INVITATION TO MATHEMATICAL CONTROL THEORY

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PREFACE

These notes are based on the series of lectures which I delivered at the summer school "Differential Equations" held in Mala Moravka (Moravia), May 22-27, 2006. Since the problem of an appropriate control of a system appears in technology, economics, natural sciences and, of course, in mathematical setting, the subject of control theory is enormously vast. This means that only few probes could be done in my lectures. I decided to concentrate mainly on linear systems. In spite of the clarity of the mathematical notion of linearity it is not simple to cover all systems appearing in practise. For this reason the first section is included though it can seem to be superfluous. One of its main message is a link between two different description of a linear system. Namely in the state space setting which is actually a part of the theory of ordinary partial differential equations on one side and in the frequency domain that has its origin in electroengineering on the other side. The frequency domain approach to stability is briefly describe in Section 1,5. With exceptions of Section 1,3, 1,4, we study finite dimensional systems to avoid undue technicalities joined with partial differential equations which could coincide with control theory procedures. But I assume that the reader is acquinted with basic facts on C_0 -semigroups. To this purpose the text [M] is highly sufficient. Control problems can be formulated in many occasions in language of operator theory, so I also suppose certain proficiency in linear functional analysis. The preliminary chapters in [D-M] cover almost all needed facts. Since the frequency analysis of signals comes from their integral transforms, mainly the knowledge of the Laplace transform is probably the last prerequisity.

As it has been mentioned the main bulk of these notes (Chapter 1) concerts with linear theory. The smaller Chapter 2 is devoted to nonlinear theory. The exposition uses mainly various aspects of linearization. The bang-bang principle is introduced in Section 2,1 and a necessary condition for optimal control known as the (Pontryagin) maximum principle and a sufficient condition for optimal control due to R. Bellman are described in a simplified version in Section 2,2 and 2,3. These conditions have their origin in classical mechanics in the Lagrange and the Hamilton formulation.

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INTRODUCTION

The first example of a control device is the well-known Watt regulator which acting is decribed in various books on ordinary differential equation. The requirements of affectioning of operations of machines, e.g. aircrafts, electronical equipments, etc. is easily realized. In spite that the following example is oversimplified, its solution is not easy.

Example (The moon landing problem)
We wish to manage a spacecraft to make a soft landing on the moon surface using the minimum amount of fuel. Let M denote the mass of the spacecraft without fuel, m denote the mass fuel. Assume that at the moment $t = 0$ the spacecraft is at the height x_0 from the moon and has the velocity v_0. Let g denote the gravitational acceleration (assuming that it is constant) given by the moon. The Newton second law yields the equations

$$(M + m)\dot{v} = (M + m)g - F$$
$$\dot{x} = v,$$

where F is the retarding thrust of the spacecraft's engines. We add the equation for consumption of fuel for the slowing down of spacecraft:

$$\dot{m} = -kF.$$

For construction reasons there is apriori limitations of F, namely $F \in [0, F_0]$. The initial conditions are

$$m(0) = m_0$$
$$x(0) = x_0$$
$$v(0) = v_0.$$

The problem is to find time $T > 0$ such that the end conditions

$$x(T) = 0$$
$$v(T) = 0$$

are satisfied and $m(T)$ is maximized. For a solution see e.g. [F-R],Chapter II,6.

Since many control tasks lead to minimization of some functional one can get an opinion that the control theory is actually a part of calculus of variations. But it is far from being true as the previous example partly shows - one of its typical feature is that the end time T is unknown in advance. If the end point is fixed, a typical constraint in the control theory is given by a system of differential equations.

Example (The Ramsey model in microeconomy)
Let c denote the amount of utilizing capital, let f, g be a production and a consumption function, respectively. Assume that the capital evolves according to the differential equation

$$\dot{c}(t) = f(c(t)) - u(t)g(c(t))$$
$$c(t_0) = c_0,$$

where $u : [t_0, t_1] \to [0, 1]$ is a control function. We wish to maximize the performance index (or cost functional)

$$J(u) = \int_{t_0}^{t_1} h(u(t))g(c(t))dt, \tag{1}$$

where h is a given increasing function on $[0, 1]$.

Many natural systems like ecosystems have their own self regulation. A typical example is the famous Voltera predator-prey model. The self regulation means that a system can return to its normal mode after small disturbaces. In technology we have to implement such regulations. It is convenient if such control equipment uses informations on the current state of system. Such controls are called the feedback controls and they can be designed to perform various tasks not only to stabilize a system. Here is one important point: Even if one has a quite good model for a real system there are still various variations during production. A good control has to eliminate all these aberrations, e.g. it has to stabilize all members of a given set of systems (the robust stability problem).
A system which should be regulated is very often very complicated (e.g. an aircraft) and it is practically impossible to write all equations describing its behaviour. Instead of solving these equations we can tested the bahaviour by analyzing the dependence of several outputs on incoming inputs (e.g. testing a profile of wing). In another words, we look at a system as a 'black box" without interests what actually happened inside it and try to get desirable information by investigation responses to inputs. Since this idea comes from electroengineering, it often uses harmonic (more generally, spectral) analysis of inputs and outputs. We will say more about this in Sections 1,1, and 1,5.

Chapter 1

LINEAR THEORY

1.1 WHAT IS A LINEAR SYSTEM

We have said in Introduction that a (linear) system is like "a black box" which sends an input u to an output y. We will suppose that $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. These assumptions on finite dimensions of a control space ($\mathbb{R}^m$) and on an observation space ($\mathbb{R}^p$) are mainly from practical point of view, since it is hardly to design a system with infinitely many outputs or inputs. We can have a model describing processes inside the black box (e.g. a ca r) but it is only a model which more or less differs from the real situation. Because of that we are interested in the operator $\mathcal{F} : u \to y$. We assume that $\mathcal{F}$ is linear (since the system is linear), time-invariant, i.e. $\mathcal{F}$ commutes with all right shifts τ_t, where

$$\begin{aligned} \tau_t f(s) &= f(t-s) & t \leq s \\ &= 0 & 0 \leq s < t \end{aligned}$$

for a function f defined on $\mathbb{R}^+$. We consider functions on $\mathbb{R}^+$ since we assume that the system is in rest until time $s = 0$. Supposing certain continuity, $\mathcal{F}$ is an operator of convolution type (its kernel need not be a function). In view of this result, such operator $\mathcal{F}$ is often called an **integral representation** of a linear system.

A simple model for such situation is based on an idea that the state of a black box is described by an internal variable $x \in \mathbb{R}^n$ which is ruled by an input u with help of a system of linear autonomous differential equations (for continuous time) or difference equations (for discrete time). Since differential equations are simpler for an investigation than difference ones, we suppose that

$$\begin{aligned} \dot{x} &= Ax + Bu \\ x(0) &= 0 \\ y &= Cx + Du, \end{aligned} \tag{1,1,1}$$

where A, B, C, D are matrices of appropriate orders. Denoting e^{tA} the fundamental matrix we have

$$x(t) = \int_0^t e^{(t-s)A} Bu(s) ds$$

and

$$y(t) = C \int_0^t e^{(t-s)A} Bu(s) + Du(t) := \mathcal{F}u(t) \tag{1,1,2}$$

If $D \neq 0$ then $\mathcal{F}$ is a convolution operator with kernel $Ce^{\cdot A}B + \delta D$, where δ is the Dirac measure (distribution). It is possible to avoid the use of distributions by

considering the Laplace transform (for functions on $\mathbb{R}^+$) or the Fourier transform (for $\mathbb{R}$).

In our case we get

$$\widehat{y}(z) := \int_0^{+\infty} e^{-zt} y(t) dt = [C(z-A)^{-1}B + D]\widehat{u}(z) \qquad (1,1,3)$$

for $z \in \mathbb{C}_\omega^+ := \{z \in \mathbb{C};\ \mathrm{Re}z > \omega\}$ where ω is the growth rate of e^{tA}, i.e. $\|e^{tA}\| \le Ce^{\omega t}$. More generally, the following theorem holds.

Theorem 1.1. *Let $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}^+, \mathbb{C}^m), L^2(\mathbb{R}^+, \mathbb{C}^p))$ be time-invariant. Then there exists a matrix-valued function*

$$H : \mathbb{C}^+ = \{z \in \mathcal{C};\ Re z > 0\} \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^p)$$

which is holomorphic and bounded on $\mathbb{C}^+$ (i.e. $H \in H^\infty(\mathbb{C}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^p)))$ such that

$$\widehat{\mathcal{F}u}(z) = H(z)\hat{u}(z) \qquad for\ z \in \mathbb{C}^+. \qquad (1,1,4)$$

Conversely, if $\mathcal{F}$ is defined by (1,1,4) then $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}^+, \mathbb{C}^m), L^2(\mathbb{R}^+, \mathbb{C}^p))$ and is time-invariant.

Proof. For a proof and more details see e.g.[W2].The converse statement is based on the fact that the Laplace transform is an isomorphism of $L^2(\mathbb{R}^+)$ onto $H^2(\mathbb{C}^+)$ (the Hardy space) - the Paley - Wiener theorem. $\qquad \square$

The function H from (1,1,4) is called the **transfer function** of a system with an input-output function $\mathcal{F}$. The Laplace transform is an injective linear continuous mapping of $L^1(\mathbb{R}^+)$ into $H^\infty(\mathbb{C}^+)$ but unfortunately it is not surjective, (e.g.$\widehat{\delta}(z) = 1$). This means that $\mathcal{F}$ need not be the convolution with L^1-kernel (cf.(1,1,2)).

For the special system (1,1,2) its transfer function (see (1,1,3)) is a rational matrix-valued function with no poles in certain right half -plane ($\mathbb{C}_\omega^+$) and which is proper (i.e. bounded on a neighborhood of infinity). Conversely, we can ask whether any such rational function R is a transfer function of some system (1,1,1). In terms of control theory, the question is: Has R a realization of the form (1,1,2)? The case $R : \mathbb{C} \to \mathbb{C}$ (the so-called SISO system) is solvable by the decomposition into elementary fractions.

The reader can get a better imagination considering a simpler case of $\mathcal{F} \in \mathcal{L}(L^2(T))$ (T is the unit circle) which commutes with the right shifts. Using Fourier series one gets

$$\widehat{\mathcal{F}u}(n) = f_n \widehat{u}(n), \qquad n \in \mathbb{Z}, \qquad (1,1,5)$$

with $(f_n) \in l^\infty(\mathbb{Z})$. Here $\widehat{u}(n)$ denotes the n-th Fourier coefficient of u. Since $(\hat{u}(n))_{n \in \mathbb{Z}}$ and, more generally, $\hat{u}(z)$ are "frequencies" of u, the representation of $\mathcal{F}$ by its transfer function H is called the **frequency domain representation** of $\mathcal{F}$.

Remark 1.2. If one has to use another L^p spaces (i.e. $p \ne 2$) then he/she encounters with difficulties occuring in the converse problem: Which sequences $(f_n)_{n \in \mathbb{Z}}$ yield continuous operators $\mathcal{F} : L^p(T) \to L^q(T)$ via (1,1,5)? This is the problem of multipliers (see any more advanced book on harmonic analysis).

Theorem 1.3. *Let $H : \mathbb{C} \to \mathbb{C}^{p \times m}$ be a rational matrix-valued function with real entries which is bounded on a neighborhood of ∞. Then there exist $n \in \mathbb{N}$ and matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{m \times p}$ such that*

$$H(z) = C(z - A)^{-1}B + D$$

for sufficiently large $\operatorname{Re} z$.

Proof. See e.g.[Z],Part I, Chapter 3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 1.4. There is no uniqueness theorem for realization of $\mathcal{F}$ in the form (1,1,1) as it can by easily seen for a SISO system. If such a realization exists then there is a realization with minimal n (the dimension of a state space).

Example 1.5.
1) The delay line, i.e.

$$\mathcal{F}u(t) = \begin{array}{ll} u(t-1), & t \geq 1 \\ 0, & 0 \leq t \leq 1 \end{array}$$

has transfer function $H(z) = e^{-z}$, which belongs to $H^\infty(\mathbb{C}^+)$. But $\mathcal{F}$ is not a convolution with a kernel given by some function. Indeed, $\mathcal{F}u(t) = \delta_1 * u(t)$, where δ_1 is the Dirac measure concentrated at the point 1.
2) A differential equation with delay, e.g. $\dot{x} = x(t-1) + u(t)$, $x(t) = 0$ for $-1 \leq t \leq 0$ and $y(t) = x(t)$. The Laplace transform yields

$$z\hat{x}(z) = e^{-z}\hat{x}(z) + \hat{u}(z),$$

i.e. the corresponding transfer function of this system is $H(z) = \frac{1}{z - e^{-z}}$. Again, it is not difficult to see that the input-output map is a convolution with a distributive kernel.

Hint. If

$$\begin{array}{ll} H_1(t) = 0 & t < 1 \\ \quad\quad\;\; = 1 & t > 1 \end{array}$$

then $H_1(z) = \frac{e^{-z}}{z}$
$\quad$ If

$$\mathcal{F}u(t) = \int_0^t k(t - s)u(s)ds \qquad\qquad\qquad (1,1,6)$$

where k is a function, it is possible to find a realization of the form (1,1,1) provided that $\mathbb{R}^n$ is replaced by the infinite-dimensional Hilbert space and A, B, C are generally unbounded operators. More precisely, the following simple result holds.

Proposition 1.6. *Let (1,1,6) hold with $k \in L^1(\mathbb{R}^+)$. Then $\mathcal{F}$ is an input-output map for the system*

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} + k(x), \\ w(0, x) &= 0, \qquad x \in (0, \infty) \\ y(t) &= w(t, 0) \end{aligned} \qquad\qquad (1,1,7)$$

in the space $L^2(0, \infty)$.

