JACOBIAN OF WEAK LIMITS OF SOBOLEV HOMEOMORPHISMS

STANISLAV HENCL AND JANI ONNINEN

Abstract. Let Ω be a domain in $\mathbb{R}^n$, $n = 2, 3$. Suppose that a sequence of Sobolev homeomorphisms $f_k \colon \Omega \to \mathbb{R}^n$ with positive Jacobian determinants, $J(x, f_k) > 0$, converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, for some $p \geqslant 1$, to a mapping f. We show that $J(x, f) \geqslant 0$ a.e. in Ω.

1. Introduction

The main goal of this note is to establish when the sign of the Jacobian is preserved under $W^{1,p}$-weak convergence. Such a question pops out naturally in the variational approach to Geometric Function Theory (GFT) [3, 12, 23] and Nonlinear Elasticity (NE) [1, 4, 5, 18, 21, 22]. Both theories GFT and NE deal with minimizing sequences of Sobolev homeomorphisms. In the context of NE, one typically deals with $2D$ or $3D$ models and require that the deformation gradients belong to $M_+^{n\times n}$, where $M^{m\times n} = \{\text{real } m\times n \text{ matrices}\}$, and $M_+^{n\times n} = \{A \in M^{n\times n} \colon \det A > 0\}$. It is certainly unrealistic to require that the infimum energy of a given stored energy functional be attained within the class of homeomorphisms; interpenetration of matter may occur. Even in a special case of Dirichlet energy injectivity is often lost when passing to the weak limit of the minimizing sequence, [2, 13, 14, 15]. Further examinations are needed to know the properties of such singular minimizers.

Throughout this text Ω will be a domain in $\mathbb{R}^n$. The class of Sobolev mappings $f \colon \Omega \to \mathbb{R}^n$ with nonnegative Jacobian determinant, $J(x, f) = \det Df(x) \geqslant 0$, almost everywhere, is closed under the weak convergence in $W^{1,p}(\Omega, \mathbb{R}^n)$ provided $p \geqslant n$, [12, Theorem 8.4.2]. However, if $p < n$, passing to the weak $W^{1,p}$-limit of a sequence with nonnegative Jacobians one may loose the sign of the Jacobian. Indeed, there exists a sequence of Sobolev mappings $f_k \colon \Omega \to \mathbb{R}^n$ with $J(x, f_k) > 0$ a.e. such that the sequence converges weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p < n$, to the mapping $f(x) = (-x_1, x_2, \dots, x_n)$, see [12, page 181]. Moreover, following the construction in [17] such mappings f_k can be made continuous. However, it is not obvious at all as to whether one can make a similar example with f_k to be

Date: October 3, 2013.

2000 *Mathematics Subject Classification.* Primary 26B10; Secondary 46E35.

Key words and phrases. Sobolev Homeomorphism, Weak Limits, Jacobian.

S. Hencl was supported by by the ERC CZ grant LL1203 of the Czech Ministry of Education. J. Onninen was supported by the NSF grant DMS-1301570.

homeomorphisms. This is the subject of our result here. Here $[\frac{n}{2}]$ denotes the integer part, i.e. $[\frac{2}{2}]=1$, $[\frac{3}{2}]=1$ and so on.

Theorem 1. *Let $\Omega\subset\mathbb{R}^n$ be a domain and let $p\geqslant 1$ for $n\in\{2,3\}$ and $p>[\frac{n}{2}]$ for $n\geqslant 4$. Suppose that a sequence of Sobolev homeomorphisms $f_k\colon\Omega\to\mathbb{R}^n$ with $J(x,f_k)\geqslant 0$ converges weakly in $W^{1,p}(\Omega,\mathbb{R}^n)$ to a mapping f and further assume that $J(x,f_k)>0$ on a set of positive measure. Then $J(x,f)\geqslant 0$ a.e. in Ω.*

It is worth noting that in Theorem 1 the Jacobian $J(x,f)$ can have very different behavior than the Jacobians in the sequence without knowing that $J(x,f_k)>0$ on a set of positive measure. Indeed, there exists a sequence of Sobolev homeomorphisms f_k with $J(x,f_k)=0$ a.e., converging weakly in $W^{1,p}(\Omega,\mathbb{R}^n)$, $1\leqslant p<n$, to the mapping $f(x)=x$. To obtain such a sequence we cover Ω by diamonds of diameter less than $1/k$ and on each diamond we follow the construction from [10] to obtain a homeomorphism with zero Jacobian a.e. It is possible to make the $W^{1,p}$-norm of the sequence uniformly bounded and hence find a weakly convergent subsequence. Furthermore, it follows from the construction that the sequence f_k converges uniformly to the identity. This also shows that there is a sequence with $J(x,f_k)=0$ a.e. converging weakly in $W^{1,p}(\Omega,\mathbb{R}^n)$, $1\leqslant p<n$, to $f(x)=(-x_1,x_2,\dots,x_n)$.

