A NEW ALGORITHM FOR APPROXIMATING THE LEAST CONCAVE MAJORANT

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Abstract. A step-by-step algorithm is given to approximate the least concave majorant of a function of one real variable. When the function is a piecewise cubic polynomial the result is exact, up to the error in finding roots of a degree six polynomial. Otherwise, the function may be initially approximated by a clamped cubic spline. In this case, an estimate of the error in the least concave majorant is obtained.

Key words. least concave majorant, level function, spline approximation

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1. Introduction. Fix a real interval $I = [A, B]$ and suppose F is an absolutely continuous function on I. Denote by $\hat{F}$ the least concave majorant of F, namely,

$$\hat{F}(x) := \inf\{G(x) : G \geq F, G \text{ concave}\},$$

which can be shown to be given by,

$$\hat{F}(x) = \sup\left\{\frac{\beta - x}{\beta - \alpha}F(\alpha) + \frac{x - \alpha}{\beta - \alpha}F(\beta) : A \leq \alpha \leq x \leq \beta \leq B\right\}, x \in I.$$

This concave function has application in such diverse areas as Mathematical Economics, Statistics, and Abstract Interpolation Theory. See, for example, [3], [2], [6], [11], [1], [10] and [7].

Our aim in this paper is to give a new algorithm to approximate $\hat{F}$, together with an estimate of the error entailed. If F is given as a continuous, piecewise cubic polynomial, then the algorithm gives a presentation of the least concave majorant of F as another continuous, piecewise cubic polynomial. If not, then F may be approximated by a cubic spline and the least concave majorant of the approximating function is seen to be a good approximation to $\hat{F}$. To estimate the error we use a known result for the approximation error for cubic splines, from [4], together with a new result on the level function with respect to Lebesgue measure. See Theorem 5.2 below.

Suppose f is almost everywhere equal to the derivative of an absolutely continuous function F. The level function f^o is defined to be the derivative of $\hat{F}$. Note that since $\hat{F}$ is concave, it is differentiable almost everywhere and, moreover, f^o is non-increasing. The level function has a simple structure that is the basis for our algorithm: Since F and $\hat{F}$ are continuous, the zero set, Z_F, of $\hat{F} - F$ is closed. The connected components of the complement of Z_F are intervals open in the relative topology of I, which are essentially the maximal level intervals (MLIs) of f. (See Section 3 for the precise definition of an MLI of f.) Then,

$$f^o(x) := \begin{cases} f(x), & x \in Z_F \\ \frac{1}{b-a}\int_a^b f, & x \in (a, b), \text{an MLI of } f. \end{cases}$$

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This function f^o plays a key role in the duality theory of Banach function spaces. See [5], [9], [12], and [8]. The algorithm presented here may also be used to approximate the level function of a given function f.

We find the least concave majorant of F by finding all the MLIs for $f = F'$. This enables us to compute f^o and then, by integration, $\hat{F}$. The approach is to use necessary conditions on MLIs to identify a small set of intervals that includes every MLI. These intervals are tested and non-MLIs are discarded. The algorithm to do this is given in the next section and justified in the one following that. It is followed by a section devoted to proving three lemmas. In Section 5 the error estimates are established.

2. The Algorithm. Suppose F is a continuous piecewise cubic polynomial on the interval $I = [A, B]$. Our object is to compute $\hat{F}$, the least concave majorant of F, and to present it in a similar fashion.

Although we only require it for continuous, piecewise cubic polynomials, the algorithm will work for continuous piecewise polynomials of higher order or even more general functions. It is sufficient that F be continuous, and that the formulas for each piece of F permit evaluation of $F(x)$, $F'(x-)$, $F'(x+)$, $F''(x-)$, and $F''(x+)$ at any point in the interior of I. Note that $F'(x-) = F'(x+)$ and $F''(x-) = F''(x+)$ except at the partition points.

To avoid having to treat the endpoints of $I \equiv [A, B]$ differently than the other partition points it is convenient to assign $F'(A-) = \infty$ and $F'(B+) = -\infty$.

Step 1. Solve $F''(x) = 0$ to determine all points of inflection of each piece of F and insert them into the partition. Now F is either convex or concave on each partition subinterval.

