Lattice-ordered groups of real continuous functions

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Abstract

We provide an internal characterization of the sets $C(X)$ of continuous real-valued functions on topological spaces X as real l-groups.

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1. Introduction

In spite of the numerous contributions to characterizing the set of real continuous functions on a certain topological space, and while this theory has undergone a considerable development in the framework of linear spaces, a very little effort has been made to extend these results at less restrictive levels.

Birkhoff [3, problem 81] and Kaplansky [13] proposed independently in 1948 the problem of characterizing the lattice $C(K)$ for K a compact Hausdorff space, which is popularly known as the problem 81 of Birkhoff since it belongs to a list of open problems proposed in his fundamental text *Lattice Theory.* That problem was solved (see [1], [8], [18]).

The problem for noncompact spaces is still open, if we do not consider the result by Jennsen [10] – her characterization is not an inner one since bigger sets of maps are involved. There are some solutions for vector lattices or lattice-ordered algebras (see [16] and [17]) inspired in the seminal works of Kakutani [12], Kreins [14], and Henriksen and Johnson [9] for compact spaces.

For uniform spaces a kind of approximation results for lattices $U(X)$ were proved, e.g., by Fenstad [6] and Mokhova [15]. Császár and Czipszer [4] generalized in 1963 the Kakutani-Stone theorem for those sublattices of $C(X)$, X compact, that contain the constant functions and are closed under subtraction. If a subset of some $C(X)$ contains the constant functions and is closed under

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subtractions, it is a subgroup of $C(X)$. So, the mentioned result by Császár and Czipszer deals with lattice-ordered groups containing the reals numbers, shortly real l-groups.

In this paper we provide an internal characterization of $C(X)$ for a completely regular Hausdorff space X as a real l-group.

Such a task has two steps. The first one consists of finding a convenient space X for a given real l-group L and conditions under which L embeds into $C(X)$. In the second step we must find conditions under which $L = C(X)$. We shall use a kind of ideals and their systems that allow us to describe separation of closed sets in X.

In the second step one must approximate every continuous function on a given space by functions from L and to ensure that L is closed under some limits. We cannot use known l-group versions of Stone-Weierstrass theorem for lattices since they use conditions that would be difficult to transform into inner conditions (see Fenstad [6] and Mokhova [15]). We found other (inner) conditions for lattices of functions (a kind of completeness) that allow separation of disjoint zero-sets and existence of special suprema sufficient for generating all continuous functions on the given set.

2. Real l-groups and semisimplicity

Although the concept of l-group is defined for general groups, we shall use it for Abelian groups only (so we use addition $+$ for the group operation). Readers may find some details of the next explanation in various books on l-groups, e.g. in [2] or [7].

Definition 2.1. *A lattice L that is also a commutative group is called* `l-group` *if $f + g \leq f + h$ whenever $f, g, h \in L$ and $g \leq h$.*

In l-groups, one can define positive and negative parts of their elements ($f^+ = f \vee 0, f^- = (-f) \vee 0$) and the absolute value $|f| = f^+ + f^- = f^+ \vee f^-$. There is a real l-subgroup L^* of L consisting of bounded elements $f \in L$ defined by $|f| \leq r$ for some $r \in \mathbb{R}$.

Clearly, $C(X)$ (for any topological space X) is an l-group under its operations $+, -, \sup, \inf$ defined pointwise. It has one more important feature, namely it contains an l-subgroup of constant functions that is isomorphic to $\mathbb{R}$ (as an l-group). Most of theorems asserting that $C(X)$ is approximated by its subset S assumes that S contains all (or many) of those constant functions. For simplicity we shall assume that our l-group contains all of them:

Definition 2.2. *An l-group L is said to be a* `real l-group` *provided there is an injective l-group morphism $i : \mathbb{R} \to L$.*

In the sequel, L will be a fixed real l-group; we shall not make distinction between $\mathbb{R}$ and $i(\mathbb{R})$. Morphisms between our objects of real l-groups are l-group homomorphisms that are fixed on reals. The set $\mathrm{Hom}(L, \mathbb{R})$ of all morphisms will be denoted by X_L. Every $f \in L$ is a mapping $X_L \to \mathbb{R}$ by the rule $f(x) = x(f)$.