Proof. If $u \in W^{1,2}(\mathbb{R}^+)$ and $k \in L^1(\mathbb{R}^+) \cap C(\mathbb{R}^+)$ then

$$w(t, x) = \int_x^{t+x} k(\xi)u(t + x - \xi)d\xi \tag{1,1,8}$$

solves the partial differential equation on $(0, \infty) \times (0, \infty)$ and satisfies the boundary condition. For $u \in L^2(\mathbb{R}^+)$, $k \in L^1(\mathbb{R}^+)$, the integral in (1,1,8) is equal to $\int_0^t k(t + x - s)u(s)ds$ and can be considered as a generalized solution to (1,1,7). Further $w(t, .) \in \mathbb{C}(\mathbb{R}^+)$ and $y(t) = w(t, 0) \in L^2(\mathbb{R}^+)$. $\qquad\square$

The realization of (1,1,7) we can write in an abstract form

$$\begin{aligned} \dot{w} &= Aw + Bu \\ w(0) &= 0 \\ y &= Cw \end{aligned} \tag{1,1,9}$$

where A is a generator of the left-shift semigroup $S(.)$ on $X = L^2(\mathbb{R}^+)$, i.e. $\mathcal{D}(\mathcal{A}) = W^{1,2}(0, \infty)$ and $Aw = \frac{\partial w}{\partial \xi}$. Further, $Bu = k.u$ and $B \in \mathcal{L}(\mathbb{R}, X)$ provided $k \in L^2(\mathbb{R}^+)$ and $C = \delta_0$, $\mathcal{D}(C) = C(\mathbb{R}^+) \cap X$, $C : \mathcal{D}(C) \subset X \to \mathbb{R}$.
From (1,1,8) we get

$$w(t) = \int_0^t S(t - s)Bu(s)ds := [\mathcal{B}u](t) \tag{1,1,10}$$

where $\mathcal{B} \in \mathcal{L}(L^2(\mathbb{R}^+, U), X)$ and $y(t) = CS(t)x =: [\mathcal{C}x](t)$,
where $\mathcal{C} \in \mathcal{L}(X, L^2(\mathbb{R}^+, U), X)$, $(U = Y = \mathbb{R})$. Operators $S, \mathcal{B}, \mathcal{C}, \mathcal{F}$ form a more determining representation of a linear system.

Theorem 1.7. *(realization theory - D.Salamon, G. Weiss,1987). Let U, Y be Banach spaces and let $\mathcal{F}$ be a linear continuous operator such that*

$$\mathcal{F} : L_\omega^2(\mathbb{R}^+, U) \to L_\omega^2(\mathbb{R}^+, Y)$$

for some exponential weight function $\omega(t) = e^{-\omega t}$. If $\mathcal{F}$ is time-invariant and causal (i.e. $u(t) = 0$ on $(0, T) \Rightarrow \mathcal{F}u(t) = 0$ on $(0, T)$) then there exists a Banach space X and C_0-semigroup $S(\cdot)$ on X and two continuous operators $\mathcal{C}, \mathcal{B}$ (as above) such that $S(t)\mathcal{B} = \mathcal{B}\tau_t$, $\mathcal{C}S(t) = \tau_t\mathcal{C}$, $\mathcal{C}\mathcal{B} = \mathcal{F}$.

Proof. For the proof one can consult either papers of [Sa1],[W1], or the recent monograph [St2]. $\qquad\square$

Remark 1.8. The existence of B, C such that (1,1,9) - (1,1,11) hold is another problem. To get B one has to extend the semigroup S into the extrapolation space X_{-1} (in concrete cases X_{-1} is a space of distribution). Then $B \in \mathcal{L}(U, X_{-1})$ exists such that (1,1,10) holds at least for sufficiently smooth u. Similarly, $C \in \mathcal{L}(X_1, Y)$ (X_1 is the domain of generator of $S(.)$ equipped with the graph norm) and (1,1,11) holds for $x \in X_1$. For details see e. g. the book [St2]. The role of interpolation spaces in control theory is briefly described in Section 1,6.

1.2 BASIC CONTROL PROBLEMS - FINITE DIMENSIONAL CASE

In this section we will suppose that a linear system is represented by a system of ODEs (1,1,1), where A, B, C, D are matrices. The corresponding spaces will be denoted by X $(= \mathbb{R}^n$, state space), U $(= \mathbb{R}^m$, control space) and Y $(= \mathbb{R}^p$, observation space). The following control problems are basic:

(P1) For given points $x_0, x_1 \in X$ find time $T > 0$ and a control $u \in L^2(0, T; U)$ such that a solution x to

$$\dot{x} = Ax + Bu \qquad (1,2,1)$$
$$x(0) = x_0,$$

hits the value x_1 at time T. In practise, it is often sufficient that $x(T)$ belongs to a prescribed neighborhood of x_1. This is the so-called **controllability problem**.

(P2) Consider a system

$$\dot{x} = Ax$$
$$x(0) = x_0 \qquad (1,2,2)$$
$$y = Cx.$$

The **observability problem** demands the existence of time T such that the operator $\mathcal{C} : x_0 \to u \in C([0, T]; Y)$ is injective. In more informal way: can we recognized the state of a system by observation during some time?

(P3) A matrix A can have an eigenvalue with positive real part, i.e. the system $\dot{x} = Ax$ is not stable. Since we want to manage a system in such way that is takes its values near to certain equilibrium (here 0), we can ask whether it is possible to find $u \in C(\mathbb{R}^+, U)$ (or in $L^2(\mathbb{R}^+, U)$) such that a solution of (1,2,1) tends to 0 if $t \to \infty$. Again from the practical point of view it would be desirable to design u working in various situations. In other words, we wish to find a **feedback** $F \in \mathbb{R}^{m \times n}$ so that the equation

$$\dot{x} = (A + BF)x \qquad (1,2,3)$$

is stable. This is the **stabilization problem**.

(P4) As we mentioned in (P1) it is often sufficient to send x_0 into a neighborhood of 0 in the prescribed time $T > 0$. But the closer $x(T)$ to 0 is the "price" for such control u can be higher. In these situations the optimality problem occurs:
Find $u \in L^2(0, T; U)$ such that the cost

$$J(u) = \int_0^T f(t, x(t), u(t))dt + g(x(T)), \qquad (1,2,4)$$

where x is a solution to (1,2,1), is minimal. Here a function g represents a "penalization" for $x(T)$ not being 0. Of course, this problem is not a linear one.

In case that f describes a quantity like energy, i.e.

$$f(t, x, u) = < x, x > + < Ru, u >$$

(R is a positive definite matrix) and

$$g(\xi) = < M\xi, \xi >$$

with a non-negative matrix M, we will show in section 1,6 that an optimal u depends linearly on x_0 and is of feedback type. For a general f the problem of minimization of (1,2,4) has many common features with calculus of variations as can by seen considering

$$
\begin{aligned}
\dot{x} &= u \\
x(0) &= x_0 \\
x(T) &= x_1.
\end{aligned}
$$

Now we turn to the controllability problem. Variation of constants formula gives a solution of (1,2,1) in the form

$$x(T) = e^{TA}x_0 + \int_0^T e^{(T-s)A} Bu(s)ds.$$

Denoting the convolution term by $\mathcal{B}_T u$ ($\mathcal{B}_T \in \mathcal{L}(L^2(0, T); U), X)$) we see that the controllability problem has a solution at time T for any x_0, x_1 if and only if $\mathcal{R}(\mathcal{B}_T) = X$. The matrix

$$Q_T = \mathcal{B}_T \mathcal{B}_T^*$$

$\mathcal{B}_T^*$ is the adjoin to $\mathcal{B}_T$ i.e. $< \mathcal{B}_T^* x, u >_{L^2} = < x, \mathcal{B}_T u >_X$

is a controllability matrix.

Proposition 1.9. *The controllability matrix Q_T is regular if and only if $\mathcal{R}(\mathcal{B}_T) = X$. Moreover*

$$\hat{u} = \mathcal{B}_T^* Q_T^{-1}(x_1 - e^{TA}x_0)$$

is a control which sends x_0 into x_1 and has the minimal L^2-norm.

Proof. Since $< Q_T x, x >_X = |\mathcal{B}_T^* x|^2_{L^2(0,T)}$

$$Q_T \text{ is regular}$$

$$\Updownarrow$$

$$\mathcal{B}_T^* \text{ is injective}$$

$$\Updownarrow$$

$$\mathcal{R}(\mathcal{B}_T)(= \overline{\mathcal{R}(\mathcal{B}_T)}) = X.$$

The control $\hat{u}$ has the desired property $\mathcal{B}_T \hat{u} = x_1 - e^{TA}x_0$ and $\hat{u}$ is perpendicular to the kernel of $\mathcal{B}_T$, i.e.

$$|\hat{u}| = \min |\hat{u} + Ker\mathcal{B}_T|.$$

$\square$

A deeper and more comfortable result is the following one.

Theorem 1.10. *(Kalman)*
$\mathcal{R}(\mathcal{B}) = X$ *if and only if the rank of matrix*

$$(B, AB, ..., A^{n-1}B)$$

is equal to n (=dim X).

Proof.
1) Rank $(B, AB, ..., A^{n-1}B) = \dim\mathcal{R}(L)$, where $L : U^n \to X$ is defined as

$$L(u_0, ..., u_{n-1}) = \sum_{k=0}^{n-1} A^k B u_i.$$

2) We have (finite dimensions)

$$\mathcal{R}(L) = \mathcal{R}(\mathcal{B}_T) \Leftrightarrow \mathcal{R}(L)^\perp = \mathcal{R}(\mathcal{B}_T^\perp)$$

and

$$x \in \mathcal{R}(L)^\perp \Leftrightarrow B^*(A^*)^k x = 0, \quad k = 0, ..., n - 1.$$

On the other hand

$$< x, \mathcal{B}_T u >= \int_0^T < B^* e^{tA^*} x, u(T - t) > dt.$$

By the Hamilton-Calley theorem $A^n = \sum_{i=0}^{n-1} \alpha_i A^i$ and, therefore, if $x \in \mathcal{R}(L)^\perp$ then
$B^* e^{tA^*} x = 0$ for all $t \geq 0$, i.e. $x \in \mathcal{R}(\mathcal{B}_T)^\perp$.
If $x \in \mathcal{R}(\mathcal{B}_T)^\perp$ then $B^* e^{tA^*} x = 0$ for all $t \geq 0$, and taking derivatives at $t = 0$ we obtain
$B^*(A^*)^k X = 0$, i.e. $x \in \mathcal{R}(L)^\perp$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 1.11. *Solvability of the controllability problem does not depend on T.*

From the proof of Theorem 1,10 we can deduce that $\mathcal{R}(\mathcal{B}) \subset \mathcal{R}(\mathcal{B}_T)$ and also that
$\mathcal{R}(\mathcal{B}_T)$ is A-invariant (again by the Hamilton-Calley theorem). This means that (1,2,1)
has the form

$$\begin{aligned} \dot{y} &= A_m y + A_{12} z + B u \qquad\qquad\qquad (1,2,5) \\ \dot{z} &= A_{22} z \end{aligned}$$

with respect to the decomposition $X = \mathcal{R}(\mathcal{B}_T) \oplus X_1$ ($x = y + z, y \in \mathcal{R}(\mathcal{B}_T), z \in X_1$.
The system (A_{11}, B) is controllable in $\mathcal{R}(\mathcal{B}_T)$. We will use the Kalman decomposition
(1,2,5) for solving the stabilizability problem.

Theorem 1.12. *(Wonham)*
*A system (1,2,1) is controllable if and only if for any $\omega \in \mathbb{R}$ there is $F \in \mathcal{L}(X, U)$ such
that $Re\sigma(A + BF) < \omega$. In particular, a controllable system is stabilizable by feedback.*

Proof. Proof of the necessity part is done in two steps.
(1)First we prove the statement for $\dim U = 1$, i.e. if there is $b \in X$ such that $Bu = ub$.
By the rank condition from Theorem 1,10, the elements

$$b_k = A^{n-k} b, \quad k = 1, ..., n$$

form a basis of X. Having a cyclic vector B the system (1,2,1) can be rewriten as a differential equation of the n-th order:

If $x = \sum_{k=1}^{n} y_k b_k$, then

$$\dot{x} = \sum \dot{y}_k b_k = A(\sum y_k b_k) + ub_n = \sum_{k=2}^{n} y_k b_{k-1} + y_1(-\sum_{j=1}^{n} \alpha_j A^{n-j} b) + ub_n,$$

where $A^n + \sum_{j=1}^{n} \alpha_j A^{n-j} = 0$ (Hamilton-Calley). This means that $y = y_1$ satisfies the equation

$$y^{(n)} + \sum_{k=1}^{n} \alpha_k y^{(n-k)} = u.$$

Let us choose a polynomial

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_n$$

and put $u = \sum_{k=0}^{n-1}(a_k + \alpha_k)y^{(k)} =: \mathcal{F}(y_1, ..., y_n) =: Fx$. Then the matrix $A + BF$ has P as its characteristic polynomial.

2) The reduction of general case to an one-dimensional case is based on the following lemma.

Lemma 1.13. *Let a system (A, B) is controllable. Then there exists $L \in \mathcal{L}(X, U)$, $w \in U$ such that the system*

$$\dot{y} = (A + BL)y + uBw \tag{1,2,6}$$

is controllable.

Proof. Let $w \in U$ be such that $Bw = b_1 \neq 0$. By induction, we can find $u_1, ..., u_{n-1} \in U$ such that $b_{k+1} = Ab_k + Bu_k$ is linearly independent on $b_1, ..., b_k, k = 1, ..., n - 1$. It is sufficient to put $Lb_k = u_k, k = 1, ..., n - 1, Lb_n = 0$ and use Theorem 1,10 to show that the system (1,2,6) is controllable.

The sufficiency of the condition of theorem can be easily shown by contradiction using the decomposition (1,2,5). □

□

Remark 1.14. It follows from the first part of the proof that a linear equation of the n-th order

$$x^{(n)} + \sum_{k=0}^{n-1} \alpha_k x^{(k)} = u, \quad \alpha_0, ..., \alpha_{n-1} \in \mathbb{R},$$

is always controllable.

Example 1.15. The system

$$\dot{x} = x + y + u$$
$$\dot{y} = -y$$

is not controllable and it is stabilizable , e.g. by $u = -2x - y$. This example shows how convenient is the decomposition (1,2,5) for the problem of stabilizability.

Remark 1.16. If an equation $\dot{x} = Ax$ is not exponentially stable then

$$\sigma^+ := \{\lambda \in \sigma(A); \ Re\lambda \geq 0\} \neq \emptyset.$$

Put $\sigma^- = \sigma(A) \setminus \sigma^+$. Then there is a direct decomposition $\mathbb{R}^n = X^- \oplus X^+$ with corresponding projections P^-, P^+ where both X^- and X^+ are A-invariant and the restriction $A^+ := A \mid_{X^+}$ has spectrum $\sigma(A^+) = \sigma^+$ and, similarly, $\sigma(A^-) = \sigma^-$. Assume that the partial system (A^+, P^+B) is controllable. Then there exists $F^+ \in \mathcal{L}(X^+, U)$ such that the matrix $A^+ + P^+BF^+$ is exponentially stable. Defining $Fx = F^+P^+x$ we get that $A + BF$ is also exponentially stable. By Theorem 1,10, the system (A^+, P^+B) is controllable provided $X^+ = \mathcal{R}(L^+)$, where

$$L^+(u_1, ..., u_k) = \sum_{j=1}^{k} (A^+)^{j-1} P^+ B u_j, \qquad k = \dim X^+.$$

If

$$X = \mathcal{R}(\lambda - A) + \mathcal{R}(B) \qquad \text{for all } \lambda \in \sigma^+, \tag{1,2,7}$$

then, by transforming A into its Jordan canonical form, it can be shown that all generalized eigenvectors of A^+ belongs to $\mathcal{R}(L^+)$. Since these vectors generate X^+, the condition (1,2,7) (the so-called **Hautus condition**) is sufficient for exponential stabilization of (1,2,1). If (1,2,1) is exponentially stabilizable by a feedback F then $(\lambda - A - BF)^{-1}$ exists for $\lambda \in \sigma^+$ and (1,2,7) follows.