It is not known if the Jacobian of Sobolev homeomorphism can change sign for $p\leqslant[\frac{n}{2}]$ and $n\geqslant 4$, see [11]. Especially, we do not know if Theorem 1 holds under this lower regularity assumption.

2. Preliminaries

2.1. **Degree and Jacobian.** There are two basic approaches to the notation of local degree for a mapping, the algebraic (see e.g. Dold [7]) and the analytic (see e.g. Lloyd [16]). Both of these notions try to capture the idea of counting the preimages of a target point. For a continuous mapping $f\colon\Omega\to\mathbb{R}^n$ and $y_\circ\in\mathbb{R}^n\setminus f(\Omega)$ the degree of f at $y_\circ$ with respect to Ω is denoted by $\deg(f,\Omega,y_\circ)$. If $f\colon\Omega\to\mathbb{R}^n$ is a homeomorphism, then $\deg(f,\Omega,y_\circ)$ is either 1 or -1 for all $y_\circ\in f(\Omega)$, see e.g. [16, IV.5] or [20, II.2.4. Theorem 3]. We say that a homeomorphism f is *sense-preserving* if $\deg(f,\Omega,y_\circ)\equiv 1$. For a linear map $A\colon\mathbb{R}^n\to\mathbb{R}^n$ with $\det A\neq 0$, it is easy to check from the definition that

$$\deg(A,\Omega,y_\circ)=\operatorname{sgn}\det A\ . \tag{1}$$

We recall the following corollary [3, Corollary 2.8.2]. Given a homeomorphism $f\colon\Omega\to\mathbb{R}^n$ suppose that f is differentiable at $x_\circ$ with $J(x_\circ,f)\neq 0$. Then we have

$$\deg(f,\Omega,f(x_\circ))=\operatorname{sgn}J(x_\circ,f)\,. \tag{2}$$

We will use the fact that the topological degree is stable under homotopy. That is for every continuous mapping $H\colon\overline{\Omega}\times[0,1]\to\mathbb{R}^n$ and $y_\circ\in\mathbb{R}^n$ such that $y_\circ \notin H(\partial\Omega, t)$ for all $t \in [0,1]$ we have

$$\deg(H(\cdot,0),\Omega,y_\circ) = \deg(H(\cdot,1),\Omega,y_\circ)\,. \tag{3}$$

2.2. **Differentiability of Sobolev mappings.** A Sobolev homeomorphism $f \in W^{1,p}(\Omega,\mathbb{R}^n)$ is differentiable almost everywhere if $p > n-1$, $n \geqslant 3$, and $p \geqslant 1$ for $n = 2$, see [9, 19, 25]. We will also need a generalization of the concept of differentiability, which is obtained by replacing the ordinary limit by an approximate limit, see e.g. [8, §6.1.3]. It is known that a Sobolev mapping $f \in W^{1,1}_{\text{loc}}(\Omega,\mathbb{R}^n)$ is *approximatively differentiable* almost everywhere, see e.g. [8, 6.1.2, Theorem 2]. Moreover, such a mapping is L^1*-differentiable* almost everywhere [26]; that is, for almost every $x_\circ \in \Omega$ we have

$$\lim_{r\to 0} \fint_{B(x_\circ,r)} \Big| \frac{f(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)}{r} \Big| \,\mathrm{d}x = 0\,. \tag{4}$$

Hereafter, the notation $\fint_{B(x_\circ,r)}$ means the integral average over the n-dimensional ball $B(x_\circ,r) = \{x \in \mathbb{R}^n \colon |x - x_\circ| < r\}$.

In order to illustrate our ideas, we first prove Theorem 1 in the cases $n = 2$ and $p > n-1$.

3. Proof of Theorem 1 for $p > n-1$, $n \geqslant 3$, and $p \geqslant 1$, $n = 2$

Each homeomorphism f_j is either sense-preserving or sense-reversing. Under our assumptions there exists a point x_j such that f_j is differentiable at x_j, see Subsection 2.2, and $J(x_j, f_j) > 0$. By (2) we know that the degree of f_j is one and hence each f_j is sense-preserving.