That assignment defines a morphism $e_L : L \to F(X_L)$ of L into the real l-group $F(X)$ of all real-valued functions on X_L. If we take the weak topology on X_L generated by $e_L(L)$, then the image $e_L(L)$ is a part of the real l-group $C(X_L)$ of all real-valued continuous functions on X_L. This assignment $e_L : L \to C(X_L)$ is known as a *spectral representation* and X_L is called *spectrum* of L. Our task is to find conditions on L so that e_L becomes a bijection. In this section we shall find conditions under which e_L is an injection.

Morphisms on L are in 1-1 correspondence with certain ideals in L. So, to find convenient conditions for morphisms $L \to \mathbb{R}$ (to ensure injectivity of e_L), we can try to find conditions on ideals in L. Recall that an l-subgroup M of L is said to be an *l-ideal* if it is convex ($f \in M$ provided $|f| \leq |g|$ for some $g \in M$). The l-ideals are precisely kernels of homomorphisms.

An example of l-ideal is L^* containing $\mathbb{R}$. The corresponding homomorphism is not real. To exclude such a situation we must assume ideals to meet $\mathbb{R}$ in 0 only. Moreover, the factor group must be linearly ordered, so the ideals M should be prime (i.e., if $f \wedge g = 0$ then either $f \in M$ or $g \in M$). Together with the previous condition we may assume that M is a maximal l-ideal not containing 1 (which is, for maximal l-ideals, equivalent to $M \cap \mathbb{R} = \{0\}$). Such l-ideals are called values of 1, or regular ideals, and are prime.

For prime l-ideals meeting $\mathbb{R}$ in 0, the corresponding homomorphic images are linearly ordered l-groups but need not be isomorphic to $\mathbb{R}$. To ensure the image to be $\mathbb{R}$, one must add, e.g., that the image is archimedean (thus for every positive $f, g \in L$ there is $n \in \mathbb{N}$ and $f \in M$ such that $g \leq nh + f$). We shall use a simpler condition.

Definition 2.3. *An l-ideal M in L will be called* real ideal *if*

(i) $M \cap \mathbb{R} = \{0\}$*;*
(ii) for every $f \in L$ there exists $r \in \mathbb{R}$ such that $f + r \in M$.

The condition (i) follows from a modified condition (ii) if we suppose there that the number r is unique (for given f).

Observe that every real ideal is a maximal l-ideal not containing 1. Indeed, let M be a real ideal and $N \supset M$ be an l-ideal not containing 1. By (ii) there exists $r \in \mathbb{R}$ such that $f + rM \subset N$. Since $-f \in N$, we have $r = (-f) + (f + r) \in N$, which means $r = 0$ and thus $f \in M$.

Since the kernel of a morphism $L \to \mathbb{R}$ is, clearly, a real ideal, we have the following result.

Proposition 2.4. *$M \subset L$ is a real ideal iff it is the kernel of a morphism of L onto $\mathbb{R}$.*

Another bad situation occurs, e.g., in the real l-group $\mathbb{R}^2$ with the real part $\mathbb{R} = \{(x, 0); x \in \mathbb{R}\}$, where the only nontrivial real ideal is $\{0\} \times \mathbb{R}$, which is not enough. To solve this situation we must add a condition ensuring that there is enough ideals.

Definition 2.5. *L is called* `semisimple` *if the intersection of all the real ideals of L is* 0.

Semisimplicity means that morphisms into $\mathbb{R}$ distinguish elements of L. Because of previous Proposition, we have the following result.

