Two remarks on remotality

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Abstract

We prove that there exists a weakly closed and bounded subset E of c_0 which is not remotal from 0, and such that $\overline{\mathrm{co}}\,(E)$ is remotal from 0. This answers a question of M. Martín and T.S.S.R.K. Rao. We also present a simple proof of the fact that in every non-reflexive Banach space there exists a closed convex bounded set which is not remotal.

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Let X be a Banach space (all spaces throughout the paper are considered to be real) and $E \subset X$ be a bounded set. If $x \in X$, we define $D(x, E) := \sup\{\|x - z\| : z \in E\}$. We say that the set E is *remotal* from a point $x \in X$ if there exists a point $e \in E$ such that $\|x - e\| = D(x, E)$. The set E is said to be *remotal* if it is remotal from all $x \in X$.

Consider the following problem: characterize those Banach spaces in which every closed convex bounded set is remotal. Clearly in finite-dimensional spaces every closed bounded set is remotal. M. Sababheh and R. Khalil claimed in [4, Theorem A] that among reflexive spaces, those spaces in which every closed convex bounded set is remotal are precisely the finite-dimensional ones. However, their proof was not entirely correct. Later, T.S.S.R.K. Rao in [3, Theorem 2.3] proved the assertion of [4, Theorem A] by showing that even in every Banach space which fails the Schur property, there exists a closed convex bounded set which is not remotal. M. Martín and T.S.S.R.K. Rao in [2, Theorem 7] then solved the problem completely by showing that in every infinite-dimensional Banach space there exists a closed convex bounded set which is not remotal. Their method was (as well as the method of the previous works [3] and [4]), roughly speaking, the following. First, they proved that if E is a bounded subset of a Banach space, then, under some additional assumptions on the set E, the remotality of $\overline{\mathrm{co}}\,(E)$ from a point $x \in X$ implies the remotality of E from x. Then they constructed an appropriate bounded set E (considering separately the spaces which fail the Schur property, reproving [3, Theorem 2.3], and the others) which is not remotal from 0, and therefore also $\overline{\mathrm{co}}\,(E)$ is not remotal from 0.

In this connection, they asked in [2, Remark 6] whether the remotality of $\overline{\mathrm{co}}\,(E)$ from a point $x \in X$, where E is a weakly closed and bounded subset of a Banach space X, implies the remotality of E from x. Example 1 below answers this question in the negative.

The second purpose of this note is to present an alternative proof of [2, Theorem 7]. To prove that in every non-reflexive Banach space there exists a closed convex bounded set which is not

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remotal, we use a simple construction using James' characterization of reflexivity. The case of reflexive spaces is covered by [3, Theorem 2.3] or by [2, Remark 3].

It should be noted that the statement of [2, Theorem 7] has also been proved by L. Veselý in [5, Remark 2.10].

Let us first summarize some notation. Let X be a Banach space. The topological dual of X is denoted by X^*. The weak closure of a subset E of X is denoted by $\overline{E}^w$, and the weak convergence in X is denoted by $\xrightarrow{w}$. The convex hull and the closed convex hull of a subset E of X are denoted by $\operatorname{co}(E)$ and $\overline{\operatorname{co}}(E)$ respectively. The symbol c_0 stands for the space of all real sequences vanishing at infinity, equipped with the supremum norm. If $x \in c_0$, we write x^k for the k-th coordinate of x.

Example 1. *There exists a weakly closed and bounded subset E of c_0 which is not remotal from 0, and such that $\overline{\operatorname{co}}(E)$ is remotal from 0.*

Construction. Define vectors $x_n \in c_0$, $n \in \mathbb{N}$, as

$$x_n := \left(2 - \tfrac{1}{n}, (-1)^n, (-1)^n, \dots, (-1)^n, 0, 0, \dots\right),$$

where the number of nonzero coordinates of x_n is $n+1$. Now, define $E := \{x_n : n \in \mathbb{N}\}$. Then E is a weakly closed and bounded subset of c_0 which is not remotal from 0, while $\overline{\operatorname{co}}(E)$ is remotal from 0.

Clearly the set E is bounded and not remotal from 0. Let us show that E is weakly closed. Assume for the contradiction that there exists $x \in \overline{E}^w \setminus E$. Let $k \in \mathbb{N}, k \geq 2$. We claim that $x^k \in \{-1, 1\}$. It is clear from the definition of the vectors x_n that there exists $m \in \mathbb{N}$ such that $x_n^k \in \{-1, 1\}$ for each $n > m$. And it is easy to see that $x \in \overline{E \setminus \{x_1, \dots, x_m\}}^w$. Then there exists a net $\{y_\alpha\}$ from $E \setminus \{x_1, \dots, x_m\}$ such that $y_\alpha \xrightarrow{w} x$. Applying a functional $\varphi \in (c_0)^*$ such that $\varphi(z) = z^k, z \in c_0$, we see that $y_\alpha^k \to x^k$. Since $y_\alpha^k \in \{-1, 1\}$ for all α, it follows that $x^k \in \{-1, 1\}$. But this is a contradiction with the fact that $x \in c_0$. Hence E is weakly closed.