We fix $\varepsilon > 0$ and for $p > n-1$ we set $\delta = \varepsilon$. For $n = 2$ and $p = 1$ we have $Df_j \to Df$ weakly in L^1 and hence the sequence Df_j is equi-integrable, see e.g. [6, page 19]. Therefore, we may, and we do, choose $0 < \delta < \varepsilon$ such that

$$\text{for all } j \text{ and every } A \subset \Omega \text{ with } |A| < 5^4\delta|\Omega| \text{ we have } \int_A |Df_j| < \varepsilon\,. \tag{5}$$

Without loss of generality we may assume that Ω is bounded and has Lipschitz boundary. For the contrary we suppose that the set

$$\Omega_1 := \{x_\circ \in \Omega \colon J(x_\circ, f) < 0 \text{ and } f \text{ satisfies (4) at } x_\circ\}$$

has positive measure. Dividing the set Ω_1 into countable many pieces we find a matrix M, a radius $r > 0$ and a set

$$\Omega_\circ = \Big\{ x \in \Omega_1 : |Df(x) - M| < \tfrac{1}{10}|M|,\ \operatorname{dist}(x,\partial\Omega) > r \text{ and } \fint_{B(x_\circ,r)} \Big| \frac{f(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)}{r} \Big| \,\mathrm{d}x < \frac{\delta^2}{2} \Big\}$$

with positive measure. Without loss of generality we may and do assume that

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix}. \tag{6}$$

We fix a point $x_\circ \in \Omega_\circ$. Since the sequence of mappings f_j converges to f weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, we have $f_j \to f$ strongly in $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Now, we may pick up an index $j_\circ$ large enough such that

$$\int_\Omega |f(x) - f_{j_\circ}(x)| \, \mathrm{d}x < r^{n+1} \frac{\delta^2}{2} \, .$$

This and the definition of $\Omega_\circ$ imply that

$$\fint_{B(x_\circ, r)} \Big| \frac{f_{j_\circ}(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)}{r} \Big| \, \mathrm{d}x < \delta^2 \, . \tag{7}$$

Our goal is to prove that

(i) if $p > n-1$, then there exists a constant C (depending only on p and n) such that

$$\delta^{n-1-p}\, r^n \leqslant C \int_{B(x_\circ, r)} |Df_{j_\circ}|^p \qquad \text{and}$$

(ii) if $n = 2$ and $p = 1$, there exist a constant C and a set $A \subset B(x_\circ, r)$ such that

$$|A| < 5^4 \delta |B(x_\circ, r)| \qquad \text{and} \qquad r^2 \leqslant C \int_A |Df_{j_\circ}| \, .$$

These would lead to a desired contradiction. Indeed, having (i) on our hands, by the Vitali covering theorem, we find a collection of pairwise disjoint balls B_i such that $\Omega_\circ \subset \cup_i 5B_i$ and

$$|\Omega_\circ| \leqslant 5^n \sum_i |B_i| \leqslant \delta^{p-n+1}\, C \int_\Omega |Df_\circ|^p \, ,$$

which is impossible because $p > n-1$ and $\delta = \varepsilon$ is arbitrary. If (ii) holds, then we obtain in the same way $A_i \subset B_i$ such that $|A_i| < 5^4 \delta |B_i|$ and

$$|\Omega_\circ| \leqslant 5^2 \sum_i |B_i| \leqslant C \int_{\cup_i A_i} |Df|$$

which contradicts with (5).

Proof of (i). We simplify the notation and write

$$\varphi_j(x) = |f_j(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)| \quad \text{and} \quad B_s = B(x_\circ, s) \, .$$

It follows from (7) that the set of radii

$$I_G = \big\{ s \in [0, r] \colon \mathcal{H}^{n-1}(\{x \in \partial B_s \colon \varphi_{j_\circ}(x) \geqslant \delta\, r\}) < 5^n \delta\, \mathcal{H}^{n-1}(\partial B_s) \big\}$$

has measure at least $\frac{3r}{4}$ i.e. $|I_G| \geqslant \frac{3r}{4}$. Hereafter, the notation $\mathcal{H}^k(A)$ stands for the k-dimensional Hausdorff measure of the set A.

On the other hand, the key point in our argument is that for $x_\circ \in \Omega_\circ$ and for every $s \in (0, r)$ we can find $\beta = \beta(s) \in \partial B(x_\circ, s)$ such that

$$\varphi_j(\beta) \geqslant \frac{4}{5}\, s \qquad \text{for every } j = 1, 2, \dots . \tag{8}$$

Finding such a point β is the only place where we use the homeomorphism assumption of f_j. Suppose on the contrary that (8) fails for every $\beta \in \partial B(x_\circ, s)$ and for some $j \in \{1, 2, \dots\}$. For $x \in \partial B(x_\circ, s)$ and $t \in [0, 1]$ we consider the following homotopy

$$H(x, t) := (1-t)\big(f_j(x) - f(x_\circ)\big) + t Df(x_\circ)(x - x_\circ)\,.$$

Since $x_\circ \in \Omega_\circ$ and M is given by (6), we have $\inf_{|z|=1}|Df(x_\circ)\, z| > 4/5$. Furthermore, if (8) does not hold, then for all $x \in \partial B(x_\circ, s)$ we have

$$\begin{aligned} |H(x,t)| &\geqslant |Df(x_0)(x - x_0)| - (1-t)|f_j(x) - f(x_0) - Df(x_0)(x - x_0)| \\ &> \tfrac{4}{5}s - (1-t)\tfrac{4}{5}s \geqslant 0\ . \end{aligned}$$

It follows that $H(x, t) \neq 0$ for every $x \in \partial B(x_\circ, s)$ and all $t \in [0, 1]$. Thus, by (3) and (1), the degree of f_j at $f(x_\circ)$ equals $\operatorname{sgn}\det(Df(x_\circ)) = -1$. This contradicts with the fact that f_j is sense-preserving.