Step 2. Replace each convex part of F by the straight line connecting the endpoints of its graph. The new F has the same least concave majorant as the original one. Now F is either strictly concave or else of constant slope on each partition subinterval.
Designate each partition subinterval on which F is strictly concave as a Type I1 interval and each of the others as Type I2. Also, designate each partition point x at which $F'(x+) \le F'(x-)$ as a Type P1 point and each of the others as Type P2. Note that the endpoints of I are both Type P1 points.

Step 3. Enumerate all pairs (P, Q), where each of P and Q is either a Type I1 interval or a Type P1 point, and P is to the left of Q. For each such pair do the following:

a) If $\left\{\begin{array}{l} P = [u, v] \\ Q = [U, V] \end{array}\right\}$, set $\left\{\begin{array}{l} c = \max\{F'(v), F'(V)\} \\ d = \min\{F'(u), F'(U)\} \end{array}\right\}$.
If $\left\{\begin{array}{l} P = [u, v] \\ Q = b \end{array}\right\}$, set $\left\{\begin{array}{l} c = \max\{F'(v), F'(b+) \\ d = \min\{F'(u), F'(b-)\} \end{array}\right\}$.
If $\left\{\begin{array}{l} P = a \\ Q = [U, V] \end{array}\right\}$, set $\left\{\begin{array}{l} c = \max\{F'(a+), F'(V)\} \\ d = \min\{F'(a-), F'(U)\} \end{array}\right\}$.
If $\left\{\begin{array}{l} P = a \\ Q = b \end{array}\right\}$, set $\left\{\begin{array}{l} c = \max\{F'(a+), F'(b+)\} \\ d = \min\{F'(a-), F'(b-)\} \end{array}\right\}$.

b) If $P = [u, v]$ set $a(y) = (F_P')^{-1}(y)$, where F_P denotes the restriction of F to P. If $P = a$ set $a(y) = a$.

c) If $Q = [U, V]$ set $b(y) = (F_Q')^{-1}(y)$, where F_Q denotes the restriction of F to Q. If $Q = b$ set $b(y) = b$.

d) Set $\varphi(y) = F(b(y)) - F(a(y)) - y(b(y) - a(y))$.

e) If $c > d$, $\varphi(c) < 0$ or $\varphi(d) > 0$, then skip 3f and 3g for the current pair and continue with the next pair at Step 3.
f) Solve $\varphi(y) = 0$ for $c \leq y \leq d$.
g) Save the triple $(a(y), b(y), y)$ for the y found in 3f and continue with the next pair at Step 3.

Step 4. For each saved triple (a, b, y) check that $F(x) \leq y(x - a) + F(a)$, as x runs through all partition points and all solutions to $F'(x) = y$. Discard those triples that fail.

Step 5. Discard any triple (a, b, y) such that $[a, b] \subseteq [\bar{a}, \bar{b}]$ for some other triple $(\bar{a}, \bar{b}, \bar{y})$.

Step 6. Modify the partition of I: For each (remaining) triple (a, b, y), insert a and b into the partition and delete any partition points that lie in (a, b). On the interval $[a, b]$, set $\hat{F}(x) = y(x - a) + F(a)$. On every other interval in the modified partition, set $\hat{F}(x) = F(x)$.

3. Justification. Here we explain and expand on each of the steps of the algorithm, and prove that the algorithm does indeed generate the least concave majorant.

About Step 1: Inserting all solutions of $F''(x) = 0$ as new partition points will ensure that F does not change concavity on any partition subinterval. If F is piecewise cubic there will be at most one new point in each of the original partition subintervals.

About Step 2: If F is convex on an interval $[u, v]$ then F lies below the line segment joining the points $(u, F(u))$ and $(v, F(v))$. Since $\hat{F}$ is concave and passes above the points $(u, F(u))$ and $(v, F(v))$, it lies above the line segment joining them. Thus, replacing F by this line segment has no effect on its least concave majorant $\hat{F}$. Note that the new F is still continuous and if F was piecewise cubic then it is still piecewise cubic. (However, if F was a spline, it may lose smoothness at partition points.)

Before we continue with Step 3, we state three lemmas that are needed to justify the step. They will be proved in Section 4.