Remark 1.17. From the point of view of applications the stabilization problem by feedback should by strengthened as follows: Does there exists a matrix $K \in \mathbb{R}^{m \times p}$ such that the equation

$$\dot{x} = (A + BKC)x$$

is stable? For more information see e.g. [B].

Now we turn to the observability problem and show that there is a duality between observability and controllability. For a solution of (1,2,2) we have

$$y(t) = Ce^{tA}x_0 =: [\mathfrak{C}_T x_0](t), \quad t \in [0, T]$$

Further, $Ker\mathfrak{C}_T = 0$ if and only if $\mathcal{R}(\mathfrak{C}_T^*) = X$, where

$$\mathfrak{C}_T^* y = \int_0^T e^{tA^*} C^* y(t) dt.$$

This means that we have

Proposition 1.18. *A system(1,2,2) is observable at time $T > 0$ if and only if the system (A^*, C^*) is controllable at time $T > 0$, i.e. the rank of*

$$(C^*, A^*C^*, ..., (A^*)^{n-1}C^*)$$

is equal to n ($= \dim X$).

Similarly as observability is in duality with controllability, the dual notion to stabilizability is detectability. A system (1,2,2) is detectable if there is $L \in \mathcal{L}(Y, X)$ such that the system $\dot{x} = (A + LC)x$ is stable. Obviously, (1,2,2) is detectable if and only if (A^*, C^*) is stabilizable.

1.3 BASIC CONTROL PROBLEMS - INFINITE DIMENSIONAL CASE

First we will consider the so-called distributed parameter system which abstract formulation is given by an evolution equation

$$\dot{x} = Ax + Bu$$
$$x(0) = x_0 \qquad\qquad (1,3,1)$$

with an observation

$$y = Cx. \qquad\qquad (1,3,2)$$

Here A is a generator of a C_0-semigroup $S(t)$ on a Hilbert (not always necessarily) space X, $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, where U, Y are also Hilbert spaces which have often finite dimensions. A solution (the so-called mild) of (1,3,1) for $u \in L^2(0, T; U)$ is $x \in C([0, T]; X)$ given by a generalization of variation of parameters formula

$$x(t) = S(t)x_0 + \int_0^t S(t - s)Bu(s)ds, \qquad\qquad (1,3,3)$$

where the integral is Bochner's. Since x is continuous, (1,3,2) holds pointwise.

The basic control problems (P1) - (P4) have been formulated in Section 1,2. The following example shows that the motion of controllability as well as observability should be modified.

Example 1.19. Let $\dim U = 1$ and let $Bu = ub$ for some $b \in X$. Then the (1,3,1) is not controllable in an infinite dimensional space X. Indeed, the operator

$$\mathfrak{B}_T u = \int_0^T S(t - s)bu(s)ds$$

is compact and thus $\mathcal{R}(\mathfrak{B}_T) \neq X$. The same result holds also for any finite-dimensional space U.

A system (1,3,1) is **approximate controllable** at time $T > 0$ if for any $x_0, x_1 \in X$ and a neighborhood V of x_1 there is $u \in L^2(0, T; U)$ such that $S(T)x_0 + \mathfrak{B}_T u \in V$, i.e. $\mathcal{R}(\mathfrak{B}_T)$ is dense in X. Since $\overline{\mathcal{R}(\mathfrak{B}_T)} = (Ker T^*)^\perp$ for any continuous linear operator $T \in \mathcal{L}(X, Y)$, $\overline{\mathcal{R}(\mathfrak{B}_T)} = X$ if and only if $\mathfrak{B}_T^*$ is injective. By the definition,

$$\mathfrak{B}_T^* x(s) = B^* S^*(T - s)x.$$

Since the adjoint semigroup $S^*(t)$ is strongly continuous in a Hilbert space, we have the following simple criterion.

Proposition 1.20. *A system (1,3,1) is approximate controllable if and only if $B^* S^*(t)x = 0$ for all $t \in [0, T]$ implies $x = 0$, i.e. the system (A^*, B^*) is observable at time T.*

Example 1.21. Assume that A is a self-adjoint operator bounded from above and has a compact resolvent (typically: Ω is a bounded domain with Lipschitz boundary, $X = L^2(\Omega)$, $Ax = \triangle x$, $\mathcal{D}(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, i.e. $\dot{x} = Ax$ is the heat equation in Ω with the Dirichlet boundary condition). Then there is an orthonormal basis (e_n) of X consisting in eigenfunctions of A

$$Ae_n = \lambda_n e_n, \quad \lambda_1 \geq \lambda_2 \geq ..., \quad \lambda_n \to -\infty.$$

Suppose further that $\dim U = 1$, i.e. there is a $b \in X$ such that $Bu = ub$.
Let

$$B^* S^*(t) = < S(t)b, x > = \sum_{n=1}^{\infty} e^{\lambda_n t} < b, e_n > < x, e_n > = 0 \qquad \text{for } t \in [0, T].$$

To obtain $x = 0$ it is necessary that $< b, e_n > \neq 0$ and $\lambda_n \neq \lambda_m$ for all $n, m \in \mathbb{N}$, $n \neq m$. These two conditions are also sufficient. Indeed, the function

$$\varphi(z) = \sum e^{\lambda_n z} < b, e_n > < x, e_n >$$

is holomorphic in $\mathbb{C}^+ := \{z \in \mathbb{C}; \; Re\, z > 0\}$. Being zero on $[0, T]$ it vanishes everywhere in $\mathbb{C}^+$. It follows that

$$0 = \lim_{t \to \infty} e^{\lambda_1 t} \varphi(t) = < b, e_1 > < x, e_1 >, \qquad i.e. \; < x, e_1 > = 0.$$

Similarly, $< x, e_n > = 0$ for all n.

Besides approximate controllability, the concept of null-controllability is also used. A system (1,3,1) is null-controllable at time $T > 0$ if for any x_0 there is $u \in L^2(0, T; U)$ such that

$$0 = e^{TA} x_0 + \int_0^T S(T - s) Bu(s) ds. \tag{1,3,4}$$

Example 1.22. Let A, b be as in the previous example. Then the condition (1,3,4) reads as follows

$$< x_0, e_n > = < b, e_n > \int_0^T e^{-\lambda_n s} u(s) ds \qquad \text{for all } n \in \mathbb{N}.$$

Finding such $u \in L^2(0, T)$ is not easy (it is a variant of classical momentum problem). For more information see e.g. [A-I].

Roughly speaking the lack of controllability for parabolic equations is due to the smoothing property of the corresponding semigroup which is analytic. Hyperbolic equations have no such property since the corresponding semigroup is actually a group. So we can expect some positive results on controllability. First of all we need to rewrite an equation of the second order

$$\ddot{x} = Ax + Bu$$
$$x(0) = x_0 \qquad\qquad (1,3,5)$$
$$\dot{x}(0) = w_0$$

in the form(1,3,1).
Since a standard example is

$$A = \triangle \text{ with } \mathcal{D}(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$$

we can assume that $-A$ is positive self-adjoint operator in $X(= L^2(\Omega))$. Then $(-A)^{\frac{1}{2}}$ is defined and it is a closed operator. Denote

$$X^\alpha = \mathcal{D}(-A)^\alpha$$

with the graph norm, $\alpha = \frac{1}{2}, 1$, and put $H = X^{\frac{1}{2}} \times X$. If $\mathcal{D}(\mathcal{A}) = X^1 \times X^{\frac{1}{2}}$ and

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \qquad\qquad (1,3,6)$$

then the equation (1,3,5) can be writen in the form

$$\dot{z} = \mathcal{A}z + \tilde{\mathcal{B}}u$$
$$z(0) = z_0$$

where

$$z = \begin{pmatrix} x \\ w \end{pmatrix} \qquad \tilde{\mathcal{B}}u = \begin{pmatrix} 0 \\ Bu \end{pmatrix}$$

The operator $\mathcal{A}$ is skew-symmetric ($\mathcal{A}^* = -\mathcal{A}$). If A has a compact resolvend end (e_n) is an orthonormal basis in X, $Ae_n = \lambda_n e_n$, $0 > \lambda_1 \geq \lambda_2 \geq ... \lambda_n \to -\infty$, then $\pm i\sqrt{-\lambda_n}$ are eigenvalues of $\mathcal{A}$ with eigenfunctions

$$\begin{pmatrix} e_n \\ \pm i\sqrt{-\lambda_n}e_n \end{pmatrix}$$

In order to find conditions for exact conatrollability (i.e. $\mathcal{R}(\mathfrak{B}_T) = H$) we need to describe $\mathcal{R}(\mathfrak{B}_T)$. The following simple result is useful.

Proposition 1.23. *Let $S \in \mathcal{L}(X, Z)$, $T \in \mathcal{L}(Y, Z)$, where X, Y, Z are Hilbert spaces. Then $\mathcal{R}(S) \subset \mathcal{R}(T)$ if and only if there is $C > 0$ such that*

$$|S^*z| \leq C|T^*z| \qquad\qquad for\ all\ z \in Z.$$

Proof. Sketch of the proof:
(i)
$$\mathcal{R}(S) \subset \mathcal{R}(T) \Leftrightarrow \exists C > 0 \quad \{F(x); \quad |x| \leq 1\} \subset \{G(y); \quad |y| \leq C\}.$$

Only necessity part is to prove. Denote $\widehat{G} = G \mid_{KerG}$. By the closed graph theorem $\widehat{G}^{-1}F$ is continuous and the inclusion follows.

(ii) Since Z is reflexive and G sends weakly convergent sequences into weakly convergent, the set $\{G(y); \quad |y| \leq C\}$ is closed.

(iii)

$$\{F(x); |x| \leq 1\} \subset \{G(y); |y| \leq C\} \iff |F^*(z)| \leq C|G^*(z)| \quad \text{for all } z \in Z.$$

The if part follows from (ii) by the Hahn-Bahn separation theorem. The only if part is a consequence of the definition of norm $|F^*(z)|$. $\qquad \square$

Let $Q_T = \mathcal{B}_T \mathcal{B}_T^*$. Then Q_T is a positive definite self-adjoint operator on X and thus there is a positive square root $Q_T^{\frac{1}{2}}$.

Corollary 1.24.
$$\mathcal{R}(\mathcal{B}_T) = \mathcal{R}(Q_T^{\frac{1}{2}}).$$

Proof. For $x \in X$ we have

$$|Q_T^{\frac{1}{2}}x|^2 = <Q_Tx, x> = |\mathcal{B}_T^* x|^2.$$

$\qquad \square$

Corollary 1.25. $\mathcal{R}(\mathcal{B}_T) = X$ *if and only if there is $C > 0$ such that*

$$|\mathcal{B}_T^* x|_{L^2(0,T;U)} \geq C|x|_X \quad \text{for all } x \in X.$$

Proof. Choose $S = I, T = \mathcal{B}_T$ in Proposition 1,23. $\qquad \square$

Example 1.26. Let $U = X$ and assume that $\mathcal{A}$ has properties as above. Then

$$\dot{z} = \mathcal{A}z + \begin{pmatrix} 0 \\ b \end{pmatrix}$$
$$z(0) = z_0$$

is exact controllable at any time $T > 0$. If $Bu = ub$ and $<b, e_n> \neq 0$ for all $n \in \mathbb{N}$ then the equation

$$\dot{z} = \mathcal{A}z + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

is approximately controllable at any time $T > 0$. Proofs of both statements need long calculations using the series expansions of the group $S(t)$ generated by $\mathcal{A}$. For more details see e.g. [B-DP-D-M].

1.4   STABILIZABILITY IN INFINITE DIMENSIONS

The asymptotic (equivalently exponential) stability for a finite system of linear ODEs

$$\dot{x} = Ax$$
$$x(0) = x_0 \tag{1,4,1}$$

means that a solution tends to zero at infinity. This is ultimately connected with location of eigenvalues of A. More precisely

$$s(A) := \sup\{Re\lambda; \ \lambda \in \sigma(A)\} < 0 \Leftrightarrow (1,4,1) \text{ is exponentialy stable}$$

The quantity $s(A)$ is called the spectral bound of A. If a state space X has infinite dimension then there are several notions of stability. We present at least two of them.

Definition 1.27. Let S be a C_0-semigroup on a Banach space X.
(i) The quantity

$$\omega_0(S) := \inf\{\alpha, \exists C > 0; \ \|S(t)\| \le Ce^{\alpha t}\}$$

is called the growth bound of S. A semigroup S is exponentially stable if $\omega_0(S) < 0$.
(ii) A semigroup S is strongly stable if

$$\lim_{t \to \infty} S(t)x = 0 \text{ for all } x \in X.$$

Theorem 1.28. *(Datko) A C_0-semigroup S is exponential stable in a Banach space X if and only if for some/all $p \in [1, \infty)$ one has $S(t)x = \in L^p(\mathbb{R}^+, X)$ for all $x \in X$.*

Proof. See e.g. [E-N], Theorem V.1.8. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For a system (1,4,1) of ODES we have $\omega_0(S) = s(A)$. This fact can be justified either by solving (1,4,1) or from the so-called spectral mapping theorem which says that

$$\sigma(s(T)) = e^{t\sigma(A)} \qquad \text{for } t \ge 0.$$

Unfortunately. this theorem does not hold even in the generalized form

$$\sigma(S(t))\backslash\{0\} = \overline{e^{t\sigma(A)}}\backslash\{0\} \tag{1,4,2}$$

(the weak spectral theorem) for any semigroup. We notice that the validity of (1,4,2) is sufficient for the equality $s(A) = \omega_0(S)$. So we need either special properties of semigroup (e.g. analycity of S, which holds for parabolic equations) or another properties of generator to be able to conclude stability of S from a quality of A.

Theorem 1.29. *(i) (Gearhart-Greiner-Prüss) A C_0-semigroup S in a Hilbert space X is exponentially stable if and only if $s(A) < 0$ and the resolvent*

$$\lambda \to (\lambda - A)^{-1} \in H^\infty(\mathbb{C}^+, \mathcal{L}(X)).$$

(ii) (Arendt-Batty-Lyubic-Vu) A bounded C_0-semigroup X with a generator A in a Banach space is strongly stable provided the following two conditions are satisfied:
(a) $\sigma_p(A) \cap i\mathbb{R}$ is countable
(b) $\sigma_p(A^*) \cap i\mathbb{R} = \emptyset$.