According to the Sobolev embedding theorem [12, Lemma 7.4.2] for almost every $s \in (0, r_\circ)$ and for all $z_1, z_2 \in \partial B(x_\circ, s)$ we have

$$|f_{j_\circ}(z_1) - f_{j_\circ}(z_2)| \leqslant C(n,p)\,\big(\operatorname{dist}_{\partial B_s}(z_1, z_2)\big)^{1-\frac{n-1}{p}} \Big(\int_{\partial B_s} |Df_{j_\circ}|^p\Big)^{\frac{1}{p}} \tag{9}$$

where $\operatorname{dist}_{\partial B_s}(z_1, z_2)$ stands for the distance between z_1 and z_2 along the sphere $\partial B_s = \partial B(x_\circ, s)$.

Now let us fix $s \in I_G$ so that (9) is satisfied on the sphere ∂B_s. Since $s \in I_G$ we find $\alpha = \alpha(s) \in \partial B_s$ satisfying

$$\varphi_{j_\circ}(\alpha) < \delta\, r \qquad \text{and} \qquad \operatorname{dist}_{\partial B_s}(\alpha, \beta) \leqslant 5^{n+2}\,\delta\, s\,. \tag{10}$$

Combining this with (8) we have found $\alpha, \beta \in \partial B_s$ such that

$$\tfrac{4}{5}\, s - \delta r - 5^{n+3}\delta s \leqslant |\varphi_j(\beta)| - |\varphi_j(\alpha)| - 2\operatorname{dist}_{\partial B_s}(\alpha, \beta) \leqslant |f_{j_\circ}(\alpha) - f_{j_\circ}(\beta)|\,.$$

This together with (9) implies that for $s \in I_G \cap [r/2, r]$ and δ small enough

$$Cs^p \leqslant \big(\tfrac{4}{5}s - \delta r - 5^{n+3}\delta s\big)^p \leqslant C(n,p)(\delta\, s)^{p-n+1} \int_{\partial B_s} |Df_{j_\circ}|^p\,. \tag{11}$$

Integrating the inequality (11) over the set $I_G \cap [\frac{r}{2}, r]$ we obtain (i), finishing the proof of Theorem 1 in the case $p > n - 1$.

Proof of (ii). We proceed as above. For $s \in I_G$ we obtain $\alpha = \alpha(s) \in \partial B_s$ and $\beta = \beta(s) \in \partial B_s$ so that (8) and (10) hold. We may choose these points the way that the functions $s \to \alpha(s)$ and $s \to \beta(s)$ are measurable on I_G. There are two arcs on ∂B_s with endpoints α and β. We denote the shorter one by $\mathcal{I}_s$. Now, for almost every $s \in I_G \cap [\frac{r}{2}, r]$ we have

$$s \leqslant C \int_{\mathcal{I}_s} |Df_{j_\circ}|\,.$$

Integrating this over the $I_G \cap [\frac{r}{2}, r]$ we obtain

$$r^2 \leqslant C \int_A |Df_{j_\circ}|$$

where

$$A = \{z \in \mathcal{I}_s \colon s \in I_G \cap [\tfrac{r}{2}, r]\} \quad \text{and} \quad |A| \leqslant 5^4 \delta |B_r|\,,$$

finishing the proof of (ii). □

The above proof was based on Sobolev embedding theorem on spheres and therefore does not work for $p < n-1$. To overcome these difficulties we follow Hencl and Malý [11] and use the theory of linking numbers and its topological invariance. For the convenience of the reader we recall the needed properties of linking numbers here.

4. Linking number

Let n, t, q be positive integers with $t + q = n - 1$. Let us consider the mapping $\Phi(\xi, \eta) \colon \overline{\mathbb{B}}_{t+1} \times \overline{\mathbb{B}}_{q+1} \to \mathbb{R}^n$ defined coordinatewise as $\Phi(\xi, \eta) = x$, where

$$\begin{aligned} x_1 &= (2 + \eta_1)\xi_1, \\ &\dots \\ x_{t+1} &= (2 + \eta_1)\xi_{p+1}, \\ x_{t+2} &= \eta_2, \\ &\dots \\ x_{t+q+1} &= \eta_{q+1}. \end{aligned}$$

Denote by $\mathbb{A}$ the anuloid

$$\Phi(\mathbb{S}_t \times \mathbb{B}_{q+1}) = \Big\{x \in \mathbb{R}^n : \big(\sqrt{x_1^2 + \ldots + x_{t+1}^2} - 2\big)^2 + x_{t+2}^2 + \ldots + x_n^2 < 1\Big\}.$$

Of course given $x \in \overline{\mathbb{A}}$ we can find a unique $\xi \in \mathbb{S}_t$ and $\eta \in \overline{\mathbb{B}}_{q+1}$ such that $\Phi(\xi, \eta) = x$. We will denote these as $\xi(x)$ and $\eta(x)$.