Lemma 3.1. *If F is a continuous function on a finite interval I then $\hat{F}$ is continuous, it agrees with F on the endpoints of I, and it has constant slope on each component of the open subset $\{x \in I : F(x) \neq \hat{F}(x)\}$.*

In view of Lemma 3.1, to describe $\hat{F}$ completely it is enough to determine all maximal intervals (of positive length) on which $\hat{F}$ coincides with some straight line. We call such intervals maximal level intervals (MLIs). It is clear that each of the components from Lemma 1 is contained in an MLI. Continuity ensures that each MLI is a closed interval. Moreover, no two MLIs can intersect in more than a single point; otherwise $\hat{F}$ would coincide with a straight line on their union, contradicting maximality. The next lemma gives some additional properties of MLIs. Recall that $F' = f$.

Lemma 3.2. *If $[a, b]$ is an MLI, then $F(a) = \hat{F}(a)$, $F(b) = \hat{F}(b)$,*

$$f(a+) \leq \frac{F(b) - F(a)}{b - a} \leq f(a-) \text{ and } f(b+) \leq \frac{F(b) - F(a)}{b - a} \leq f(b-).$$

The final lemma shows that the search for (endpoints of) MLIs can be restricted to Type P1 points and Type I1 intervals.

Lemma 3.3. *Each endpoint of an MLI is either equal to a Type P1 point or interior to a Type I1 interval.*

About Step 3. According to Lemma 3.3, each MLI must have its left endpoint on some P and its right endpoint on some Q. Lemma 3.2 provides strong conditions on these endpoints. Specifically, if $[a, b]$ is an MLI there must be a y (the slope of the line ℓ) satisfying,

$$y = \frac{F(b) - F(a)}{b - a}, \; f(a+) \leq y \leq f(a-), \; f(b+) \leq y \leq f(b-). \tag{3.1}$$

3a) These definitions restrict the range of possible slopes to $y \in [c, d]$. If $P = [u, v]$, then $[u, v]$ is a Type I1 interval, so F' is strictly decreasing on (u, v). Since $a \in (u, v)$, (3.1) implies that $y = f(a)$ and hence $f(v) \leq y \leq f(u)$. If $P = a$ we have $f(a+) \leq y \leq f(a-)$, from (3.1). Similar comments applied to Q show that either $f(V) \leq y \leq f(U)$ or $f(b+) \leq y \leq f(b-)$. In all four cases, $[c, d]$ represents the range of possible y values.

3b) If P is a Type I1 interval let F_P denote the piece of F on P. The equation $y = F_P'(a)$ and the monotonicity of F_P' implies that $a(y) = (F_P')^{-1}(y)$ is well-defined on $[c, d]$. If $P = a$, then a is independent of y so $a(y) = a$.

If F is piecewise cubic, then F_P' is a decreasing quadratic and one can easily give a formula for $a(y)$: If $y = F_P'(a) = r_1 a + r_0$ then $a = (r_0 - y)/r_1$. If $y = F_P'(a) = r_2 a^2 + r_1 a + r_0$ then $a = (-r_1 - \sqrt{r_1^2 - 4r_2(r_0 - y)})/(2r_2)$. This choice of solution in the quadratic formula always gives the root that is on the decreasing portion of the quadratic.

In the absence of a formula for $a(y)$, the function $a(y)$ may be evaluated by solving the equation $F_P'(a) = y$. Since F_P' is strictly decreasing the solution will be unique for any $y \in [c, d]$.

3c) Similar comments to those in 3b apply to Q and the function $b(y)$.

3d) Observe that if y is a root of φ, then it will satisfy the first condition of (3.1).

If F is piecewise cubic, then the formulas for $a(y)$ and $b(y)$ from 3b ensure an algebraic formula for $\varphi(y)$. However, it is not necessary to obtain an algebraic formula for $\varphi(y)$ in order to carry out the algorithm.

Observe that for $y \in [c, d]$, $f(b(y)) = f(a(y)) = y$, so the derivative of φ satisfies

$$\begin{aligned} \varphi'(y) =& f(b(y))b'(y) - f(a(y))a'(y) - (b(y) - b(a)) - y(b'(y) - a'(y)) \\ =& a(y) - b(y) < 0. \end{aligned}$$

Thus, φ is continuous and strictly decreasing on $[c, d]$.

3e) If $c > d$ there is no possible y value and if the interval $[\varphi(d), \varphi(c)]$ does not contain zero there is no solution to $\varphi(y) = 0$. Thus there is no MLI with its left endpoint in P and its right endpoint in Q.