The reader can consult for proofs and more information on asymptotic behaviour of semigroups e.g. [E-N] or [A-B-H-N].

Because of several notions of stability there are also various notions of stabilizability.

Definition 1.30. A linear systam (1,3,1) is **strongly stabilizable,** respectively **exponentially stabilizable,** if there exists a feedback $F \in \mathcal{L}(X, U)$ such that the semigroup generated by $(A + BF)$ is exponentially stable and strongly stable.

If $s(A) > 0$ then (1,4,1) is unstable and we need to construct a feedback F so that it shifts the "bad" part od $\sigma(A)$ to the left half plane $\mathbb{C}^-$. If $\dim U < \infty$ we can expect only finite number of eigenvalues can be shifted.

We say that A satisfies the decomposition property if $\sigma(A) = \sigma^+ \cup \sigma^-$, where $\sigma^- = \{\lambda \in \sigma(A); Re\lambda < 0\}$ is closed and $\sigma^+ = \{\lambda \in \sigma(A); Re\lambda > 0\}$ is finite and consists in eigenvalues of finite multiplicity only. In particular, σ^- and σ^+ are separated and there is a projection $P : X \to X^+$, where $\dim X^+ < \infty$, $APx = PAx$ for all $x \in X$ and the restriction $A^+ := A \mid_{X^+}$ has spectrum $\sigma(A^+) = \sigma^+$ and the restriction $A^- := A \mid_{X^-}$, $X^- = Ker\, P$, has spectrum $\sigma(A^-) = \sigma^-$.

Theorem 1.31. *Let $dim\, U < \infty$ and let A be a generator of C_0-semigroup. Then the system (1,3,1) is exponentially stabilizable if and only if A satisfies the decomposition property and the system(A^+, PB) is controllable.*

Proof. 1) Sufficiency: Let $F^+ \in \mathcal{L}(X^+, U)$ be a stabilizing feedback, i.e. $s(A^+ + PBF^+) < 0$. Put $Fx = F^+Px$. Since $A + BF$ has in $X = X^- \oplus X^+$ the decomposition

$$\begin{pmatrix} A^- & (I - P)BF^+ \\ 0 & A^+ + PBF^+ \end{pmatrix}$$

the growth bound of the semigroup generated by $A + BF$ is the maximum of that of the semigroups generated by A^- and $A^+ + PBF^+$ and, therefore, strictly negative.
2) The proof of the necessity is more involved and it is based on a perturbation result which is due to [D-S]. See also [C-Z], §5.2. $\qquad\square$

Example 1.32. Let A be as in Example 1,21 Assume that $\lambda_1 > \lambda_2 > ... > \lambda_n \geq 0 > \lambda_{n+1} > ...$ are eigenvalues of A with finite multiplicity $k_j, j = 1, ...$ (A has a compact resolvent). Let $e_1^j, ..., e_{k_j}^j$ be pairwise orthogonal eigenfunctions of A corresponding to λ_j. Put $X^+ = Lin\{e_i^j, i = 1, ..., k_j, j = 1, ..., n\}$. Assume that $\dim U = m$ and $Bu = \sum_{i=1}^m u_i b_i, b_1, .., b_m \in X$. Since the orthogonal projectin P onto X^+ and thus also A^+, PB are explicitely given, we can use Theorem 1,10 to obtain the condition for controllability of (A^+, PB). After some computations this condition reads as follows

$$rank \begin{pmatrix} <b_1, e_1^j>, ..., <b_m, e_1^j> \\ . \\ . \\ . \\ <b_1, e_{k_j}^j>, ..., <b_m, e_{k_j}^j> \end{pmatrix} = k_j, \qquad j = 1, ...n.$$

Cf. Remark 1.16.

Theorem 1.33. *Let A be a generator of a C_0-semigroup S of contractions (i.e.$\|S(t)\| \leq 1$) in a Hilbert space X and let A have a compact resolvent. Then the feedback $f = -B^*$ is strongly stabilizable for (1,3,1) if and only if*

$$Ker(\lambda - A^*) \cap KerB^* = \{0\} \ \text{for all } \lambda \in i\mathbb{R} \cap \sigma_p(A^*)$$

Proof.
1) Let $A - BB^*$ is strongly stable. If $\widetilde{x} \in Ker(\lambda - A^*) \cap KerB^*$, $\widetilde{x} \neq 0$, then $(\lambda - A^* - BB^*)\widetilde{x} = 0$ and for the semigroup S_F generated by $A - BB^*$ one has $S_F^*(t)\widetilde{x} = e^{\lambda t}\widetilde{x}$ (by the spectral mapping theorem for point spectrum) and, therefore,

$$0 = \lim_{t \to \infty} < S_F(t)x, \widetilde{x} > = \lim_{t \to \infty} e^{\overline{\lambda}t} < x, \widetilde{x} > \ \text{for any } x \in X.$$

This means that $\widetilde{x} = 0$ a contradiction.
2) Beeing a generator of a semigroup of contractions, A is a dissipative operator $(Re < Ax, x >\leq 0$ for all $x \in \mathcal{D}(A))$ and $\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda}$ for $\lambda > 0$ (the Hille-Yosida theorem). It follows that

$$(\lambda - A + BB^*)^{-1} = [I + BB^*(\lambda - A)^{-1}]^{-1}(\lambda - A)^{-1}$$

exists for λ sufficiently large and it is a compact operator. Moreover, $A - BB^*$ is dissipative and thus it generates a bounded semigroup. The assumption (a) of Theorem 1,29 (ii) is satisfied by compactness of resolvent for $A - BB^*$. Assume that

$$(A^* - BB^*)x = i\mu x \ \text{for } \mu \in \mathbb{R} \text{ and some } x \neq 0.$$

Then

$$< A^*x, x > -|B^*x|^2 = i\mu|x|^2.$$

Taking the real part we get $B^*x = 0$ and $A^*x = i\mu x$. By the assumption of Theorem, $x = 0$. This contradiction shows that the assumptions (b) of Theorem 1.29 (ii) is also satisfied and, therefore, $A - BB^*$ generates a strongly stable semigroup. $\qquad \square$

Corollary 1.34. *Let A satisfy the assumptions of Theorem 1,33 and let (1,3,2) be approximately controllable. Then (1,3,2) is strongly stabilizable.*

Proof. Let $x \in Ker(\lambda_0 - A^*) \cap KerB^*$ for some $\lambda_0 \in i\mathbb{R}$. For any $\lambda \notin \sigma(A^*)$ we have

$$(\lambda - \lambda_0)B^*(\lambda - A^*)^{-1}x = B^*x = 0.$$

Since $(\lambda - A^*)^{-1}$ is the Laplace transform of $S^*(t)x$, the uniqueness theorem for this transform yields $B^*S^*(t)x = 0$ for all $t \geq 0$ and, by Proposition 1.20, $x = 0$. $\qquad \square$

Example 1.35. Theorem 1.33 and its Corollary can by used for parabolic and hyperbolic equations as well. If $\mathcal{A}$ has the form (1,3,6) and $-A$ is a positive operator, then $\sigma(\mathcal{A}) \subset i\mathbb{R}$. Since

$$(\lambda - \mathcal{A})^{-1} = \begin{pmatrix} \lambda & A \\ I & \lambda \end{pmatrix}(\lambda^2 - A)^{-1} \ \text{for } \lambda \notin i\mathbb{R},$$

$(\lambda - \mathcal{A})^{-1}$ is compact. Moreover, $\mathcal{A}$ is a generator of C_0-group of contractions.

1.5 STABILITY IN FREQUENCY DOMAIN

In Section 1,1 we discussed the description of a linear system by its transfer function H. Now we will continue this study with examination of stability. Throughout this section we restrict our attention to a finite dimensional control space U (dim $U = m$) and a observation space Y (dim $Y = p$) so that $H : \mathbb{C} \to \mathbb{C}^{p \times m}$. If H has rational entries then (Theorem 1,3) there exist matrices A, B, C, D such that

$$H(z) = C(z - A)^{-1} B + D.$$

Moreover, the equation $\dot{x} = Ax$ is exponentially stable if and only if

$$H \in H^{\infty}(\mathbb{C}_{\beta}^{+}, \mathbb{C}^{p \times m}) \quad \text{for some } \beta < 0 \quad (\mathbb{C}_{\beta}^{+} := \{z \in \mathbb{C}; \, Re z > \beta\}),$$

since the eigenvalues of A coincide with the poles of entries of matrix-valued function H. If we want to incorporate systems like a delay line and retarded differential equations we have to extend the class of transfer functions admitting also others functions than only rational, e.g. exponentials.

We consider following classes of matrix-valued functions:
(1) $F \in \mathcal{A}(\beta)$ if there are

$$K \in L_{\beta}^{1}(\mathbb{R}^{+}, \mathbb{C}^{p \times m}) \qquad (\text{i.e. } \int_{0}^{\infty} e^{-\beta t} \|K(t)\| dt < \infty),$$

$$(t_k) \subset [0, \infty), \, (A_k) \in \mathbb{C}^{p \times m} \text{ such that } \sum_{k=1}^{\infty} e^{-t_k \beta} \|A_k\| < \infty$$

$$\text{and } F(t) = K(t) + \sum_{k=1}^{\infty} \delta(t - t_k) A_k$$

(2) $\mathcal{A}_{-} := \cup_{\beta < 0} \mathcal{A}(\beta)$
(3) $H \in \widehat{\mathcal{A}}(\beta) \, (\widehat{\mathcal{A}}_{-})$ if H is the Laplace transform of $F \in \mathcal{A}(\beta) \, (\mathcal{A}_{-})$. In particular, $\widehat{\mathcal{A}}(\beta) \subset H^{\infty}(\mathbb{C}_{\beta}^{+}; \mathbb{C}^{p \times m})$.
(4) (Callier-Desoer ring)

$$\widehat{\mathcal{B}}(\beta) := \widehat{\mathcal{A}}_{-}(\beta)[\widehat{\mathcal{A}}_{\infty}(\beta)]^{-1} \text{ (quotient ring)},$$

where

$$\widehat{\mathcal{A}}_{\infty}(\beta) = \{H = (h_{ij}) \in \widehat{\mathcal{A}}_{-}(\beta); \exists \rho > 0 : \inf\{|h_{ij}(z)|; \, Re z \geq \beta, |z| \geq \rho\} > 0$$

$$\text{for } i = 1, ..., p, \, j = 1, .., m\}.$$

Remark 1.36. In case $m = p = 1$ it can be shown that functions in $\widehat{\mathcal{B}}(\beta)$ are meromorphic in $\mathbb{C}_{\beta}^{+}$ and have only finitely many singularities in $\mathbb{C}_{\beta}^{+}$ which are poles. More precisely, $H \in \widehat{\mathcal{B}}(\beta)$ if and only if it has the representation $H = H_1 + H_2$, where $H_1 \in \widehat{\mathcal{A}}_{-}(\beta)$ and H_2 is a rational function with all its poles in $\overline{\mathbb{C}_{\beta}^{+}}$ and $\lim_{z \to \infty} H_2(z) = 0$. A similar characterization holds also for a matrix-valued $\widehat{\mathcal{B}}(\beta)$.

Example 1.37.

1) $H(z) = \frac{1}{z}$ is the transfer function of the integrator $\mathcal{F}u(t) = \int_0^t u(s)ds$. Also $H(z) = \frac{1}{z+a}(\frac{z}{z+a})^{-1} \in \widehat{\mathcal{B}}(\beta)$ for $\beta > -a$. Notice that $\frac{1}{z+a}$ and $\frac{z}{z+a}$ are coprime over $\widehat{\mathcal{A}}(\beta)$. Indeed, for $X(z) = a$, $Y(z) = 1$, it is

$$X(z)\frac{1}{z+a} + Y(z)\frac{z}{z+a} = 1 \quad X, Y \in \widehat{\mathcal{A}}(\beta).$$

Notice that the characterization mentioned in the previous remark implies immediately that $H \in \widehat{\mathcal{B}}(\beta)$.

2) Consider the retarded equation

$$
\begin{aligned}
\dot{x} &= x(t-1) + u(t) \quad t \geq 0 \\
x(s) &= 0 \qquad\qquad\quad s \in [-1, 0].
\end{aligned}
$$

The Laplace transform yields

$$z\hat{x}(z) = e^{-z}\hat{x}(z) + \hat{u}(z)$$

and the transfer function of this system is $H(z) = \frac{1}{z - e^{-z}}$. Since $H(z) = \frac{1}{z+a}(\frac{z-e^{-z}}{z+a})^{-1}$, $H \in \widehat{\mathcal{B}}(\beta)$ for $\beta > -a$.

Definition 1.38. A transfer function H (or a system with this transfer function) is input-output stable (i-o stable) if $H \in \widehat{\mathcal{A}}_-$.

Remark 1.39. The input-output stability implies that the corresponding input-output operator

$$\mathcal{F}u(t) = \int_0^t K(t-s)u(s)ds + \sum_{k=1}^{\infty} A_k u(t - t_k)$$

sends an input $u \in L^2(\mathbb{R}^+, \mathbb{C}^m)$ into an output $y \in L^2(\mathbb{R}^+, \mathbb{C}^p)$. In many applications the L^2-norm of u and y represents an energy of u and y. So, roughly speaking, tho i-o stability means that the system with this i-o operator can not explode.