A *link* is a pair (φ, ψ) of parametrized surfaces $\varphi \colon \mathbb{S}_t \to \mathbb{R}^n$, $\psi \colon \mathbb{S}_q \to \mathbb{R}^n$. The *linking number* of the link (φ, ψ) is defined as the topological degree

$$\pounds(\varphi, \psi) = \deg(L, \mathbb{A}, 0),$$

where the mapping $L = L_{\varphi,\psi} : \overline{\mathbb{A}} \to \mathbb{R}^n$ is defined as

$$L(x) = \varphi(\xi(x)) - \bar{\psi}(-\eta(x)) \text{ or equivalently}$$

$$L(\Phi(\xi, \eta)) = \varphi(\xi) - \bar{\psi}(-\eta), \qquad \xi \in \mathbb{S}_t,\ \eta \in \mathbb{B}_{q+1},$$

where $\bar{\psi}$ is an arbitrary continuous extension of ψ to $\overline{\mathbb{B}}_{q+1}$ (of course, the degree does not depend on the way how we extend ψ, it depends only on the values on the boundary $\partial\mathbb{A} = \Phi(\mathbb{S}_t \times \mathbb{S}_q)$). Geometrically speaking, for $t = q = 1$, the linking number is the number of loops of a curve φ around a curve ψ counting orientation into account as $+1$ or -1. For the introductions to the linking number in $\mathbb{R}^3$ and its application to the theory of knots see [24].

The *canonical link* is the pair (μ, ν), where

$$\begin{aligned} \mu(\xi) &= \Phi(\xi, 0), \qquad && \xi \in \mathbb{S}_t, \\ \nu(\eta) &= \Phi(\mathbf{e}_1, \eta), \qquad && \eta \in \mathbb{S}_q. \end{aligned} \tag{12}$$

For example in dimension $n = 3$ we get that

$$\begin{aligned} &\mu(\mathbb{S}_1) = \{x \in \mathbb{R}^3 : \ x_3 = 0,\ x_1^2 + x_2^2 = 4\} \text{ and} \\ &\nu(\mathbb{S}_1) = \{x_2 = 0,\ (x_1 - 2)^2 + x_3^2 = 1\}. \end{aligned}$$

It is well known, that the linking number is a topological invariant. The simple proof of the following proposition can be found in [11].

Proposition 2. *Let n, t, q be positive integers with $t + q = n - 1$. Let $f : \mathbb{B}_n(4) \to \mathbb{R}^n$ be a homeomorphism. Then $\pounds(f \circ \mu, f \circ \nu)$ is 1 if f is sense preserving and -1 if f is sense reversing.*

Analogously we can pick $a \in \overline{\mathbb{B}}_{q+1}(0, \frac{1}{10})$ and $b \in \overline{\mathbb{B}}_{t+1}(\mathbf{e}_1, \frac{1}{10}) \cap \overline{\mathbb{B}}_{t+1}$ and consider the pair

$$\begin{aligned} \mu_a(\xi) &= \Phi(\xi, a), \qquad && \xi \in \mathbb{S}_t, \\ \nu_b(\eta) &= \Phi(b, \eta), \qquad && \eta \in \mathbb{S}_q. \end{aligned} \tag{13}$$

Similarly to the previous proposition we have.

Proposition 3. *Let n, t, q be positive integers with $t + q = n - 1$, $a \in \overline{\mathbb{B}}_{q+1}(0, \frac{1}{10})$ and $b \in \overline{\mathbb{B}}_{t+1}(\mathbf{e}_1, \frac{1}{10}) \cap \overline{\mathbb{B}}_{t+1}$. Let $f : \mathbb{B}_n(4) \to \mathbb{R}^n$ be a homeomorphism. Then $\pounds(f \circ \mu_a, f \circ \nu_b)$ is 1 if f is sense preserving and -1 if f is sense reversing.*

5. Proof of Theorem 1 for $p > [\frac{n}{2}]$, $n \geqslant 3$, and $p = 1$, $n = 3$

Our argument is similar to the proof given in Section 3 and therefore some details are only sketched.