3f) Continuity and monotonicity of φ ensure that there is a unique $y \in [c, d]$ satisfying $\varphi(y) = 0$. This may be found by a technique adapted to the functions involved. In the most general case, monotonicity ensures that a binary search may be carried out, even if an algebraic formula for φ is not available. This method does not require formulas for $a(y)$ and $b(y)$, only the ability to evaluate them at each $y \in [c, d]$.

Of course, in many cases $a(y) = a$ and $b(y) = b$ are both constant and then $y = (F(b) - F(a))/(b - a)$ is immediate. If F is a piecewise cubic, then the formulas for a and b given in 3b,c ensure that the equation $\varphi(y) = 0$ may be translated to a polynomial equation of degree at most six and solved using standard numerical methods. To see that the solution of $\varphi(y) = 0$ is also the root of a polynomial of degree at most six, let $\mathbf{R}_k = \mathbf{R}_k[y]$ denote the collection of real polynomials in the variable y, of degree at most k. Then there exist $\alpha, \beta \in \mathbf{R}_1$ such that $a(y) \in \mathbf{R}_0 + \mathbf{R}_0\sqrt{\alpha}$ and $b(y) \in \mathbf{R}_0 + \mathbf{R}_0\sqrt{\beta}$. Since F_P and F_Q are polynomials of degree at most 3,

$F_P(a(y)) - ya(y) \in \mathbf{R}_1 + \mathbf{R}_1\sqrt{\alpha}$ and $F_Q(b(y)) - yb(y) \in \mathbf{R}_1 + \mathbf{R}_1\sqrt{\beta}$ so $\varphi(y) = 0$ is equivalent to $\mu_1 + \mu_2\sqrt{\alpha} + \mu_3\sqrt{\beta} = 0$ for some $\mu_1, \mu_2, \mu_3 \in \mathbf{R}_1$. Isolating μ_1 and squaring both sides gives, $\mu_1^2 = \mu_2^2\alpha + \mu_3^2\beta + 2\mu_2\mu_3\sqrt{\alpha}\sqrt{\beta}$. Isolating the term with square roots and squaring both sides again gives, $(\mu_1^2 - \mu_2^2\alpha - \mu_3^2\beta)^2 = 4\mu_2^2\mu_3^2\alpha\beta$. Each side of the last equation is a polynomial of degree at most 6, in the variable y.

As an example consider the most complicated case, when both P and Q are intervals in Step 3a and F is a genuine cubic on each interval. That is, suppose that $F_P(y) = ky^3 + ly^2 + my + n$ on $P = [u, v]$, with $k \neq 0$, and $F_Q(y) = Ky^3 + Ly^2 + My + N$ on $Q = [U, V]$, with $K \neq 0$. Then y is a root of the sextic polynomial,

$$(\mu_1^2 - \mu_2^2\alpha - \mu_3^2\beta)^2 - 4\mu_2^2\mu_3^2\alpha\beta,$$

where α, β, μ_1, μ_2, and μ_3 are given by,

$$\alpha = 3ky + l^2 - 3km, \quad \beta = 3Ky + L^2 - 3KM, \quad \mu_2 = -\frac{2\alpha}{27k^2}, \quad \mu_3 = \frac{2\beta}{27K^2},$$

and

$$\mu_1 = \frac{1}{3}\left(\frac{L}{K} - \frac{l}{k}\right)y + \left(N + \frac{2L^3}{27K^2} - \frac{ML}{3K}\right) - \left(n + \frac{2l^3}{27k^2} - \frac{ml}{3k}\right).$$

Obtaining these expressions is routine: Differentiate F_P and view $F_P'(a) - y = 0$ as a quadratic in a. The discriminant is 2α and the solution given in 3b determines a. In the same way, use $F_Q'(b) - y = 0$ to find β and b. Values for μ_1, μ_2, and μ_3 may be read off from the expanded form of $\varphi(y) = (F_Q(b(y)) - yb(y)) - (F_P(a(y)) - ya(y)) = 0$. With the coefficients determined, the sextic can be solved by standard numerical methods, aided by the knowledge that one of its roots is the unique solution to $\varphi(y) = 0$ in the interval $[c, d]$. Recall that, in this case, $c = \max\{F'(v), F'(V)\}$ and $d = \min\{F'(u), F'(U)\}$. (We do not assert that the sextic itself has a unique root in $[c, d]$ as squaring may have introduced extraneous roots.)