Theorem 1.40. *Let U, Y be finite dimensional spaces and let A be a generator of C_0-semigroups S. If a system (A, B) is exponentially stabilizable or (A, C) is exponentially detectable and its transfer function $C(z - A)^{-1}B$ is i-o stable, then S is exponentially stable.*

Proof. By Theorem 1,31 there is an A-invariant decomposition

$$X = X^- \oplus X^+, \quad \dim X^+ < \infty$$

and the semigroup generated by A^- is exponentially stable. It follows that

$$H_1(z) = C(z - A)^{-1}P^- B \in \widehat{\mathcal{A}}_-$$

(P^- is the projection onto X^-). Since $\sigma(A^+) \subset \overline{\mathbb{C}^+}$ and

$$H(z) = H_1(z) + C(z - A^+)^{-1}P^+ B =: H_1(z) + H_2(z),$$

we get $H_2 \in \widehat{\mathcal{A}}_-$ what can occur in finite dimensional case only if $A^+ = 0$. $\qquad \square$

We will further investigate the so-called **feedback configuration** (or a closed loop) consisting in a system S (we will identify the system with its transfer function) and a controller K which are connected as demonstrated in the following figure
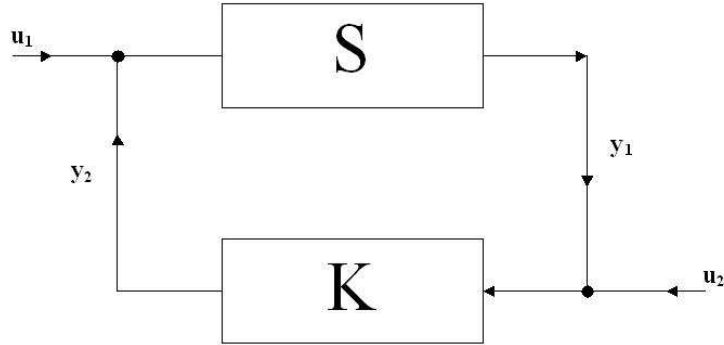
The Laplace transform of output signals y_1, y_2 are given by the following couple of relations

$$y_1 = S(u_1 + y_2), \quad y_2 = K(u_2 - y_1). \tag{1,5,1}$$

The sign minus means that the input y_1 is compared with a (reference) input u_2. If S, K are rational-valued matrix functions then the determinant $det(I + SK)$ is a meromorphic function with a finite number of poles and hence $(I + SK)^{-1}(z)$ exists for $\mathrm{Re}z$ sufficiently big. So we obtain from (1,5,1) (at least formally)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (I + SK)^{-1}S & (I + SK)^{-1}SK \\ -(I + KS)^{-1}KS & (I + KS)^{-1}K \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{1,5,2}$$

We call a feedback configuration (S, K) i-o stable if all entries in (1,5,2) are i-o stable.

The following problem has obvious meaning.

Problem 1.41. *Assume that S is not i-o stable. Is it possible to design a controller K such that (S, K) is i-o stable? Can the set of all stabilizable controller K be described? Suppose that $\mathcal{S}$ is a set of perturbation of given S (e.g. $\mathcal{S} = S + \triangle W$ -additive perturbation, W given and $\triangle$ represents a magnitude of amplitude of W, or multiplicative perturbation: $\mathcal{S} = S(1 + \triangle W)$). Is there a controller K which stabilizes all members of $\mathcal{S}$? (The so-called robust stabilization problem).*

One can formulate another problems, see e.g. [D-F-T].

Example 1.42. Let S represents an integrator, i.e. the corresponding input-output operator is given by $y(t) = \int_0^t u(s)ds$, i.e. $S(z) = \frac{1}{z}$. Put $K(z) = 1$, i.e. K is the Laplace transform of δ_0. Then the matrix (1,5,2) has form

$$\begin{pmatrix} \frac{1}{z+1} & \frac{1}{z+1} \\ -\frac{1}{z+1} & \frac{z}{z+1} \end{pmatrix}$$

Hence (S, K) is i-o stable. This K stabilizes also a derivator $S(z) = z$ ($y(t) = \dot{u}(t), u(0) = 0$). Even though $S(z) = z$ does not belong to $\widehat{\mathcal{A}}(\beta)$ for any β,

$$\frac{z}{z+1} = (e^{-t} + \delta_0)^\wedge(z) \in \widehat{\mathcal{A}}(\beta)$$

and also $\frac{1}{z+1} \in \widehat{\mathcal{A}}(\beta)$ for $\beta > -1$.

If S and K are rational functions it is not difficult to give a necessary and sufficient condition for i-o stability. To generalize such result to matrix-valued functions an approach (see e.g. [V].)

Definition 1.43. (i) Two matrix-valued functions

$$
\begin{aligned}
M : \mathbb{C} &\rightarrow \mathbb{C}^{p \times m} \\
N : \mathbb{C} &\rightarrow \mathbb{C}^{q \times m}
\end{aligned}
$$

$M, N \in \widehat{\mathcal{A}}_-(\beta)$, are called **right - coprime** over $\widehat{\mathcal{A}}_-(\beta)$ if there are $X, Y \in \widehat{\mathcal{A}}_-(\beta)$ such that $XM - YN = I$ on $\overline{\mathbb{C}^+_\beta}$. **Left-coprime** functions are defined via condition $\widetilde{M}\widetilde{X} - \widetilde{N}\widetilde{Y} = I$.
(ii) Let $H \in \widehat{\mathcal{B}}(\beta)$ and let M, N be right (left)-coprime over $\widehat{\mathcal{A}}_-(\beta)$ such that M is a square matrix with $\det M \in \widehat{\mathcal{A}}_\infty(\beta)$ and $H = NM^{-1}(\widetilde{M}^{-1}\widetilde{N})$. Then this factorization is called **right(left)-coprime factorization** of H.

Remark 1.44. The coprime factorization is purely algebraic notion and it can by defined in any ring. A necessary and sufficient condition for $f, g \in H^\infty(\mathbb{C}^+)$ to be coprime:
$$\inf_{z \in \mathbb{C}^+} (|f(z)| + |g(z)|) > 0,$$

is rather deep and it is connected with the so-called Corona problem (L.Carlesson).

Theorem 1.45. *(Vidyasagar) Any matrix-valued function $H \in \widehat{\mathcal{B}}(\beta)$ possesses both left- and right- coprime factorization over $\widehat{\mathcal{A}}_-(\beta)$.*

Proof. The proof is based on a decomposition $H = H_1 + H_2$, where $H_1 \in \widehat{\mathcal{A}}_-(\beta)$, H_2 is a rational function with poles in $\overline{\mathbb{C}^+_\beta}$ and $\lim_{z \to \infty} H_2(z) = 0$. It is not too difficult to construct a right-coprime factorization of $H_2 = NM^{-1}$ (first consider the scalar case). Then $H = (H_1 M + N)M^{-1}$ is a right-coprime factorization. □

Remark 1.46. Factorizations of functions in the Hardy spaces H^p is a classical topic in complex function theory. Their generalization to operator(matrix)-valued functions is an important part of operator theory. See e.g. [N] or in a more easy way [Pa].

Theorem 1.47. *(Youla parametrization). Let $S \in \widehat{\mathcal{B}}(0)$ and $H = NM^{-1}$ be its right-coprime factorization. Then S is i-o stabilizable and all stabilizing controllers $K \in \widehat{\mathcal{B}}(0)$*

are parametrized by $K = (Y + MQ)(X + NQ)^{-1}$ where X, Y are as in Definition 1,42 and $Q \in \widehat{\mathcal{A}}_-$ is such that

$$\inf_{Rez \geq 0} |det[X(z) + N(z)Q(z)]| > 0.$$

Similarly for a left-coprime factorization of S.

Proof. Proof is by a long calculation - see e.g. [C-Z],Section 9,1. $\square$

In the robust stabilization problem, e.g. for aditive or multiplicative perturbations, one need to estimate the norm of admissible perturbations. The following theorem shows a connection with the H^∞-optimalization problem.

Theorem 1.48. *Let $S, K \in \widehat{\mathcal{B}}(0)$, $K \neq 0$ and*

$$\lim_{r \to \infty} \sup_{z \in \mathbb{C}^+, |z| \geq r} |S(z)| = 0.$$

Then K stabilizes $S + W$ for all $W \in \widehat{\mathcal{B}}(0)$ which are i-o stable and $|W|_\infty < \alpha$ if and only if K stabilizes S and

$$|K(I + SK)^{-1}|_\infty < \frac{1}{\alpha}.$$

Proof. If we factorize S, K, W we need to show the existence of the inverse $[I + (S + W)K]^{-1}$ in $\widehat{\mathcal{A}}_-$. In follows from an estimate of $|(S + W)K|_\infty$. For details see e.g. [C-Z], Section 9.2. $\square$

Corollary 1.49. *The best possible margin for additive perturbations is given by solving H^∞-optimization problem*

$$\inf\{K(I + SK)^{-1}|_\infty; \ K \ stabilizes \ S\}.$$

There is a vast number of papers devoted to $H^\infty-$control. see e.g. Chapter IX in [F-F] or [C-Z],Chapters 8,9, and [Pe] and references given there.

1.6   OPTIMAL CONTROL - FINITE HORIZON

We formulate again problem **P4** : Let A be a generator of a C_0-semigroup S on a Hilbert space X, let U be a Hilbert space and let $B \in \mathcal{L}(U,X)$. For given self-adjoints operators $Q, Q_0 \in \mathcal{L}(U)$ which are non-negative and R is positive definite (i.e.$< Ru, u >\geq a|u|^2$ for $a > 0$ and every $u \in U$) and given $x_0 \in X, T > 0$, find $u \in L^2(0,T;U)$ such that the functional

$$J_T(x_0, u) :=$$

$$\int_0^T [< Qx(t), x(t) >_X + < Ru(t), u(t) >_U]dt + < Q_0 x(T), x(T) >_X, \qquad (1,6,1)$$

where $x(\cdot)$ is a mild solution of

$$\dot{x} = Ax + Bu \qquad (1,6,2)$$
$$x(0) = x_0,$$

takes its minimal value.

Around 1960 this problem was solved for finite-dimensional spaces X, U.

Theorem 1.50. *Let $dim\, X$ and $dim\, U$ be finite. Under the above assumptions there exists a unique $\hat{u} \in L^2(0,T;U)$ which minimizes (1,6,1). This $\hat{u}$ is given by the feedback formula*

$$\hat{u} = -R^{-1}B^*P(T-t)\hat{x}(t), \qquad (1,6,3)$$

where $\hat{x}$ is a solution to the equation

$$\dot{x} = [A - BR^{-1}B^*P(T-t)]x(t), \qquad x(0) = x_0, \qquad (1,6,4)$$

and P is a solution to the so-called **Riccati differential (matrix) equation**

$$\dot{P}(t) = A^*P(t) + P(t)A + Q - P(t)BR^{-1}B^*P(t) \qquad (1,6,5)$$
$$P(0) = Q_0.$$

Proof. The existence and uniqueness is easy since $[J_T(x_0, u)]^{\frac{1}{2}}$ is the distance of the point $(Q^{\frac{1}{2}}S(\cdot)x_0, 0, Q_0^{\frac{1}{2}}S(T)x_0)$ from the closed subspace

$$\mathcal{M} = \{-Q^{\frac{1}{2}}\mathcal{B}u, -R^{\frac{1}{2}}u, -Q_0^{\frac{1}{2}}\mathcal{B}_T);\ u \in L^2(0,T;U)\}$$

in the Hilbert space $L^2(0,T;X) \times L^2(0,T;U) \times X$. We notice that the closedness of $\mathcal{M}$ follows from the positive definitness of R which implies that $(R^{\frac{1}{2}})^{-1} \in \mathcal{L}(U)$. The feedback form of $\hat{u}$ and the global existence on $[0,T]$ of a solution of the quadratic differential equation (1,6,5) is more involved and we will discussed those in infinite dimensional case. The reader can consult e.g.[L-M]. $\qquad \square$

If dim$X = \infty$ then we encounter several difficulties arising from the fact that A is no more continuous. This means that either P should have some smoothing property (to get $P(t) \in \mathcal{D}(A)$) or a solution to (1,6,5) should be considered in some weaker sense, e.g. as a week solution:
$P : [0,T] \to \mathcal{L}(X)$ is strongly continuous and for all $x, y \in \mathcal{D}(A)$ the function $t \to< P(t)x, y >$ is absolutely continuous on $[0,T]$, $P(0)x = Q_0x$, and

$$\frac{d}{dt} < P(t)x, y >= \qquad (1,6,6)$$

$$< P(t)x, Ay > + < P(t)Ax, y > + < Qx, y >$$
$$- < P(t)BR^{-1}B^*P(t)x, y >$$

holds for a.a. $t \in [0,T]$.

There are several approaches to the feedback formula (1,6,3). Two of them based on the dynamic programming (due to R. Bellman) and on the minimum principle (due to L.Pontryagin), respectively, will be discussed in Chapter 2. Here we present the so-called direct approach due to G.De Prato. As the name indicates the direct way consists in solving the Riccatti equation (1,6,5) in the weak sense (the first step) and showing that the equation (1,6,4) possesses an evolution family (the second step) and the formula (1,6,3) yields a unique optimal control (the third step).

1st **step**

Lemma 1.51. *If $Q(\cdot) : X \to X$ is strongly continuous on an interval $[0,\tau]$, then a weak solution to*

$$\dot{P} = A^*P(t) + P(t)A + Q(t)$$
$$P(0) = Q_0$$

exists and it is given by

$$P(t)x = S^*(t)Q_0 S(t)x + \int_0^t S^*(t-s)Q(s)S(t-s)x ds,$$

for $x \in X$ and $t \in [0,\tau]$.

Proof. Straighforward. $\qquad\square$

By this lemma, we can solve the integral equation

$$P(t)x = S^*(t)Q_0 S(t)x + \int_0^t S^*(t-s)[Q - P(s)BR^{-1}B^*P(s)]S(t-s)x ds \qquad (1,6,7)$$

using the contraction principle in the Banach space

$$C([0,\tau]; \mathcal{L}_{sa}(X))$$

where $\mathcal{L}_{sa}(X)$ denotes the space of all continuous self-adjoint operators on X. As a computation shows the length of the time interval depends on $\|Q_0\| = \|P(0)\|$. In order to continue this process (notice that the Riccati equation is quadratic and the blow-up of a solution has to be excluded) we need an estimate of the final value $\|P(\tau)\|$.

Lemma 1.52. *Let $P(\cdot)$ be a strongly continuous solution of (1,6,7) on an interval $[0,\tau]$ and let $u \in L^2(0,\tau;U)$. Then*

$$J_\tau(x_0, u) = < P(\tau)x_0, x_0 >_X + \int_0^\tau |R^{\frac{1}{2}}u(s) + R^{-\frac{1}{2}}B^*P(\tau-s)x(s)|_U^2 ds \qquad (1,6,8)$$

where $x(\cdot)$ is a mild solution to (1,6,2) on the interval $[0,\tau]$.

Proof. First. let $\xi \in \mathcal{D}(A)$ and $u \in C^1([0,\tau];U)$. Then a mild solution of (1,6,2) is actually a classical one,i-e. $x(\cdot) \in C^1([0,T]; X_0 \cap C([0,T]; \mathcal{D}(A))$. From Lemma 1,51 and (1,6,6) we have for $t \in [0,\tau]$

$$\frac{d}{dt} < P(\tau-t)x(t), x(t) >= \ldots =- < Qx(t), x(t) > +|R^{\frac{1}{2}}u(t) + R^{-\frac{1}{2}}B^*P(\tau-t)x(t)|^2$$
$$- < Ru(t), u(t) > .$$

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By integrating, we obtain (1,6,7). The general case follows now from the density argument. □

Corollary 1.53. *Let P be a strongly continuous solution to (1,6,7) on an interval $[0, \tau]$. Then*

$$< P(\tau)\xi, \xi > \le J_\tau(\xi, 0).$$

In particular,

$$\|P(\tau)\| \le [\|Q_0\| + \frac{\|Q\|}{2\alpha}]e^{2\alpha\tau},$$

where α is the growth bound of S.