Now, let us verify that $\overline{\operatorname{co}}(E)$ is remotal from 0. Clearly $D(0, \overline{\operatorname{co}}(E)) = D(0, E) = 2$ (for the first equality see [4, Lemma 2.1]). Let us show that $(2, 0, 0, \dots) \in \overline{\operatorname{co}}(E)$, which clearly implies the remotality of $\overline{\operatorname{co}}(E)$ from 0. To this end, we will show that if

$$a_n := \sum_{i=1}^{n} \frac{1}{n} x_i \in \operatorname{co}(E),$$

then $a_n \to (2, 0, 0, \dots)$.

First, it is easy to see that if $t_n, t \in \mathbb{R}$ and $t_n \to t$, then also

$$\sum_{i=1}^{n} \frac{1}{n} t_i \xrightarrow{n \to \infty} t.$$

Then

$$a_n^1 = \sum_{i=1}^{n} \frac{1}{n} x_i^1 \to 2,$$

since $x_n^1 = 2 - \frac{1}{n} \to 2$.

Further, let $k \in \mathbb{N}, k \geq 2$. It is clear from the definition of the vectors x_n that

$$\left(x_1^k, x_2^k, x_3^k, \dots\right) = \left(0, \dots, 0, (-1)^{m+1}, (-1)^m, (-1)^{m+1}, (-1)^m, \dots\right),$$

where the number $l \in \mathbb{N} \cup \{0\}$ of zero coordinates of the vector on the right hand side and the number $m \in \{0, 1\}$ depend on k (the precise values of l and m are not important for us). Then

$$\left|a_n^k\right| = \left|\sum_{i=1}^{n} \frac{1}{n} x_i^k\right| \leq \frac{1}{n}.$$

Hence

$$\|a_n - (2, 0, 0, \dots)\| \leq \max\left\{2 - a_n^1, \frac{1}{n}\right\} \to 0,$$

as desired. ■

Let us now present the promised proof of [2, Theorem 7].

Theorem 2. *Let X be an infinite-dimensional Banach space. Then there exists a closed convex bounded subset of X which is not remotal.*

Proof. If X is in addition reflexive, then it fails the Schur property, and therefore we may apply the argument from [2, Remark 3] or follow [3, Theorem 2.3].

Suppose that X is not reflexive. By James' theorem (see [1, p. 12]), there exists $\varphi \in X^*$ such that $\|\varphi\| = 1$ and φ is not norm-attaining, i.e. there exists no $x \in X$ such that $\|x\| \leq 1$ and $\varphi(x) = 1$. Define

$$K := \left\{x \in X : \|x\|^2 \leq \varphi(x)\right\}.$$

Then K is a closed convex bounded set which is not remotal from 0.

The set K is closed, because the functions $\|.\|^2$ and φ are continuous. To prove the convexity of K, let $x, y \in K$ and $\lambda \in [0, 1]$. Then (we use the fact that the function $t \mapsto t^2, t \in \mathbb{R}$, is convex and non-decreasing on $[0, \infty)$)

$$\begin{aligned}\|\lambda x + (1-\lambda)y\|^2 &\leq (\lambda\|x\| + (1-\lambda)\|y\|)^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\ &\leq \lambda\varphi(x) + (1-\lambda)\varphi(y) = \varphi(\lambda x + (1-\lambda)y).\end{aligned}$$

Hence K is convex.

Further, $\sup_{x\in K} \|x\| = 1$. Indeed, if $x \in K$, then $\|x\|^2 \leq \varphi(x) \leq \|\varphi\|\|x\| = \|x\|$, and therefore $\|x\| \leq 1$. On the other hand, if $\varepsilon > 0$, then, since $\|\varphi\| = 1$, there exists $y \in X$ such that $\|y\| = 1$ and $|\varphi(y)| > 1 - \varepsilon$. Let $x := \varphi(y)y$. Then $x \in K$, since $\|x\|^2 = \|\varphi(y)y\|^2 = \varphi(y)^2 = \varphi(\varphi(y)y) = \varphi(x)$, and $\|x\| = |\varphi(y)| > 1 - \varepsilon$.

Finally, let us show that there exists no $x \in K$ such that $\|x\| = 1$. Assume for the contradiction that there exists $x \in K$ such that $\|x\| = 1$. Then $1 = \|x\|^2 \leq \varphi(x) \leq \|\varphi\|\|x\| = 1$. Hence $\varphi(x) = 1$, a contradiction with the fact that φ is not norm-attaining. □

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