Theorem 2.6. *The map $e_L : L \to C(X_L)$ is an injection iff L is semisimple.*

We shall now look closer at the topology of X_L. By the definition of the weak topology (the coarsest topology on X_L making every $f \in L$ continuous), a set $U \subset X_L$ is a neighborhood of its point x iff there are $f_1, ..., f_n \in L$ and intervals $G_i = (f_i(x) - r, f_i(x) + r)$ in $\mathbb{R}$ such that $\bigcap f_i^{-1}(G_i) \subset U$. By shifting each f_i by $-f_i(x)$ we may assume that $f_i(x) = 0$ for each i. Taking sufficiently large $n \in \mathbb{N}$ and defining $f = \sum n|f_i|$ we get $f \in L$ such that $f \geq 0, f(x) = 0, f(y) \geq s$ for every $y \in X_L \setminus U$, where s is any given positive real number. Using $g = f \wedge s$ we can get the values on $X_L \setminus U$ to be exactly s. If one takes $h = s - g$ then the set $\mathrm{coz}(h)$ (where $\mathrm{coz}(f) = \{x \in X_L; x(h) \neq 0\}$) is an open set containing x and contained in U. Consequently, the sets $\{\mathrm{coz}(f) : f \in L\}$ form an open base for the topology of X_L, and their complements (zero-sets $\mathrm{zero}(f)$) form a closed base.

If one regards X as the set of real ideals, then a closed base is formed by the sets corresponding to $\mathrm{zero}(f)$, which are all the real ideals containing f. That topology is often called a *hull-kernel topology*.

Proposition 2.7. *The space X_L is a realcompact space. Conversely, if X is a realcompact space, then $X_{C(X)} = X$.*

Proof. Since X_L has a weak topology generated by maps in $\mathbb{R}$, it is completely regular. Since L separates the points of X_L, the space X_L is Hausdorff. Let $\xi \in \upsilon(X_L)$. Every $f \in L$ has a continuous extension $\widetilde{f} : \upsilon(X_L) \to \mathbb{R}$. Then the map $\widetilde{\xi} : L \to \mathbb{R}$, defined by $\widetilde{\xi}(f) = \widetilde{f}(\xi)$ is a homomorphism and so, it belongs to X_L.

Conversely, supposing that X is a realcompact space it remains the question whether, every $h \in X_{C(X)}$ is determined by a point of X. Define $Y = X \cup h$ endowed with the weak topology with respect to $C(X)$ (considering $f(h) = h(f)$ for every $f \in C(X)$). Then X is a subspace of Y and every $f \in C(X)$ can be continuously extended to Y. Supposing $h \notin \overline{X}$, there exists $f \in C(X)$, $0 \leq f \leq 1$ such that $h(f) = 1$ and $f(X) = 0$. But $h \in X_{C(X)}$ which implies $h(0) = 0$. Since X is realcompact, there does not exist Y containing X as a proper dense C-embedded subset, which is a contradiction □

As well, $X_{C(X)} = \upsilon X$, the Hewitt-Nachbin realcompactification of X. Furthermore, the spectral representation $C(X) \to C(\upsilon X)$ becomes an isomorphism of real l-groups. Thus $C(X)$ is semisimple.

3. Complete separation

In this section we shall assume that L is a semisimple real l-group.

We shall need to separate disjoint zero subsets of X_L by elements of L. It is a difficult task that needs a description of such pair of zero-sets by means of L. Then we must find a convenient condition for L to allow such a separation.

Definition 3.1. *A subset J of L is said to be a vanished ideal if it is an intersection of real ideals. We shall denote $\mathcal{V}(L) = \{$vanished ideals of $L\}$.*

There is a one-to-one correspondence between $\mathcal{V}(L)$ and the set consisting of nonempty closed subsets of X_L by identifying any nonempty closed subset F of X_L with the vanished ideal $I_F = \bigcap_{x \in F} M_x$, and any vanished ideal $J \in \mathcal{V}(L)$ with the nonempty closed subset $F_J = \bigcap_{f \in J} \mathrm{zero}(f)$. We may agree that L is a vanished ideal for which $F_L = \varnothing$ and $I_\varnothing = L$. So, $\mathcal{V}(L)$ is a lattice having the smallest and the largest element, similarly as the lattice of closed subsets of X_L (under inclusion). The correspondence between them reverses the inclusion.