3g) The interval $[a, b]$ is the only possible candidate for an MLI with its left endpoint in P and its right endpoint in Q. It is convenient to save the slope y as well as the endpoints, although the y value could be recovered from a and b using (3.1).

About Step 4: If $[a, b]$ is an MLI, then $\hat{F}$ will coincide on $[a, b]$ with the line passing through $(a, F(a))$ and $(b, F(b))$. Since $\hat{F}$ is concave, the entire line will lie above the graph of $\hat{F}$ and hence above the graph of F. If this line does not lie above the graph of F then $[a, b]$ may be discarded. A bit of calculus shows that it is enough to consider only partition points and solutions to $F'(x) = y$ when checking that the line lies above the graph of F.

About Step 5. Discard triples for which $[a, b]$ is not maximal. The remaining triples correspond exactly to the MLIs: Lemma 3.3 ensures that the search in Step 3 finds a triple corresponding to every MLI. On the other hand, for each remaining triple (a, b, y), Step 4 guarantees that $\hat{F}$ coincides with the line $y(x - a) + F(a)$ on $[a, b]$ and this step ensures maximality.

About Step 6. Since the remaining triples correspond to the MLIs, no two intervals $[a, b]$ obtained from remaining triples can overlap. Lemma 3.1 ensures that this piecewise presentation of $\hat{F}$ gives the least concave majorant of F.

4. Proofs of Three Lemmas. *Proof of Lemma 3.1.* Write $I = [A, B]$. Since $\hat{F}$ is concave, it is continuous on (A, B). The continuity of F on I ensures that for all $\varepsilon > 0$, there exists a slope m such that the graph of F lies under the line $\ell_A(x) =$

$m(x-A)+F(A)+\varepsilon$. But ℓ_A is a concave majorant of F, so $F(x) \le \hat{F}(x) \le \ell_A(x)$ for all $x \in I$. Since $\varepsilon > 0$ was arbitrary, $\hat{F}$ is continuous at A and $\hat{F}(A) = F(A)$. A similar argument shows that $\hat{F}$ is continuous at B and $\hat{F}(B) = F(B)$.

Since F and $\hat{F}$ are continuous, $\{x \in I : F(x) \ne \hat{F}(x)\}$ is an open subset of (A,B) and its connected components are open intervals. Let (a,b) be one such component and note that $F(a) = \hat{F}(a)$ and $F(b) = \hat{F}(b)$. Let ℓ be the line through the points $(a, F(a))$ and $(b, F(b))$ and let y be the point at which the continuous function $F - \ell$ achieves its maximum value on $[A,B]$. Since F lies below the line $\ell + F(y) - \ell(y)$, so does $\hat{F}$. In particular, $\hat{F}(y) \le F(y)$ so $\hat{F}(y) = F(y)$ and hence $y \notin (a,b)$. But $F(a) = \hat{F}(a)$ and $F(b) = \hat{F}(b)$ so, by concavity, $\hat{F}$ lies above ℓ on $[a,b]$ and below ℓ off (a,b). Thus $F(y) - \ell(y) \le \hat{F}(y) - \ell(y) \le 0$ and so $\hat{F} \le \ell + F(y) - \ell(y) \le \ell$. In particular, $\hat{F}$ lies below ℓ on $[a,b]$. It follows that $\hat{F}$ coincides with the line ℓ on $[a,b]$ and hence has constant slope there. ∎

Proof of Lemma 3.2. Suppose $F(a) < \hat{F}(a)$. Then a is in the interior of I and there exists a positive ε such that $a - \varepsilon$ is in I, $a + \varepsilon < b$, and both $\hat{F}(a-\varepsilon)$ and $\hat{F}(a+\varepsilon)$ are larger than the maximum value of $F(x)$ for $x \in [a-\varepsilon, a+\varepsilon]$. The line through $(a-\varepsilon, \hat{F}(a-\varepsilon))$ and $(a+\varepsilon, \hat{F}(a+\varepsilon))$ lies above F and below $\hat{F}$ on $[a-\varepsilon, a+\varepsilon]$ and it lies above $\hat{F}$, and hence above F, off $(a-\varepsilon, a+\varepsilon)$. This line is therefore a concave majorant of F and so lies above $\hat{F}$. In particular, it coincides with $\hat{F}$ on $[a-\varepsilon, a+\varepsilon]$ This contradicts the maximality of the MLI $[a,b]$. Thus $F(a) = \hat{F}(a)$. The equation $F(b) = \hat{F}(b)$ follows in the same way.