By C_1 and C_2 we denote a fixed constants whose exact value will be determined later. We fix $\varepsilon > 0$ and for $p > [\frac{n}{2}]$ we set $\delta = \varepsilon$. For $n = 3$ and $p = 1$ we choose $0 < \delta < \varepsilon$ such that (5) holds with constant C_2 instead of 5^4. For the contrary we again suppose that the set

$$\Omega_1 := \{x_\circ \in \Omega \colon J(x_\circ, f) < 0 \text{ and (4) holds for } f \text{ at } x_\circ\}$$

and the set

$$\Omega_\circ = \Big\{x \in \Omega_1 : |Df(x) - M| < \tfrac{1}{10}|M|, \operatorname{dist}(x, \partial\Omega) > r \text{ and} \\ \fint_{B(x_\circ, 4r)} \Big| \frac{f(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)}{r} \Big| \, \mathrm{d}x < C_1 \frac{\delta^{[\frac{n}{2}]+1}}{2} \Big\}$$

have positive measure, and M is given by (6).

We fix $\varepsilon > 0$ and a point $x_\circ \in \Omega_\circ$. Again we can find $j_\circ$ such that

$$\fint_{B(x_\circ,4r)} \Big| \frac{f_{j_\circ}(x) - f(x_\circ) - Df(x_\circ)(x - x_\circ)}{r} \Big| \, \mathrm{d}x < C_1 \delta^{[\frac{n}{2}]+1} . \tag{14}$$

We fix $t, q \geqslant [\frac{n}{2}]$ such that $t + q = n - 1$ (i.e. $t = q = \frac{n-1}{2}$ for n odd and $t = \frac{n-2}{2}$, $q = \frac{n}{2}$ for n even). Our goal is to prove that

(i) if $p > [\frac{n}{2}]$ and $n \geqslant 3$, then there exists a constant C (depending only on p and n) such that

$$\delta^{\min\{t,q\}-p} \, r^n \leqslant C \int_{B(x_\circ,4r)} |Df_{j_\circ}|^p \qquad \text{and}$$

(ii) if $p = 1$ and $n = 3$ we have $A \subset B(x_\circ, 4r)$ such that

$$|A| < C_2 \delta |B(x_\circ, 4r)| \quad \text{ and } \quad r^3 \leqslant C \int_A |Df_{j_\circ}| \ .$$

Having these and using $p > [\frac{n}{2}]$ or (5), it is again a simple application of the Vitali covering theorem to obtain a desired contradiction.

Proof of (i). Without loss of generality we will assume that $x_\circ = 0$. We write

$$\varphi_j(x) = |f_j(rx) - f(x_\circ) - Df(x_\circ)rx| \ .$$

Let us fix $y \in \mu_a(\mathbb{S}_t)$ and denote

$$B_{\mu_a(\mathbb{S}_t)}(y, \delta) = \{x \in \mu_a(\mathbb{S}_t) : \mathrm{dist}_{\mu_a(\mathbb{S}_t)}(x, y) < \delta\}$$

the ball of radius δ on the link $\mu_a(\mathbb{S}_t)$. We can clearly choose a constant C_1 big enough at the beginning of the proof so that (14) implies that the set of good links

$$I_A = \big\{a \in \overline{\mathbb{B}}_{q+1}(0, \tfrac{1}{10}) : \ \mathcal{H}^t(x \in \mu_a(\mathbb{S}_t) : \varphi_{j_\circ}(x) \geqslant \delta) < \mathcal{H}^t(B_{\mu_a(\mathbb{S}_t)}(y, \delta))\big\} \text{ and}$$
$$I_B = \big\{b \in \overline{\mathbb{B}}_{t+1}(\mathbf{e}_1, \tfrac{1}{10}) \cap \overline{\mathbb{B}}_{t+1} : \ \mathcal{H}^q(x \in \nu_b(\mathbb{S}_q) : \varphi_{j_\circ}(x) \geqslant \delta) < \mathcal{H}^q(B_{\nu_b(\mathbb{S}_q)}(y, \delta))\big\}$$

has measure at least

$$\mathcal{H}^{q+1}(I_A) > \frac{1}{2} |\mathbb{B}_{q+1}(0, \tfrac{1}{10})| \text{ and } \mathcal{H}^{t+1}(I_B) > \frac{1}{2} |\mathbb{B}_{t+1}(\mathbf{e}_1, \tfrac{1}{10}) \cap \mathbb{B}_{t+1}| \ .$$