Given $I, J \in \mathcal{V}(L)$ we set

$$I \prec J \text{ in case there exists } H \in \mathcal{V}(L) \text{ such that } I \wedge H = 0 \text{ and } H \vee J = L.$$

It is easy to see that $I \wedge H = 0$ if and only if $F_H \cup F_I = X_L$, and $H \vee J = L$ if and only if $F_J \cap F_H = \varnothing$. Hence, the motivation for this new order on $\mathcal{V}(L)$ is the fact that

$$I \prec J \text{ if and only if } F_J \subseteq \overset{\circ}{F_I}.$$

Given $J \in \mathcal{V}(L)$ we set

$$J^\perp = \{f \in L : |f| \wedge |g| = 0 \text{ for all } g \in J\}.$$

Since $|f| \wedge |g| = 0$ if and only if $x(f) = 0$ for every $x \notin \mathrm{zero}(g)$, it is clear that $f \in J^\perp$ if and only if $x(f) = 0$ for every $x \notin F_J$, and therefore $F_{J^\perp} = \overline{X_L \setminus F_J}$.

Definitions 3.2. *A separating chain in L is a countable chain $\mathscr{S}$ in $\mathcal{V}(L)$ which satisfies*

(i) $\bigwedge \mathscr{S} = 0$ *and* $\bigvee \mathscr{S} = L$;
(ii) if $I, J \in \mathscr{S}$ and $I \subset J$, then there exists $H \in \mathscr{S}$ such that $I \prec H \prec J$.

L is said to be `completely separating` *in case for each two members I, J belonging to a separating chain in L, the inclusion $I \subset J$ implies that for any $h \in L$, there exists $f \in J$ and $g \in I^\perp$ such that $f + g = h$.*

To functionally separate two closed sets A and B (i.e., to find some $f \in C(X)$ such that $f(A) = 0$ and $f(B) = 1$), Urysohn noticed that it suffices to construct a countable chain of sets $\{F_r\}$ (ordered by a dense order, e.g. by rational or dyadic rational numbers in [0,1)) such that for $r < s$ one has $\overline{F_r} \subset \overset{\circ}{F_s} \subset X \setminus B, F_0 = A$. Using that he proved his famous Urysohn lemma. The procedure is described in the following lemma due to Johnson and Mandelker [11].

Lemma 3.3. *Two subsets A and B of a topological space X are functionally separated if and only if there exists a countable chain $\mathscr{F}$ of closed subsets of X satisfying the following conditions:*

(i) $\bigcap \mathscr{F} = \varnothing$ *and* $\bigcup \mathscr{F} = X$*;*

(ii) if $F, G \in \mathscr{F}$ *and* $F \subset G$*, then there exists* $W \in \mathscr{F}$ *such that* $F \subseteq \overset{\circ}{W} \subseteq W \subseteq \overset{\circ}{G}$*,*

(iii) there exist $C, D \in \mathscr{F}$ *such that* $A \subseteq C \subset D \subseteq X \setminus B$*.*

If $\mathscr{S}$ denotes a separating chain in $C(X)$ and $I, J \in \mathscr{S}$ are such that $I \subset J$, then $F_J \subset F_I = \upsilon X \setminus (\upsilon X \setminus F_I)$. According to the previous Lemma F_J and $\upsilon X \setminus F_I$ are functionally separated in υX. Taking into account the isomorphism $C(X) = C(\upsilon X)$, there exists $\tilde{f} \in C(X)$ such that $\tilde{f}(F_I) = 0$ and $\tilde{f}(\upsilon X \setminus F_J) = 1$. Then $\tilde{f}(\overline{\upsilon X \setminus F_J}) = 1$ because $\tilde{f}$ is continuous. Given $h \in C(X)$ if we consider $f = h\tilde{f}$ we derive that $f \in I$ and $g = h - f \in J^{\perp}$. Thus, $C(X)$ is completely separating.