This corollary shows that there is no blow-up in the Riccati equation and hence this has a (unique) weak solution on any finite interval $[0, T]$.

2nd step
Since the perturbation $-BR^{-1}B^*P(T - \cdot)$ in (1,6,4) is pointwise bounded and strongly continuous, the equation (1,6,4) posses an evolution family $U(t, s)$, see e.g. [E-N],Chapter VI,9.

3rd step
In follows from the formulae (1,6,8) that

$$J_T(x_0, \hat{u}) =< P(T)x_0, x_0 >= \inf_{u \in L^2(0,T;U)} J_T(x_0, u),$$

i.e. $\hat{u}$ is an optimal control. If u is another optimal control and x is a corresponding solution to (1,6,2) then, again by (1,6,8) and invertability of R,

$$u(s) = -R^{-1}B^*P(T - s)x(s) \quad \text{for a.a. } s \in (0, T),$$

i.e. x solves (1,6,4) and $u = \hat{u}$ in $L^2(0, T; U)$. Therefore, the following theorem has been proved.

Theorem 1.54. *Let X, U be Hilbert spaces and let Q, Q_0 be self-adjoint operators on X an let R be a positive-definite operator on U. Then for any $x_0 \in X$ and $T < \infty$ there exists a unique $\hat{u} \in L^2(0, T; U)$ which minimizes (1,6,1) over $L^2(0, T; U)$. Moreover, this $\hat{u}$ is given by (1,6,3),(1,6,4), whereP($\cdot$) is a weak solution to (1,6,5).*

For more details the reader consult [B-DP-D-M],Vol II. pp 133-193.

Example 1.55. (Boundary control)
We can look at Bu in an equation $\dot{x} = Ax + Bu$ either as an external force or as heating or cooling which effects the whole body that occupies an region Ω. But this is sometimes impossible and a control can operate only on the (part or) boundary $\partial\Omega$. To be more concrete, let A_m be a closed differential operator defined on a dense subset of

a Banach space X, e.g $L^2(0,T)$, and assume that $\Gamma \in \mathcal{L}(X_m, U)$ is a boundary operator (a restriction of the trace operator, e.g. $\Gamma x = x(0)$). Consider a process given by

$$\dot{x} = A_m x$$
$$x(0) = x_0$$
$$\Gamma x = u. \qquad (1,6,10)$$

We can formulate for (1,6,10) all problems (P1)-(P4). It would be convenient to rewrite (1,6,10) in a more convenient form (1,6,2). We can reach this goal as follows: Suppose that $A = A_m \mid_{Ker\Gamma}$ generates a C_0-semigroup S on X and $\mathcal{R}(\Gamma) = U$. Then for $\lambda \in \rho(A)$ the operator Γ is injective on $ker(\lambda - A_m)$ and hence there is (the so-called Dirichlet operator) $D(\lambda) \in \mathcal{L}(U, X_m)$. Since $X_m \subset X = \mathcal{D}(A_{-1})$, where A_{-1} is a generator of the extrapolation semigroup S_{-1} which acts on the extrapolation space X_{-1} (see Remark 1.8), the operator

$$Bu := (\lambda - A_{-1})D(\lambda)u$$

is well defined (in particular, it does not depend on the choice $\lambda \in \rho(A)$ and the problem (1,6,10) can be rewritten in the form

$$\dot{x} = A_{-1}x + Bu \qquad (1,6,11)$$
$$x(0) = x_0$$

(for detail see e.g.[D-M-S]). A mild solution to (1,6,11) is given by

$$x(t) = S(t)x_0 + \int_0^t S_{-1}(t-s)Bu(s)ds$$

for $x_0 \in X$. To get well-defined cost functional (1,6,1) we need that the convolution term has an continuous extension on $L^2(0,T;U)$ into X (to be defined the term $< Q_0 x(T), x(T) >$) and into $L^2(0,T;U)$. More generally, we can interpret the term $< Qx(t), x(t) >= |Q^{\frac{1}{2}}x(t)|^2$ as a contribution of an observation $C = Q^{\frac{1}{2}}$ and it is reasonable to assume that C is defined only on a subset of X (e.g. to include also a point observation like $Cx = x(\xi_0)$, $\xi_0 \in \overline{\Omega}$). In this case we need that $C(S_{-1} * B)$ has a continuous extension on $L^2(0,T;U)$. We notice that $S_{-1} * Bu \in C([0,T];X)$ for $u \in W^{1,2}(0,T;U)$. For an exhaustive discussion the reader can consult the recent monograph [L-T2]. See also [B-DP-D-M],Vol II, Part II, and [L-Y]. The case of an unbounded observation is discussed in [L-T1] and in a more abstract setting in [Sa1]. See also [D-F-M-S].

If A is a self-adjoint operator bounded from above in a Hilbert space X with a compact resolvent then there is an orthonormal basis (e_n) of X such that $Ae_n = \lambda_n e_n$, where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \to -\infty$.

The spaces

$$X_\alpha := \{x \in X; \sum |\lambda_n|^{2\alpha}| < x, e_n > |^2 < \infty\}$$

from an interpolation scale between X and $X_1 = [\mathcal{D}(A)]$ and the spaces

$$X_{-\beta} := \{(\xi_m) \subset \mathbb{C}; \sum |\lambda_n|^{-2\beta}|\xi_n|^2 < \infty\}, \qquad \beta \in [0,1],$$

form an interpolation scale between X_{-1} and X. If $C \in \mathcal{L}(X_\gamma, X)$, $\gamma < \frac{1}{2}$ and Γ is such that $D(\lambda) \in \mathcal{L}(U, X_\alpha)$, where $\alpha > \gamma$ (i.e. $B \in \mathcal{L}(U, X_{\alpha-1})$), and $Q_0 = 0$, then the cost functional is well-defined and there is a unique optimal control. This optimal control is of feedback type (1,6,3)provided $\alpha > \gamma + \frac{1}{2}$. These results follow

from computations with series and solving a minimization problem in a Hilbert space. Notice that $B^* \in \mathcal{L}(X_{1-\alpha}, U)$ and a problem where the operator $B^* P(T-t)$ in (1,6,3) is defined.

We notice that for the Neumann boundary operator (i.e. $\Gamma_N x = \frac{\partial x}{\partial n}$ on $\partial\Omega$) and elliptic operator A_m ($X_m = W^{2,2}(\Omega)$ we get $D_N(\lambda) \in \mathcal{L}(U, X_\alpha)$ for $\alpha < \frac{3}{4}$. For the Dirichlet boundary operator (i.e. $\Gamma_D x = x|\partial\Omega$) we obtain much worse result, namely $D_D(\lambda) \in \mathcal{L}(U, X_\alpha)$ for $\alpha < \frac{1}{4}$. This means that we cannot assure a feedback (1,6,3) even for $Q \in \mathcal{L}(X)$ (an optimal control $\hat{u}$ still exists in this case) and approximations of $\hat{u}$ are desirable.

1.7 OPTIMAL CONTROL - INFINITE HORIZON

If T is infinite then we consider a cost functional in the form

$$J(x_0, u) = \int_0^\infty [< Qx(t), x(t) > + < Ru(t), u(t) >]dt, \qquad (1,7,1)$$

where Q and R are as in Section 1,6 and x is a solution to (1,6,2). It is easy to see that $J((x_0, u)$ need not be finite for any $u \in L^2(\mathbb{R}^+, U)$ when the semigroup S generated by A is not exponentially stable. We notice that $J(x_0, u) < \infty$ for all $x_0 \in X$ and $u \in L^2(\mathbb{R}^+, U)$ whenever S is exponentially stable. If not we need some stabilizability by B: we say that (A, B, Q) is **open-loop stable** if for any $x_0 \in X$ there is $u \in L^2(\mathbb{R}^+, U)$ such that $J(x_0, u) < \infty$.

Since P_{T_2} is an extension of P_{T_1} for $T_1 < T_2$ (by (1,6,6)) we can define a self-adjoint operator valued function $P(t) = P_T(t)$ for each $t \geq 0$ and any $T \geq t$. If $Q_0 = 0$ we call this P differential Riccati operator.

Proposition 1.56. *Let (A, B, Q) be open-loop stable and let $P(\cdot)$ be a differential Riccati operator. Then the limit*

$$\lim_{t \to \infty} P(t)x := Px$$

exists for all $x \in X$ and $P \in \mathcal{L}_{sa}(X)$ and P solves the so-called **algebraic Riccati equation**

$$< Px, Ay >_X + < Ax, Py >_X + < Qx, y >_X - < R^{-1}B^*Px, B^*Py >_U = 0$$
$$\text{for } x, y \in \mathcal{D}(A) \quad (1,7,2)$$

Proof. By (1,6,9),l we have

$$< P(T)x_0, x_0 >= \inf_{u \in L^2(0,T;U)} J_T(x_0, u) \leq J_T(x_0, \widetilde{u}) \leq J(x_0, \widetilde{u}) < \infty$$

for $\widetilde{u} \in L^2(\mathbb{R}^+, U)$ which existence is guaranted by the open-loop stability. This means that $T \to< P(T)x_0, x_0 >$ is an nondecreasing bounded function for any $x_0 \in X$. Denote its limit by $f(x_0)$ and define (the polarization identity)

$$< Px, y >= \frac{1}{4}[f(x + y) - f(x - y)].$$

Then $P \in \mathcal{L}_{sa}(X)$ (by the uniform boudnedness principle) and $Px = \lim_{t \to \infty} P(t)x$. The right-hand side in (1,6,6) has the finite limit for $t \to \infty$ which is equal to the left-hand side in (1,7,2). Since $t \to< P(t)x, y >$ is a bounded function and its derivative has a finite limit at ∞, this limit has to be zero. $\qquad \square$

The algebraic Riccati equation (1,7,2) can have many self-adjoint solutions. It can by shown that P stated in Proposition 1,55 is a minimal non-negative one, i.e. $< Px, x > \leq < \widetilde{P}x, x >$ for any non-negative self-adjoint solution $\widetilde{P}$ of (1,7,2) and all $x \in X$.

Theorem 1.57. *Let $Q \in \mathcal{L}_{sa}(X)$, $R \in \mathcal{L}_{sa}(U)$ and let Q be non-negative and R be positive-definite. Assume that (A, B, Q) is open-loop stable and $J(x_0, u)$ is given by (1,7,1). Then for any $x_0 \in X$ there is $\hat{u} \in L^2(\mathbb{R}^+, U)$ such that*

$$J(x_0, \hat{u}) = \inf_{u \in L^2(\mathbb{R}^2, U)} J(x_0, U).$$

Moreover, this $\hat{u}$ is given by

$$\hat{u}(t) = -R^{-1}B^*P\hat{x}(t), \tag{1,7,3}$$

where $\hat{x}$ is a solution to

$$\dot{x} = [A - BR^{-1}B^*P]x \tag{1,7,4}$$
$$x(0) = x_0$$

Here P is as in Propositon 1,56.

Proof. The operator P solves the equation (1,6,6) on any interval $[0, T]$ with the initial condition $P(0) = P$. Theorem 1,53 shows that $\hat{u}$ given by (1,7,3), (1,7,4) minimizes the functional

$$\int_0^T [< Qx(t), x(t) > + < Ru(t), u(t) >]dt + < Px(t), x(T) >$$

that has the minimal value $< Px_0, x_0 >$. It follows that

$$\int_0^T [< Q\hat{x}(t), \hat{x}(t) > + < R\hat{u}(t), \hat{u}(t)]dt \leq < Px_0, x_0 >$$

for any T and, therefore, $J(x_0, \hat{u}) \leq < Px_0, x_0 >$. On the other hand, for any $u \in L^2(\mathbb{R}^+, U)$ and $T > 0$, we have (see(1,6,9))

$$< P(T)x_0, x_0 > \leq \int_0^T [Qx(t), x(t) > + < Ru(t), u(t) >]dt \leq J(x_0, u)$$

and, by taking limit,

$$< Px_0, x_0 > \leq J(x_0, u).$$

$\square$

Remark 1.58. The proof of Theorem 1,57 depends essentialy on boundedness of both operators B and C (cf. Example 1,55). The infinite horizon case for a bounded operator C is fully discussed in [L-T2]. See also [B-DP-D-M],Vol.II.PartIII and [L-Y]. The case of an unbounded operator C is discussed in [D-F-M]. The frequency domain approach is used for generally unbounded operators B and C in recent papers of O.Staffans and G.Weiss. The interested reader can consult their webb pages.

Chapter 2

NONLINEAR THEORY

In this part we restrict our attention to systems of nonlinear ordinary differential equations, i.e. $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$. Including of nonlinear partial differential equations would require many prerequisities. The interested reader can consult e.g. [L-Y].

2.1 CONTROLLABILITY FOR LINEAR SYSTEMS WITH CONSTRAINTS

Assume that we need to control an actual linear systems given by (1,2,1) from an initial state x_0 to another state x_1 in given time T and these two points are distant from each other. If there is a control satisfying this requirement then its size has to be large and this fact can cause problems in realization. This means that there are often apriori limitations on the size of admissible controls. We can express such restrictions by requirement that controls take their values in a bounded set $M \subset \mathbb{R}^M$. The additional assumption that M is also a symmetric and convex neighborhood of 0 does not seem to be too restrictive. The case $x_1 = 0$ is more transparent. We denote

$$M_T = L^\infty(0, T; M)$$

$$\mathcal{C}_T = \{x_0 \in \mathbb{R}^n; \exists u \in M_T \text{ such that } x_0 + \int_0^T e^{-sA} Bu(s)ds = 0\}$$

$$\mathcal{C} = \bigcup \mathcal{C}_T.$$

Theorem 2.1.
(i)$\mathcal{C}_T$ is a neighborhood of origin if and only if the system (1,2,1) is controllable.
(ii)$\mathcal{C} = \mathbb{R}^n$ if and only if the system (1,2,1) is controllable and $\operatorname{Re}\sigma(A) \leq 0$.

