method of the proof worked also for an operator with more general kernel

$$K(y) = \sum_{j\in\mathbb{Z}} \chi_{(2^j,2^{j+1}]}(|y|)b(|y|)\frac{\Omega_j(y/|y|)}{|y|^n},$$

where each $\Omega_j \in L^q$. We see that our example gives for such kernel the range $0 < \beta < 1$. Nevertheless, there remains a gap of the size $\log^{1/2}$ between the positive and negative result.

3. Example related to Theorem 2

In this section, we give an example showing that the interplay of p and γ in the Theorem 2 is optimal. This provides us with a simple way to construct a singular integral operator bounded precisely in some range (p, p'). The proof of the Theorem 2 relies on certain l^2 vector valued estimates. Therefore the example centers around showing that if the hypotesis of the Theorem is not satisfied, the vector valued estimate also no longer holds.

Theorem 4. *For any $1 \leq \gamma < 2$, any constant $C > 0$ and any $1 < p < 2$ such that $|1/2 - 1/p| > 1/\gamma'$ there is a function $\|b\|_{\Delta_\gamma} \leq 1$ and a function Ω integrable on the sphere $\mathbf{S}^1$ with mean value zero such that $\|\Omega\|_{H^1} \leq 1$ and $\|T\|_p > C$.*

Proof. We choose $\Omega((y_1, y_2)) = (\chi_{(0,\epsilon)} - \chi_{(-\epsilon,0)})(y_1)\chi_{\{y_2>0\}}(y_2)/\epsilon$ for $(y_1, y_2) \in S^1$. We specify the precise value of $0 < \epsilon < 1$ later. Clearly, $\|\Omega\|_{H^1} \approx 1$. Let us take $m \in \mathbb{N}$ and define the set

$$R = \{(y_1, y_2), y_2 \in \mathbb{N} + (0, 1/(100m))\}.$$

For an index set $\{i_1, \ldots, i_m\} = I \subset \mathbb{N}$, we define the test functions

$$f_j((y_1, y_2)) = \chi_R((y_1, y_2)) \sin(2\pi 2^{i_j} y_1)$$

and put $f = \sum_{j=1}^m f_j$. We define functions b_j as

$$b_j(t) = \sum_{k=1}^{2^{-i_j+l}/100} \chi_{\{(0,1/100m)+j/m+k+2^{-i_j+l}\}}.$$

We choose l huge, in particular $l > 1000 \sup I$ and $\epsilon = 1/l$. Now, we define the kernels

$$K_j(y) = b_j(|y|)\frac{\Omega(y/|y|)}{|y|^2}$$

and define the related operators T_j. A simple integral estimate gives

$$\|T_j f_k\|_\infty \lesssim 2^{-|i_j - i_k|}/m,$$

while

$$|T_j f_j(x_1, x_2)| \gtrsim 1/m$$

for $x_1 \in 2^{-i_j}\mathbb{N} + (-2^{-i_j}/100, 2^{-i_j}/100)$ and $x_2 \in \mathbb{N} + (0, 1/(200m)) - j/m$. Let us denote this set S_j. We see that if we choose the set I such that $\min_{j \neq k} |i_j - i_k|$ is large enough, we get

$$|Tf(x)| \gtrsim 1/m,$$

where $T = \sum T_j$ and $x \in \cup S_j$. For $b = \sum b_j$ we have

$$\|b\|_{\Delta_\gamma} \approx m^{-1/\gamma}.$$

Since the kernel $K = \sum K_j$ is supported in some ball $B(0, R)$, we see that $(Tf)|_{B(0,R)} = (T(f|_{B(0,2R)}))_{B(0,R)}$. We have

$$\|f|_{B(0,2R)}\|_p \approx m^{1/2-1/p} R^{2/p},$$

while

$$\|(Tf)|_{B(0,R)}\|_p \gtrsim m^{-1} |\cup S_j \cap B(0, R)|^{1/p} \approx m^{-1} R^{2/p}.$$

Collecting this estimates finishes the proof. $\square$

We only gave examples of operators with large norms here, however the construction of unbounded operators is possible using trivial summation argument, see [3] or [4].

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