Theorem 3.4. *If* L *is completely separating, then for any pair* A, B *of functionally separated subsets of* X_L *and for any* $h \in L$*, there exists* $f \in L$ *such that* $f = 0$ *on* A *and* $f = h$ *on* B*.*

Proof. Lemma 3.3 ensures that there exists a countable chain $\mathscr{F}$ of closed subsets of X_L satisfying: $\bigcap \mathscr{F} = \varnothing$ and $\bigcup \mathscr{F} = X_L$; if $F, G \in \mathscr{F}$ and $F \subset G$, then there exists $W \in \mathscr{F}$ such that $F \subseteq \overset{\circ}{W} \subseteq W \subseteq \overset{\circ}{G}$; there exist $C, D \in \mathscr{F}$ such that $A \subseteq C \subset D \subseteq \mathrm{Spec}_e(E) \setminus B$. By virtue of the one-to-one correspondence between $\mathcal{V}(L)$ and the closed subsets of X_L, the family

$$\mathscr{S} = \{I_F : F \in \mathscr{F}\}$$

becomes a separating chain in L. Since $I_D, I_C \in \mathscr{S}$ and $I_D \subset I_C$, there exist $f \in I_C$ and $g \in I_D^{\perp}$ such that $f + g = h$. On the one hand, if $x \in A \subset C = F_{I_C}$, then $x(f) = 0$; on the other hand, if $x \in B \subset X_L \setminus D \subset \overline{X_L \setminus D} = F_{I_D^{\perp}}$, then $x(g) = 0$ and therefore $x(h) = x(f) + x(g) = x(f)$. □

4. Approximation

In this section we shall assume that L is a semisimple and completely separated real l-group. The next definition can be, however, stated for any l-group L. It says that some needed topological concepts have sense in l-groups, without changing their verbal formulation.

Definitions 4.1. *A sequence* $\{f_n\}_n$ *in* L *is said to be* uniformly Cauchy *if for each* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $|f_n - f_m| \leq \varepsilon$ *for all* $n, m > N$*. The sequence* $\{f_n\}_n$ *is* uniformly convergent *to some* $f \in L$ *if for each* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $|f_n - f| \leq \varepsilon$ *for all* $n > N$*.*

L *is said to be* `uniformly complete` *when every uniformly Cauchy sequence of* L *is uniformly convergent to some element of* L*.*

Now comes the expected approximation theorem.

Theorem 4.2. *If* L *is uniformly closed, then* L^* *is isomorphic to* $C^*(X_L)$*.*

Proof. Let $f \in C^*(X_L)$ and $\varepsilon > 0$. There exists $n \in \mathbb{N}$ for which $-n\varepsilon < f < n\varepsilon$. For any $-n \leq i \leq n$ define the zero-set $Z_i = \{x \in X_L : f(x) \leq i\varepsilon\}$ which is functionally separated from $X_L \setminus Z_{i+1}$. By hypothesis there exists $g_i \in L$ having values 0 on Z_i and $i\varepsilon$ on $X_L \setminus Z_{i+1}$. Considering $h_i = (-i\varepsilon \vee g_i) \wedge i\varepsilon \in L^*$, we derive that $h = \bigvee_i h_i \in L^*$ and therefore $|f - h| \leq \varepsilon$ in $C^*(X_L)$. Thus L^* is uniformly dense in $C^*(X_L)$. Since L is semisimple, then L^* is uniformly complete in $C^*(X_L)$ in particular, and therefore $f \in L^*$. $\square$

Next definition is due to Feldman and Porter [5], and it is close to the condition (A2) of Fenstad [6].