Let ℓ be the line through the points $(a, F(a))$ and $(b, F(b))$. This ℓ coincides with $\hat{F}$ on $[a,b]$ and lies above $\hat{F}$ on I. So it lies above F on I as well. Since the left and right derivatives of F exist everywhere,

$$F'(a+) = \lim_{x\to a+} \frac{F(x)-F(a)}{x-a} \le \lim_{x\to a+} \frac{\ell(x)-\ell(a)}{x-a} = \frac{F(b)-F(a)}{b-a},$$
$$F'(a-) = \lim_{x\to a-} \frac{F(a)-F(x)}{a-x} \ge \lim_{x\to a-} \frac{\ell(a)-\ell(x)}{a-x} = \frac{F(b)-F(a)}{b-a}$$

A similar argument proves the inequalities involving $F'(b+)$ and $F'(b-)$ and completes the proof. ∎

Proof of Lemma 3.3. Suppose $[a,b]$ is an MLI. By Lemma 2, we have $F'(a+) \le F'(a-)$ and $F'(b+) \le F'(b-)$. It follows that if a or b is a partition point then it is a Type P1 point. Suppose a is interior to a partition interval of Type I2. Then F has constant slope there. By Lemma 2, its slope is $F'(a+) = F'(a-) = (F(b) - F(a))/(b-a)$, which is also the slope of the line ℓ that coincides with $\hat{F}$ on $[a,b]$. Since $\ell(a) = F(a)$ the function F and the line ℓ coincide on the entire partition interval. But $F \le \hat{F} \le \ell$, so $\hat{F}$ also coincides with ℓ on the partition interval. This contradicts the maximality of the MLI $[a,b]$. We conclude that if a is interior to a partition interval, the interval must be of Type I1. ∎

5. Error Estimates. Given an absolutely continuous function G on a closed interval I of finite length, we choose F to be the clamped cubic spline interpolating G at the points of a partition ρ of I. This permits us to take advantage of the following special case of optimal error bounds for cubic spline interpolation obtained by Charles A. Hall and W. Weston Meyer in [4].

PROPOSITION 5.1. *Suppose $G \in C^4(I)$ and let $\rho := [x_0, \dots, x_{n+1}]$ be a partition of I. Denote by F the clamped cubic spline interpolating G at the nodes of ρ. Then,*

$$|G(x) - F(x)| \le \frac{1}{24} \|G^{(4)}\|_\infty \|\rho\|^3, \quad x \in I,$$

where $\|\cdot\|_\infty$ *denotes the usual supremum norm and*

$$\|\rho\| := \sup\{|x_k - x_{k-1}| : k = 1, \dots, n\}.$$

To estimate the error in the least concave majorant, we consider the sensitivity of the level function to changes in the original function.

THEOREM 5.2. *Suppose* F *and* G *are absolutely continuous functions defined on a finite interval* I. *Then* $\hat{F}$ *and* $\hat{G}$ *are also absolutely continuous on* I, *and*

$$\|f^o - g^o\|_\infty \le \|f - g\|_\infty.$$

Here $\hat{F}$ *and* $\hat{G}$ *denote the least concave majorants of* F *and* G, *respectively, and* $f = F'$, $g = G'$. $f^o = (\hat{F})'$, *and* $g^o = (\hat{G})'$.

Proof. Set $Z_F = \{x \in I : F(x) = \hat{F}(x)\}$, $Z_G = \{x \in I : G(x) = \hat{G}(x)\}$ and observe that $f^o = f$ almost everywhere on Z_F and $g^o = g$ almost everywhere on Z_G. By Lemma 3.1, $\hat{F}$ is continuous and is of constant slope on each component of the complement of Z_F. It follows that $\hat{F}$ is absolutely continuous on I. Since $\hat{G}$ is continuous and is of constant slope on each component of the complement of Z_G, $\hat{G}$ is absolutely continuous on I as well.