Proof.
(i) We will prove both implications by contradiction. Notice that the range of operator

$$L_T u := \int_0^T e^{-sA} Bu(s)ds \qquad (2,1,1)$$

is $\mathbb{R}^n$ if and only if (1,2,1) is controllable at T. Assume first that $\mathcal{C}_T$ is not a neighborhood of origin. Then $\mathcal{C}_T$ cannot contain a basis of $\mathbb{R}^n$ and hence there is a non-zero $\widetilde{x}$ which is perpendicular to $\mathcal{C}_T$, i.e.

$$0 = <L_T u, \widetilde{x}> = <u, L_T^* \widetilde{x}>_{L^\infty \times L^1}$$

for all $u \in M_T$. Since M_T is a neighborhood of $0 \in L^\infty(0, T; \mathbb{R}^m)$, $L_T^* \widetilde{x} = 0$ and, by controllability, $\widetilde{x} = 0$. On the other hand if there is $\overline{x} \in Ker L_T^*$, $\overline{x} \neq 0$ (i.e. (1,2,1) is

not controllable) then $\overline{x} \perp \mathcal{C}_T$ what is impossible if $\mathcal{C}_T$ is a neighborhood.

(ii) We proceed again by contradiction.

1) Assume that $\mathcal{C} = \mathbb{R}^n$ and there is $\lambda_0 \in \sigma(A)$ such that $Re\lambda_0 > 0$. Since all admissible u are bounded then an initial value x_0 has to be found such that the corresponding solution x of (1,2,1) does not tend to 0 for $t \to \infty$. But $x_0 \in \mathcal{C}_T$ for some T, i.e. $x(T) = 0$, a contradiction.

If rank $(B, AB, ..., A^{n-1}B) < n$ then it can by seen from the proof of Theorem 1.10 that $\widetilde{x}$ from the proof of (i) can be taken independent on T, i.e. $\widetilde{x} \perp \mathcal{C}$.

2) Assume that the both conditions from the statement of (ii) are satisfied and $\mathcal{C} \neq \mathbb{R}^n$. Since $\mathcal{C}$ is convex, a point $x_0 \in \mathbb{R}^n \backslash \mathcal{C}$ can by separated from $\mathcal{C}$ by some

$$\varphi \in X^*, \; \varphi \neq 0 : \quad \sup_{x \in \mathcal{C}} \varphi(x) \leq \varphi(x_0).$$

We have

$$L_T^* \varphi(s) = B^* e^{-A^* s} \varphi \text{ for } s \in [0, T].$$

Put $w(s) = B^* e^{-A^* s} \varphi$ for $s \in [0, \infty)$. By the statement (i), there is $C > 0$ such that $u \in L^\infty(0, T; \mathbb{R}^m)$ with $|u(s)| \leq C$ for a.a. $s \in [0, T]$ has to belong to M_T. In particular, $u(s) = C\text{sgn}w(s) \in M_T$ and $\varphi(L_T u) = (L_T^* \varphi)(u) = C \int_0^T |w(s)| ds \to \infty$ for $T \to \infty$, since $|w| \notin L^1(0, \infty)$ as it follows from the assumption on $\sigma(A)$. This means that $\sup_{x \in \mathcal{C}} \varphi(x) = \infty$, a contradiction. $\qquad\square$

In order to choose an appropriate M (i.e. a ball) and a time T to send x_0 to $x_1 = 0$ with help of $u_0 \in M_T$ it is convenient to know an estimate of the norm $|u_0 + KerL_T|_{L^\infty(0,T)}$. The following result is a variant of the Lagrange multiplier theorem.

Theorem 2.2. *Let F be a convex continuous weakly coercive functional on a Banach space Y. Let $\mathcal{M}$ be a non-trivial closed subspace Y. Then for $y_0 \in Y$ the following equality holds*

$$\min\{F(y); y \in y_0 + \mathcal{M}\} = \max_{f \in \mathcal{M}^\perp} \inf_{z \in Y} [F(z) + f(z) - f(y_0)]. \qquad (2,1,2)$$

Proof. The existence of minimum in the left-hand side is a standard result (see e.g. [D-M],Theorem 6.2.12). Denote the left-hand side by α and

$$\beta := \sup_{f \in \mathcal{M}^\perp} \inf_{z \in Y} [F(z) + f(z) - f(y_0)]$$

Obviously, $\alpha \leq \beta$. To prove the opposite inequality define the epigraph of F:

$$W = \{(t, z) \in \mathbb{R} \times Y; t \geq F(z), z \in Y\}.$$

This set is convex and its interior is non-empty. Since $IntW$ is disjoint with

$$V = \{(\alpha, y); y \in y_0 + \mathcal{M}\},$$

the sets W and V can be separated by some $\varphi \in (\mathbb{R} \times Y)^*, \varphi \neq 0 :$

$$\sup_V \varphi(\alpha, y) \leq \inf_W \varphi(t, z). \qquad (2,1,3)$$

Such φ has a form

$$\varphi(t, z) = at + \widetilde{f}(z), \quad \widetilde{f} \in Y^*.$$

It follows from (2,1,3) that $\widetilde{f} \in \mathcal{M}^\perp$ and $a > 0$. So we can take $a = 1$ and (2,1,3) has the form

$$\alpha + \widetilde{f}(y_0) \leq \inf_{t \geq F(z)} [t + \widetilde{f}(z)]$$

and the inequality $\alpha \leq \beta$ follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The reader should notice that Theorem 2,1 is only the existence result and it does not say anything about a construction of $u \in \mathcal{C}_T$. If M is moreover closed then M_T is w^*-compact (and convex).Therefore (the Krein-Milman theorem), M_T is a w^*-closure of the convex hull of its extremal points. We denote by $\mathcal{C}(M_T)$ the set of extremal points of M_T. Controls from $\mathcal{C}(M_T)$ are called **bang-bang** controls. For example, if $M = [-1, 1]^m$ then

$$\mathcal{C}(M_T) = \{u = (u_1, ..., u_m); |u_i(t)| = 1 \text{ for a.a. } t \in [0, T] \text{ and all } i = 1, ..., m\}.$$

Theorem 2.3. *(**Bang-bang principle**,*La Salle 1960)
Let M be a bounded, convex and closed subset of $\mathbb{R}^m$ and let L_T be defined by (2,1,1). Then

$$\mathcal{C}_T = -L_T(\mathcal{C}(M_T)) \text{ for any } T > 0.$$

Proof. To avoid technical difficulties we only consider the case $m = 1$, i.e. $Bu = ub$ for some $b \in \mathbb{R}^n$. Choose a basis $e_1, ..., e_n$ of $\mathbb{R}^n$. Then

$$L_T u = \int_0^T (\sum_{i=1}^n <b, e_i> e^{-sA}e_i)u(s)ds =: \int_0^T u(s)d\mu(s),$$

with an absolutely continuous (with respect to the Lebesgue measure) vector measure μ. This shows that L_T is w^*-continuous and hence $(L_T)_{-1}(x) \cap M_T$ is a non-empty w^*-compact convex set for $x \in \mathcal{C}_T$. By the Krein-Milman theorem, this set has an extremal point $\widetilde{u}$. Now, it is sufficient to prove that $\widetilde{u}$ is also an external point of M_T. If $n = 1$ then it is not difficult to show, by contradiction, that $\widetilde{u} \in \mathcal{C}(M_T)$. The same result can be proved for a general $n \in \mathbb{N}$ by induction. We notice that it would be also sufficient to prove that the set of values of the vector measure μ is a compact convex set (a special case of the Lyapunov theorem on vector measures). There exists a generalization to a finite system of vector measures which yields the statement for $m > 1$. For another proof the reader can consult [L-M]. $\qquad$ $\square$

If $x_0 \in \mathcal{C}$ then $x_0 \in \mathcal{C}_T$ for some time $T > 0$ and hence also $x_0 \in \mathcal{C}_\tau$ for all $\tau > T$. It follows from w^*-compactness of M_t and w^*-continuity of L_t that

$$\bigcap_{t > t_0} \mathcal{C}_t = \mathcal{C}_{t_0}.$$

This means that there exists a minimal time $\hat{t}$ for which $x_0 \in \mathcal{C}_{\hat{t}}$. A corresponding control $\hat{u}$ which steers x_0 into 0 at the minimal time $\hat{t}$ can be characterized by the following special form of the maximum principle (see Section 2,3).

Theorem 2.4. *Let M be a convex compact neighborhood of the origin. Let $x_0 \in \mathcal{C}$ and let $\hat{u} \in M_{\hat{t}}$ be a control steering x_0 to 0 at the minimal time $\hat{t}$. Then there exists a solution $\hat{y}$ to the adjoint equation*

$$\dot{y} = -A^* y \tag{2.1}$$

for which

$$< \hat{y}(t), B\hat{u}(t) > = \max_{u \in M} < \hat{y}(t), Bu > \quad \text{for a.a. } t \in [0, t^*] \tag{2,1,4}$$

Proof. Denote

$$K_{x_0}(t) = \{x; \exists u \in M_t \text{ s.t. } x = e^{tA} x_0 + \int_0^t e^{(t-s)A} Bu(s) ds\}.$$

By the definition of $\hat{t}$, $0 \in K_{x_0}(\hat{t})$. It is clear that $0 \in \partial K_{x_0}(\hat{t})$, actually. Since $K_{x_0}(\hat{t})$ is closed and convex, there exists a supporting hyperplane to $K_{x_0}(\hat{t})$ at 0. Such a hyperplane is given by its normal vector y_0 which can be chosen such that $< x, y_0 > \leq 0$ for all $x \in K_{x_0}(\hat{t})$. Put $\hat{y}(t) := e^{-tA^*} e^{\hat{t}A^*} y_0$. Then $\hat{y}$ solves the adjoint equation and

$$< e^{\hat{t}A} x_0, y_0 > + \int_0^{\hat{t}} < e^{(\hat{t}-s)A} Bu(s), y_0 > ds \leq 0$$

for all $u \in M_{\hat{T}}$. It follows that

$$\int_0^{\hat{t}} < Bu(s), \hat{y}(s) > ds \leq \int_0^{\hat{t}} < B\hat{u}(s), \hat{y}(s) > ds. \tag{2,1,5}$$

Suppose that

$$< B\hat{u}(s), \hat{y}(s) > \; < \; \max_{u \in M} < Bu, \hat{y}(s) >$$

on a subset $P \subset (0, \hat{t})$ of a positive measure. Since the function v defined on P by $v(s) = \max_{u \in M} < Bu, \hat{y}(s) >$ is measurable, we can define

$$\widetilde{u}(s) = v(s) \qquad s \in P$$

$$\hat{u}(s) \qquad s \in (0, \hat{t}) \backslash P$$

to get a contradiction with (2,1,5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 2.5. In accordance with classical mechanics the function

$$H(y, x, u) := < y, Ax + Bu >$$

is called the Hamiltonian. If $\hat{x}$ is a solution to (1,2,1) for $\hat{u}$ and $\hat{H}(y, x, t) : H(y, x, \hat{u}(t))$ then $\hat{x}$ and $\hat{y}$ solve the Hamilton equations

$$\dot{x} = \frac{\partial \hat{H}}{\partial y}, \qquad \dot{y} = -\frac{\partial \hat{H}}{\partial x}.$$

Example 2.6. Consider the equation $\ddot{x} - x = u$ with $M = [-1, 1]$. Solutions of the adjoint equation have the form

$$y(t) = \alpha e^t + \beta e^{-t}.$$

Hence an optimal $\hat{u}(t) = \text{sign}(\alpha e^t - \beta e^{-t})$ and it has at the most one switching point (cf. with the equation $\ddot{x} + x = u$). A solution $\hat{x}$ corresponding to $\hat{u}$ is a solution either to $\ddot{x} - x = 1$ or to $\ddot{x} - x = -1$ and it is convenient to find it by solving these equations backwards, i.e. with the initial condition $x(0) = \dot{x}(0) = 0$.

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2.2 CONTROLLABILITY AND STABILIZABILITY OF NONLINEAR SYSTEMS

The result of this section will be based on a method of linearization, so we restrict our attention to behaviour of nonlinear systems in a neighborhood of a stationary point. We assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is differentiable on a neighborhood of origin and $f(0,0) = 0$.

Definition 2.7. A system

$$\dot{x} = f(x, u) \tag{2,2,1}$$

is said to be locally controllable at $0 \in \mathbb{R}^n$ at time T if for any $\varepsilon > 0$ there is $\delta > 0$ such that for $a, b \in \mathbb{R}^n$, $|a| < \delta$, $|b| < \delta$ there exist $\tau \in [0, T]$ and $u \in C([0, \tau]; \mathbb{R}^m)$ for which a solution φ of the problem

$$\dot{x} = f(x, u(t)) \tag{2,2,2}$$
$$x(0) = a$$

satisfies the conditions $\varphi(\tau) = b$ and $\|\varphi\|_{C[0,\tau]} < \varepsilon$.

Denote

$$A := f_1'(0,0)$$
$$B := f_2'(0,0), \text{ i.e.}$$
$$\dot{y} = Ay + Bu \tag{2,2,3}$$

is the linearization of (2,2,1) at the origin.

Theorem 2.8. *Let the linear system (2,2,3) is controllable. Then the nonlinear system (2,2,1) is locally controllable at $0 \in \mathbb{R}^n$ at any positive time.*

Proof. For the sake of simplicity we suppose that $T = 1$ and show how $a = 0$ can by steered to any point of sufficiently small neighborhood of 0. A case of $a \neq 0$ will follow by considering the reverse system

$$\dot{x} = -f(x, \widetilde{u})$$
$$x(0) = 0$$
$$\widetilde{u}(t) = u(1 - t).$$

We proceed in several steps:
1) It follows from the variation of constants formula and the Gronwall inequality that for any $\varepsilon > 0$ there is $\triangle > 0$ such that for arbitrary $u \in C[0, T]$ with $\|u\|_C < \triangle$ a solution of (2,2,1) satisfying $x(0) = 0$ is defined on the interval $[0, 1]$ and $\|x\|_{C[0,1]} < \varepsilon$.
2) If (2,2,3) is controllable then, by the Banach open mapping theorem, there is $\delta > 0$ such that the equation $\xi = \mathcal{B}_1 u$ is solvable in $C([0, 1]; \mathbb{R}^m)$ for any $|\xi| < \delta$.
3) As a consequence of (2) we can find $u_1, ..., u_n \in C([0, 1]; \mathbb{R}^m)$ for which solutions $x_1, .., x_n$ of (2,2,3) vanishing at 0 are such that their values $x_1(1), ..., x_n(1)$ are linearly independent.
4) Put $u(t, \alpha) = \sum_{i=1}^{n} \alpha_i u_i(t)$ for $\alpha = (\alpha_1, .., \alpha_n)$ such that $\sum |\alpha_i| < \triangle$, i.e. $\|u(\cdot, \alpha)\|_{C[0,1]} < \triangle$, where $\triangle$ is as the first step. Then solutions x_α of

$$\dot{x} = f(x, u(t, \alpha))$$
$$x(0) = 0$$

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are all defined on the interval $[0, 1]$. By the inverse function theorem (see e.g. [D-M], Theorem 4.1.1), the mapping $\phi(\alpha) := x_\alpha(1)$ is a local diffeomorphism at 0 (have in view that the collumus of $\phi'(0)$ are $x_i(1)$). $\square$

Example 2.9. The linear system

$$\dot{x} = -x + u$$
$$\dot{y} = -y$$

is not controllable but the nonlinear system

$$\dot{x} = -x + u$$
$$\dot{u} = -y + x^3$$

is locally controllable at 0 for sufficiently large T. This can be shown similarly as in Example 2,6 by considering $u(t) = 1$ and $u(t) = -1$.