Definitions 4.3. *A collection $\{f_n\}_n$ in a real l-group L is called* 2-disjoint *in case for each n, $f_n \wedge f_k \neq 0$ for at most two indices k distinct from n and for every real ideal M of L, there is some m such that $f_m \notin M$.*

L is said to be `2-universally complete` *in case every 2-disjoint sequence in L has a least upper bound in L.*

In $C(X)$, if the pointwise supremum of a sequence exists and belongs to $C(X)$ then it will be the supremum of the sequence. Indeed, given a 2-disjoint collection $\{f_n\}_n$ in $C(X)$, for each $x \in \upsilon X$ there exists f_n with $|f_n(y)| \geq 0$ for all y in a x-neighbourhood U in υX. The pointwise supremum of the collection $\{f_n\}_n$ in U thus involves at most three functions; one concludes that the pointwise supremum is continuous, and thus $\bigvee_n f_n \in C(X)$. Thus $C(X)$ is 2-universally complete.

5. Main theorem

Theorem 5.1. *A real l-group L is isomorphic to $C(X)$ for some topological space X if and only if*

(i) L is semisimple;
(ii) L is completely separating;
(iii) L is uniformly complete;
(iv) L is 2-universally complete.

Proof. We know from the preceding considerations that $C(X)$ satisfies the conditions of Theorem.

We must now prove that any $f \in C(X_L)$ belongs to L, provided L satisfies the condition in Theorem. It suffices to assume that $f \geq 0$. Indeed, $f = f^+ - f^-$ and belongs to l provided both nonnegative functions f^+, f^- belong to L.

For each $n \geq 3$ we define the disjoint zero-sets

$$Z^n = \{x \in X_L : n - 2 \leq f(x) \leq n - 1/2\}$$

$$Z_n = \{x \in X_L : f(x) \leq n - 5/2\} \cup \{x \in X_L : n \leq f(x)\}\,.$$

There exists a sequence $\{f_n\}_n$ in L_+ such that $f_n = 0$ on Z_n, $f_n = f \wedge (n - 1/2)$ on Z^n (because $f \wedge (n-1/2) \in C^*(X_L) = L^* \subseteq L$). If $n \geq m+3$ and $f_m(x) \neq 0$, then $x \in Z_{m+3}$ and therefore $x \in Z_n$ for all $n \geq m+3$. Thus $f_n(x) = 0$ whenever $n \geq m+3$. On the other hand, there exists $k \geq 3$ such that $x \in Z^k$, which implies that there exists n such that $f_n(x) > n$. Then, $\{f_n\}_n$ is a 2-disjoint sequence and accordingly with 2-universal completeness $\bigvee_n f_n \in L$.

Supposing $f \lneqq \bigvee_n f_n$, there exist $x_0 \in X_L$ such that $f(x_0) \neq (\bigvee_n f_n)(x_0)$. Let $f(x_0) < \alpha < \beta < (\bigvee_n f_n)(x_0)$. Taking the disjoint zero-sets

$$A = \{x \in X_L : f(x) \leq \alpha\},\ B = \{x \in X_L : f(x) \geq \beta\}$$

there exists $h \in L_+$ such that $h = 0$ on B and $h = (\bigvee_n f_n)(x_0)$ on A. Setting the zero-set

$$C = \{x \in X_L : h(x) \leq \beta\}$$

it is clear that $B \subseteq C$ and $A \cap C = \varnothing$. Moreover, there exists $k \in L_+$ such that $k = 0$ on A and $k = \bigvee_n f_n$ on C. If $x \in B \subseteq C$, then $f(x) \leq (\bigvee_n f_n)(x) = k(x)$; if $x \in X_L \setminus B$, then $f(x) < \beta$. In both cases $f \leq k \vee \beta$, so then if we consider $s = (k \vee \beta) \wedge (\bigvee_n f_n) \in L_+$, then $f \leq s \leq \bigvee_n f_n$. Now, since $x_0 \in A$ we have that $k(x_0) = 0 \leq f(x_0) < \beta$, and therefore $k(x_0) \vee \beta = \beta$. Henceforth, $s(x_0) = \beta \wedge (\bigvee_n f_n)(x_0) = \beta < (\bigvee_n f_n)(x_0)$. We obtain that

$$f_n \leq f \leq s \lneqq \bigvee_n f_n, \text{ for every } n,$$

contrary to the definition of the suprema $\bigvee_n f_n$.

Consequently, $f = \bigvee_n f_n \in L$. □

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