We consider several cases to establish that $|f^o(x) - g^o(x)| \le \|f - g\|_\infty$ for almost every $x \in I$.

Case 1: $x \in Z_F$ and $x \in Z_G$. For almost every such x,

$$|f^o(x) - g^o(x)| = |f(x) - g(x)| \le \|f - g\|_\infty.$$

Case 2: $x \in Z_G$ but $x \notin Z_F$. Then x is in the interior of some MLI $[a, b]$ of f. By Lemma 3.2, $\hat{F}(a) = F(a)$ and $\hat{F}(b) = F(b)$.

Since $\hat{F}$ has constant slope on $[a, b]$,

$$\int_a^x f = F(x) - F(a) \le \hat{F}(x) - \hat{F}(a) = (x - a) f^o(x).$$

and

$$\int_x^b f = F(b) - F(x) \ge \hat{F}(b) - \hat{F}(x) = (b - x) f^o(x).$$

Also, since $\hat{G}(x) = G(x)$ and g^o is non-increasing,

$$\int_a^x g = G(x) - G(a) \ge \hat{G}(x) - \hat{G}(a) = \int_a^x g^o \ge (x - a) g^o(x).$$

and

$$\int_x^b g = G(b) - G(x) \le \hat{G}(b) - \hat{G}(x) = \int_x^b g^o \le (b - x) g^o(x).$$

Combining these four inequalities, we obtain,

$$\begin{aligned} -\|f - g\|_\infty &\le \frac{1}{x - a} \int_a^x (f - g) \le f^o(x) - g^o(x) \\ &\le \frac{1}{b - x} \int_x^b (f - g) \le \|f - g\|_\infty. \end{aligned}$$

Thus, $|f^o(x) - g^o(x)| \le \|f - g\|_\infty$.

Case 3: $x \in Z_F$ but $x \notin Z_G$. Just reverse the roles of F and G in Case 2.

Case 4: $x \notin Z_F$ and $x \notin Z_G$. Suppose without loss of generality that $g^o(x) \le f^o(x)$. Let a be the left endpoint of the MLI of g containing x, and let b be the right endpoint of the MLI of f containing x. By Lemma 2, $\hat{G}(a) = G(a)$ and $\hat{F}(b) = F(b)$. Since g^o is constant on (a, x) and non-increasing on (x, b) we have

$$(b - a)g^o(x) \ge \int_a^b g^o = \hat{G}(b) - \hat{G}(a) \ge G(b) - G(a) = \int_a^b g.$$

Since f^o is non-increasing on (a, x) and constant on (x, b), we have

$$(b - a)f^o(x) \le \int_a^b f^o = \hat{F}(b) - \hat{F}(a) \le F(b) - F(a) = \int_a^b f.$$

Combining these, we have

$$f^o(x) - g^o(x) \le \frac{1}{b-a}\int_a^b (f - g) \le \|f - g\|_\infty.$$

This completes the proof. $\square$

The last result can be combined with Proposition 5.1 to give the desired error estimates.

Theorem 5.3. *Let ρ be a partition of the interval $[A, B]$ and suppose $G \in C^4([A, B])$. Let F be the clamped cubic spline interpolating G on ρ. Then*

$$\|f^o - g^o\|_\infty \le \|f - g\|_\infty \le \frac{1}{24}\|F^{(4)}\|_\infty \|\rho\|^3$$

and for each $x \in [A, B]$,

$$|\hat{F}(x) - \hat{G}(x)| \le \frac{\min(x - A, B - x)}{24}\|F^{(4)}\|_\infty \|\rho\|^3.$$

Here $\hat{F}$ and $\hat{G}$ denote the least concave majorants of F and G, respectively, and $f = F'$, $g = G'$. $f^o = (\hat{F})'$, and $g^o = (\hat{G})'$.