The key point of our argument is that for every $a \in \overline{\mathbb{B}}_{q+1}(0, \frac{1}{10})$ and every $b \in \overline{\mathbb{B}}_{t+1}(\mathbf{e}_1, \frac{1}{10}) \cap \overline{\mathbb{B}}_{t+1}$ we can find $\xi \in \mathbb{S}_t$ and $\eta \in \mathbb{S}_q$ such that

$$\begin{aligned} \varphi_j(\mu_a(\xi)) &= \big|f_j(r\mu_a(\xi)) - f(x_\circ) - Df(x_\circ)r\mu_a(\xi)\big| > \frac{r}{10} \text{ or} \\ \varphi_j(\nu_b(\eta)) &= \big|f_j(r\nu_b(\eta)) - f(x_\circ) - Df(x_\circ)r\nu_b(\eta)\big| > \frac{r}{10} \ . \end{aligned} \tag{15}$$

We prove the observation by contradiction and we suppose that (15) does not hold. We define

$$f_s(x) = (1 - s)(f(x_\circ) + Df(x_\circ)rx) + sf_j(rx)$$

and we consider the homotopy $H(\overline{\mathbb{A}} \times [0,1]) \to \mathbb{R}^n$ defined as

$$H(\Phi(\xi,\eta),s) = (f_s \circ \mu_a)(\xi) - \overline{(f_s \circ \nu_b)}(-\eta) \ ,$$

where $\overline{(f_s \circ \nu_b)}$ denotes a continuous extension of $f_s \circ \nu_b$ to $\overline{\mathbb{B}}_{q+1}$ as in the definition of the linking number. From [11] we know that the mapping $f_j \in W^{1,p}$, $p > [\frac{n}{2}]$, with nonnegative and nonzero Jacobian is sense preserving. By Proposition 3 we get that

$$\deg(H(x,1),\mathbb{A},0) = 1 \ .$$

On the other hand

$$\deg(H(x,0),\mathbb{A},0) = -1$$

since the linear mapping $f(x_\circ) + Df(x_\circ)rx$ is sense reversing. To obtain a contradiction (with the preservation of the degree under homotopy) it is now enough to show that for every $\xi \in \mathbb{S}_t$, for every $\eta \in \mathbb{S}_q$ and for every $s \in [0,1]$ we have $H(\Phi(\xi,\eta),s) \neq 0$. It is easy to see that

$$\operatorname{dist}((f_0 \circ \mu_a)(\mathbb{S}_t),(f_0 \circ \nu_b)(\mathbb{S}_q)) \geqslant \operatorname{dist}((f_0 \circ \mu)(\mathbb{S}_t),(f_0 \circ \nu)(\mathbb{S}_q)) - \frac{6r}{10} \geq \frac{3r}{10} \ .$$

Since (15) does not hold we obtain from the definition of f_t that

$$\operatorname{dist}((f_s \circ \mu_a)(\mathbb{S}_t),(f_s \circ \nu_b)(\mathbb{S}_q)) \geqslant \frac{3r}{10} - \frac{r}{10} - \frac{r}{10}$$

which implies $H(\Phi(\xi,\eta),s) \neq 0$.

By (15) and the symmetry we may assume without loss of generality that

$$\tilde{I}_A = \bigl\{a \in I_A : \ \exists \xi \in \mathbb{S}_t, \varphi_j(\mu_a(\xi)) > \tfrac{r}{10}\bigr\}$$

satisfies $\mathcal{H}^{q+1}(\tilde{I}_A) > \frac{1}{4}|\mathbb{B}_{q+1}(0,\frac{1}{10})|$. Since $p > [\frac{n}{2}] \geq t$ we can use the Sobolev embedding theorem on the t-dimensional space $r\mu_a(\mathbb{S}_t)$ and we have for almost every $a \in \tilde{I}_A$ and for all $z_1, z_2 \in r\mu_a(\mathbb{S}_t)$

$$|f_{j_\circ}(z_1) - f_{j_\circ}(z_2)| \leqslant C\bigl(\operatorname{dist}_{r\mu_a(\mathbb{S}_t)}(z_1,z_2)\bigr)^{1-\frac{t}{p}}\Bigl(\int_{r\mu_a(\mathbb{S}_t)} |Df_{j_\circ}|^p\Bigr)^{\frac{1}{p}} \tag{16}$$

where $\operatorname{dist}_{r\mu_a(\mathbb{S}_t)}(z_1,z_2)$ stands for the distance between z_1 and z_2 along the t-dimensional sphere $r\mu_a(\mathbb{S}_t)$.

Now let us fix $a \in \tilde{I}_A$ so that (16) is satisfied and find $\xi \in \mathbb{S}_t$ so that for $\beta = \mu_a(\xi)$ we have $\varphi_{j_\circ}(\beta) > \frac{r}{10}$ as in the definition of $\tilde{I}_A$. Using $a \in I_A$ we find $\alpha \in \mu_a(\mathbb{S}_t)$ satisfying

$$\varphi_{j_\circ}(\alpha) < \delta \qquad \text{and} \qquad \operatorname{dist}_{\mu_a(\mathbb{S}_t)}(\alpha,\beta) \leqslant \delta\,. \tag{17}$$

Thus we have found $\alpha, \beta \in \mu_a(\mathbb{S}_t)$ such that

$$\tfrac{r}{10} - 3\delta \leqslant |\varphi_{j_\circ}(\beta)| - |\varphi_{j_\circ}(\alpha)| - 2\operatorname{dist}_{\mu_a(\mathbb{S}_t)}(\alpha,\beta) \leqslant |f_{j_\circ}(r\alpha) - f_{j_\circ}(r\beta)|\,.$$

This together with (16) implies that for almost every $a \in \tilde{I}_A$ and δ small enough we have

$$C \leqslant C\delta^{p-t}\int_{r\mu_a(\mathbb{S}_t)} |Df_{j_\circ}|^p\,. \tag{18}$$

Integrating the inequality (18) over the set $\tilde{I}_A$ we obtain (i).

Proof of (ii). If $p = 1$ and $n = 3$, then in (16) and (18) instead of integrating over the entire $r\mu_a(\mathbb{S}_t)$ we integrate only over the set $B_{r\mu_a(\mathbb{S}_t)}(r\alpha, r\delta)$. Integration over the set $\tilde{I}_A$ leads to a set A where (ii) holds with some absolute constant C_2. $\square$

References

1. S. S. Antman, *Nonlinear problems of elasticity. Applied Mathematical Sciences*, 107. Springer-Verlag, New York, 1995.
2. K. Astala, T. Iwaniec, and G. Martin, *Deformations of annuli with smallest mean distortion*, Arch. Ration. Mech. Anal. **195** (2010), no. 3, 899–921.
3. K. Astala, T. Iwaniec, and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton University Press, Princeton, NJ, 2009.
4. J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1976/77), no. 4, 337–403.
5. P. G. Ciarlet, *Mathematical elasticity Vol. I. Three-dimensional elasticity*, Studies in Mathematics and its Applications, 20. North-Holland Publishing Co., Amsterdam, 1988.
6. B. Dacorogna, *Direct methods in the Calculus of Variations*, Springer-Verlag, Berlin, 1989.
7. A. Dold, *Lectures on algebraic topology*, Springer-Verlag, New York, 1980.
8. L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, Boca Raton, 1992.
9. F. W. Gehring and O. Lehto, *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Ser. A. I. Math., **272** (1959), 1–9.
10. S. Hencl, *Sobolev homeomorphism with zero Jacobian almost everywhere*, J. Math. Pures Appl. (9) **95** (2011), no. 4, 444–458.
11. S. Hencl and J. Malý, *Jacobians of Sobolev homeomorphisms*, Calc. Var. Partial Differential Equations **38** (2010), 233-242.
12. T. Iwaniec and G. Martin, *Geometric Function Theory and Non-linear Analysis*, Oxford Mathematical Monographs, Oxford University Press, 2001.
13. T. Iwaniec, N-T. Koh, L. V. Kovalev and J. Onninen, *Existence of energy-minimal diffeomorphisms between doubly connected domains*, Invent. Math. **186** (2011), no. 3, 667–707.
14. T. Iwaniec and J. Onninen, *n-harmonic mappings between annuli: the art of integrating free Lagrangians*, Mem. Amer. Math. Soc. 218 (2012).
15. T. Iwaniec and J. Onninen, *Mappings of least Dirichlet energy and their Hopf differentials*, Arch. Ration. Mech. Anal. **209** (2013), no. 2, 401–453.
16. N.G. Lloyd, *Degree theory*, Cambridge University Press, New York, 1978.
17. J. Malý, *Examples of weak minimizers with continuous singularities*, Exposition. Math. **13**, no. 5 (1995) 446–454.
18. J. E. Marsden and T. J. R. Hughes, *Mathematical foundations of elasticity*, Dover Publications, Inc., New York, 1994.
19. D. Menchoff, *Sur les différentielles totales des fonctions univalentes*, Math. Ann. **105** (1931), no. 1, 75–85.
20. T. Rado and P. V. Reichelderfer, *Continuous Transformations in Analysis*, Springer 1955.
21. M. Šilhavý, *The mechanics and thermodynamics of continuous media*, Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.
22. C. Truesdell and W. Noll, *The non-linear field theories of mechanics*, Edited and with a preface by Stuart S. Antman. Springer-Verlag, Berlin, 2004.

23. Yu. G. Reshetnyak, *Space mappings with bounded distortion*, American Mathematical Society, Providence, RI, 1989.
24. D. Rolfsen, *Knots and links*, Berkeley, 1976.
25. J. Väisälä, *Two new characterizations for quasiconformality*, Ann. Acad. Sci. Fenn. Ser. A I Math., **362** (1965), 1–12.
26. W. P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.

Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ 186 00 Prague 8, Czech Republic
E-mail address: hencl@karlin.mff.cuni.cz

Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA
E-mail address: jkonnine@syr.edu