Proof. The first inequality is just Theorem 5.2 and the result from [HM]. For the second, observe that by Lemma 3.1, $\hat{F}(A) = F(A)$ and $\hat{G}(A) = G(A)$, and since A is in the partition ρ, $G(A) = F(A)$. Thus, $\hat{F}(A) = \hat{G}(A)$. Since both $\hat{F}$ and $\hat{G}$ are concave and hence absolutely continuous,

$$|\hat{F}(x) - \hat{G}(x)| = \left|\int_A^x f^o - g^o\right| \le \int_A^x \|f^o - g^o\|_\infty \le \frac{x - A}{24}\|F^{(4)}\|_\infty \|\rho\|^3.$$

A similar argument, using integration on $[x, B]$, shows that

$$|\hat{F}(x) - \hat{G}(x)| \le \frac{B - x}{24}\|F^{(4)}\|_\infty \|\rho\|^3$$

and completes the proof. $\square$

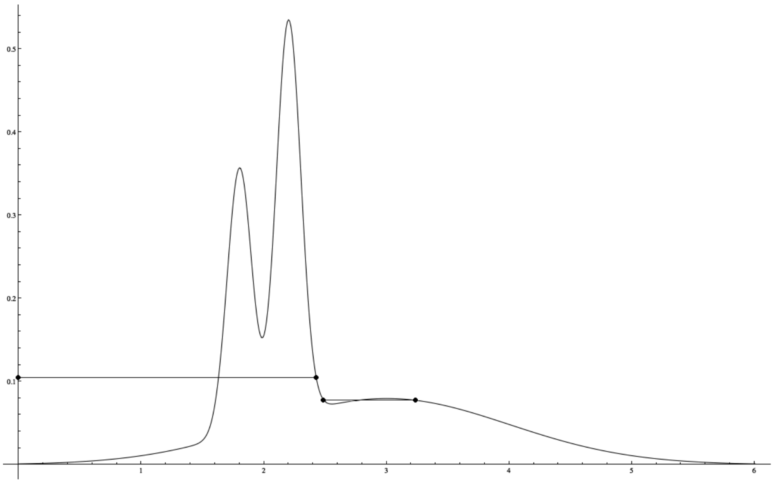

FIG. 6.1. *The function f with its level intervals*

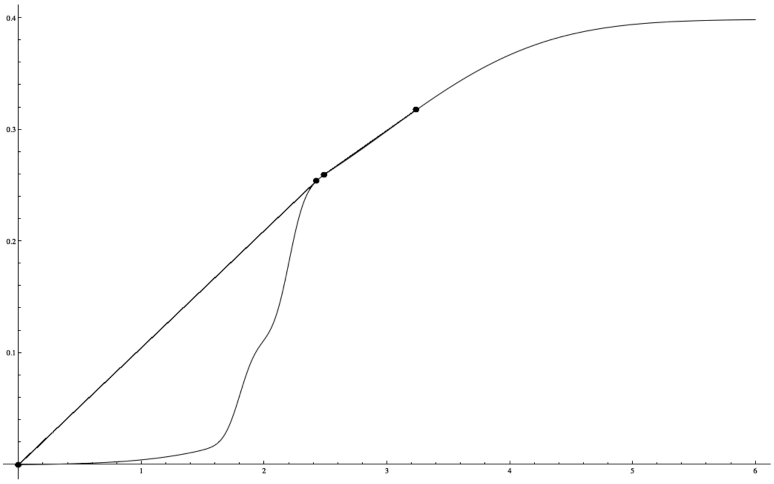

FIG. 6.2. *The function F with its least concave majorant*

6. Example. Consider the trimodal density function,

$$f(x) = 0.5\varphi(x-3) + 3\varphi(10(x-3.8)) + 2\varphi(10(x-4.2)),$$

in which $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$. We wish to approximate the least concave majorant of $F(x) = \int_0^x f(y)\,dy$ on $[0,6]$. Now, $\|F^{(4)}\|_\infty \le 700$ so to ensure that the clamped cubic spline S_F, approximating F on $[0,6]$, satisfies $|f^o(x) - (S'_F)^o(x)| \le .001$ on $[0,6]$ we solve the equation $(700/24)\|\rho\|^3 = .001$ to obtain $\|\rho\| = .03248661$. Dividing $[0.6]$ into $85 > (6/.03248661)$ equal subintervals, we apply the algorithm to identify the level intervals for S'_F and obtain $(S'_F)^o$. The approximation $\int_0^x (S'_f)^o(y)\,dy$ to $F(x)$ is accurate to within .003.

A graph of $f(x) \approx S'_F(x)$ with its level intervals is given in Figure 6.1. Figure 6.2 shows the graph of $F(y)$ and the approximation to its least concave majorant, $\int_0^x (S'_f)^o(y)\,dy$.

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