Definition 2.10. A system (2,2,1) is called exponentially stabilizable if there exists an feedback $u = \varphi(x)$, $\varphi(0) = 0$, such that the origin is exponentially stable for the equation

$$\dot{x} = f(x, \varphi(x)), \tag{2,2,4}$$

i.e. there are positive constants $\alpha, C, \triangle$ such that

$$|x(t, x_0)| \le Ce^{-\alpha t}|x_0| \text{ for } t \in [0, \infty) \tag{2,2,5}$$

provided $|x_0| < \triangle$. Here $x(\cdot, x_0)$ stands for a solution to (2,2,4) for which $x(0, x_0) = x_0$.

We recall the well-known Lyapunov-Perron theorem on the so-called linearized stability:

Theorem 2.11. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 in the neighborhood of $0 \in \mathbb{R}^n$.*
(i)If $Re\,\sigma(f'(0)) < 0$ then the origin is exponentially stable for the equation $\dot{x} = f(x)$.
(ii) If there is $\lambda \in \sigma(f'(0))$ with $Re\lambda > 0$ then the origin is not stable (in the sense of Lyapunov) for the equation $\dot{x} = f(x)$.

Corollary 2.12. *A system (2,2,1) is exponentially stabilizable if and only if its linearization (2,2,3) is exponentially stabilizable.*

Proof.
1) Assume that a feedback φ stabilizes exponentially (2,2,1) and (2,2,5) holds. Choose $\beta \in (0, \alpha)$ and put $y(t) = e^{\beta t}x(t)$, where x is a solution of (2,2,4). Then y solves the equation

$$\dot{y} = [A + B\varphi'(0) + \beta I]y + h(e^{-\beta t}y),$$

where $h(x) = o(|x|)$. It follows from Theorem 2,11 (ii) and (2,2,5) that

$$Re\sigma(A + B\varphi'(0)) \le -\beta < 0$$

i.e. (2,2,3) is exponentially stabilizable by $Fx = \varphi'(0)x$.
2) Let a feedback $u = Fx$ exponentially stabilizes (2,2,3). Theorem 2,11(i) shows that the equation $\dot{x} = f(x, Fx)$ is exponentially stable. $\square$

Now we turn to the problem of robust stability. Assume that (2,2,1) is exponentially stabilizable and let $u = Fx$ (see the second part of the proof of Corollary 2,12) stabilize (2,2,1). Can F stabilize also an equation

$$\dot{x} = f(x, u) + g(x) \tag{2,2,6}$$

provided that g is "small"? Suppose that (2,2,5) holds for solutions to $\dot{x} = f(x, Fx)$ and $g \in C^1$. If $|g'(0)| < \frac{\alpha}{C}$ then a standard using of the Gronwall lemma shows that the linear equation

$$\dot{x} = [A + BF + g'(0)]x$$

is exponentially stable. By Corollary 2,12, the equation (2,2,6) is exponentially stabilizable by $u = Fx$. Denote the stability margin of (2,2,1) by

$$s_f := \sup\{\frac{\alpha}{C}; \varphi \text{ is a stabilizing feedback with } (2,2,5)\}$$

An upper estimate of s_f can be useful.

Proposition 2.13. *Denote* $\triangle = \sup_{|x|=1} dist(Ax, \mathcal{R}(B))$. *Then*

$$s_f \leq \frac{\|A\|^2}{\triangle} \tag{2,2,7}$$

Proof. We may assume that both f and φ are linear and $\triangle > 0$. Choose $\beta \in (0, 1)$ and let z be such that $|z| = 1$ and

$$\min_{u \in \mathbb{R}^m} |Az - Bu| = |Az + B\overline{u}| > \beta\triangle.$$

Put $Az + B\overline{u} = w$. We have

$$< Ax + Bu, w > = < w, w > - < A(x - z), w > \quad \text{for all } x \in \mathbb{R}^n, u \in \mathbb{R}^m,$$

as it follows from the characterization of minimizer. In particular,

$$< Ax + Bu, w > \geq |w|[|w| - \|A\||x - z|]. \tag{2,2,7}$$

1) Let F be a stabilizing feedback for which (2,2,5) holds and let x be a solution to

$$\dot{x} = Ax + BFx - BFz$$
$$x(0) = z,$$

i.e. a solution to

$$\dot{x} = Ax + Bu$$
$$x(0) = z$$
$$\text{with } u(t) = F[x(t) - z].$$

Then

$$x(t) - z = \int_0^t e^{(t-s)(A+BF)} Az ds \text{ and}$$

$$|x(t) - z| \leq \frac{C}{\alpha}|Az| \leq \frac{C}{\alpha}\|A\|, \ t \geq 0. \tag{2,2,9}$$

2) Choose now any control $u(\cdot)$ and let x be a solution to (2,2,3) with $x(0) = z$. Then, by (2,2,7),

$$< \dot{x}(t), w > = < Ax(t) + Bu(t), w > \geq |w|[|w| - \|A\| |x(t) - z|].$$

This implies

$$|x(t) - z| \geq t|w|^2[|w| - r\|A\|]$$

whenever $|x(t) - z| \leq r < \frac{|w|}{\|A\|}$. It means that

$$\sup_{t \geq 0} |x(t) - z| \geq \frac{|w|}{\|A\|}.$$

This inequality together with (2,2,8) and the choice of w yields the desired inequality (2,2,7). □

Remark 2.14. An idea of linerization can be also used through transforming a nonlinear system in a neighborhood of stationary point into a linear one. This procedure is in certain sense similar to the notion of normal forms or to processes in differential geometry. The interested reader can consult e.g. [I].

2.3 PONTRYAGIN MAXIMUM PRINCIPLE

This section is devoted to a necessary condition for the minimization of a cost functional under constrains given by a system of (nonlinear) differential equations. First of all we remind some basic results (see e.g.[D-M]).

Proposition 2.15. *Let X be a linear space and $M \subset X$. Assume that $J : X \to \mathbb{R}$ has a minimum with respect to M at a point x_0. If for $y \in X$ there is $\delta > 0$ such that $x_0 + ty \in M$ for all $t \in [0, \delta)$ then the directional derivative*

$$\partial J(x_0, y) = \lim_{t \to 0+} \frac{J(x_0 + ty) - J(x_0)}{t} \geq 0$$

provided that it exists. Moreover, if x_0 is an algebraic interior point of M and $\partial J(x_0, y)$ exists then $\partial J(x_0, y) = 0$.

Proposition 2.16. *Let M be a convex set in X and let J be a convex functional on M. Suppose that for $x_0 \in M$ one has $\partial J(x_0, y) \geq 0$ or all $y \in X$ for which $x_0 + y \in M$. Then*

$$\min_{x \in M} J(x) = J(x_0).$$

Proof. From the convexity we have

$$J(x) - J(x_0) \geq \frac{J(x_0 + t(x - x_0)) - J(x_0)}{t} \geq \partial J(x_0, x - x_0)$$

for all $x \in M$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

We want to apply these results to the optimal control problem: For given $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, $\phi : \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$, $M \subset \mathbb{R}^n$ convex, minimize $\phi(x(t_1)$ with respect to solutions

$$\begin{aligned}
\dot{x} &= f(t, x, u) \qquad\qquad\qquad\qquad\qquad\qquad\qquad (2,3,1) \\
x(t_0) &= x_0
\end{aligned}$$

when $u \in L^\infty(t_1, t_2; M)$.

This formulation is rather vague. We admit measurable controls since we know that several solutions have a bang-bang features (cf. Section 2,1). If we consider measurable controls then solutions of (2,3,1) are only absolutely continuous functions such that the equation is satisfied almost everywhere. This means that we need to assume that $g(t, x) := f(t, x, u(t))$ satisfies the Caratheodory conditions. It is still not clear whether any solution to (2,3,1) has to be defined on the whole interval $[t_0, t_1]$. So we restrict controls to a set $\mathcal{K} \subset L^\infty(t_1, t_2; M)$ consisting of such u for which a solution to (2,3,1) exists on $[t_0, t_1]$. Moreover we assume some smoothness of f that assures the unicity of a solution.

Now we need to compute directional derivatives of ϕ with respect to an implicit variable u. To avoid difficulties which can be seen from the further computation, we compute the directional derivative for a special class of directions. Moreover, we take $\widetilde{u} \in \mathcal{K}$ which is piecewise continuous and left-continuous on (t_0, t_1). Let $\widetilde{x}$ be a solution of (2,3,1) with this $\widetilde{u}$.Define admissible directions in the following way (Pontryagin's

idea of a needle variation). Let $\widetilde{u} \in \mathcal{K}$ and $\widetilde{x}$ be a solution to (2,3,1). Choose $\tau \in (t_0, t_1]$ and $v \in M$. For sufficiently small $\varepsilon > 0$ a solution to

$$\dot{y} = f(t, x, v)$$
$$y(\tau - \varepsilon) = \widetilde{x}(\tau - \varepsilon)$$

does exist on $[\tau - \varepsilon, \tau]$. Denote this solution by y. By the continuous dependence on initial conditions a solution to

$$\dot{x} = f(t, x, \widetilde{u})$$
$$x(\tau) = y(\tau)$$

exist on $[\tau, t_1]$ (again for small ε). This means that the controls

$$u^\varepsilon(t) = v \qquad t \in [\tau - \varepsilon, \tau]$$
$$= \widetilde{u}(t) \qquad \text{for other } t$$

has the same property as $\widetilde{u}$ for all sufficiently small ε. If x^ε is a solution to (2,3,1) corresponding to u^ε then

$$x^\varepsilon = \widetilde{x}(t), \qquad t \in [t_0, \tau - \varepsilon)$$
$$= \widetilde{x}(\tau - \varepsilon) + \int_{\tau - \varepsilon}^t f(s, x^\varepsilon(s), v)ds, \qquad t \in [\tau - \varepsilon, \tau]$$
$$= x^\varepsilon(\tau) + \int_\tau^t f(s, x^\varepsilon(s), \widetilde{u}(s))ds, \qquad t \in (\tau, t_1].$$

With help of derivation with respect to initial conditions we get that

$$z(t) := \frac{\partial x^\varepsilon}{\partial \varepsilon} \mid_{\varepsilon = 0}$$

satisfies

$$z(t) = 0, \qquad 0 \le t < \tau \tag{2,3,2}$$
$$= f(\tau, \widetilde{x}(\tau), v) - f(\tau, \widetilde{x}(\tau), \widetilde{u}(\tau)) + \int_\tau^t f_2'(s, \widetilde{x}(s), \widetilde{u}(s))z(s)ds, \qquad t \in (\tau, t_1).$$

Assume that a cost functional ϕ has continuous partial derivatives and let w be a solution of the adjoint equation

$$\dot{w}(t) = -(f_2')^*(t, \widetilde{x}(t), \widetilde{u}(t))w(t) \tag{2,3,3}$$
$$w(t_1) = -\phi'(\widetilde{x}(t_1)).$$

Then

$$\frac{d}{dt} < w(t), z(t) >= 0 \text{ for all } t$$

and, therefore,

$$-\frac{d}{d\varepsilon} J(u^\varepsilon) \mid_{\varepsilon = 0} = - < \phi'(\widetilde{x}(t_1)), z(t_1) >=< w(t_1), z(t_1) >=$$
$$= < w(\tau), z(\tau) >=< w(\tau), f(\tau, \widetilde{x}(\tau), v) - f(\tau, \widetilde{x}(\tau), \widetilde{u}(\tau)) >$$

So we arrive to the so-called **maximum principle** due to Pontryagin.

Theorem 2.17. *Let* $f : \mathcal{M} := [t_0, t_1] \times G_1 \times G_2 \to \mathbb{R}^n$ *be continuous,* G_1, G_2 *be open sets in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$, *respectively. Let* f *have continuous partial derivatives with respect to* x *and* u *- variables in* $\mathcal{M}$. *Assume that* $\phi : G_1 \to \mathbb{R}$ *has continuous partial derivatives in* G_1. *Let* $\widetilde{u}$ *be piecewise continuous and left-continuous on the interval* $[t_0, t_1]$ *in which* $\phi(x(t_1))$ *takes its minimum over* $u \in L^\infty(t_0, t_1, M)$ *(where* x *is a solution to (2,3,1)). Let* w *be a solution to (2,3,3). Then for all* $v \in M$ *and* $\tau \in [0, t_1]$ *the following inequality holds*

$$< w(\tau), f(\tau, \widetilde{x}(\tau), \widetilde{u}(\tau)) > \geq < w(\tau), f(\tau, \widetilde{x}(\tau), v) >, \qquad (2,3,4)$$

where $\widetilde{x}$ *is a solution to (2,3,1) corresponding to* $\widetilde{u}$.

The function

$$H(t, x, u) := < w(t), f(t, x, u) >$$

is often called the Hamiltonian and the necessary condition reads as

$$\max_{v \in M} H(t, \widetilde{x}(t), v) = H(t, \widetilde{x}(t), \widetilde{u}(t)).$$

It is evident that one can consider another directional derivatives to obtain further necessary conditions. The reader can be confused by our choice of a cost functional $\phi(x(t_1))$. But a more general functional, e.g.

$$J(u) = \int_{t_0}^{t_1} L(s, x(s), u(s)) ds + G(x(t_1)) \qquad (2,3,5)$$

can be reduced to the previous one by adding one more equation to (2,3,1), namely

$$\dot{x}_0(t) = L(s, x_1(s), ..., x_n(s), u(s)) \qquad (2,3,6)$$
$$x_0(t_0) = 0.$$

As an application of Theorem 2,17 we present the existence of an optimal control for LQ-problem (cf. Section 1,5).

Theorem 2.18. *Let* $L : \mathcal{M} \to \mathbb{R}$ *be continuous and have continuous partial derivatives with respect to* $x - u$ *variable and let* L *be convex on the convex set* $G_1 \times G_2$ *for all* $t \in [t_0, t_1]$. *Let* $G : G_1 \to \mathbb{R}$ *has continuous partial derivatives. Let* $A, B : [t_0, t_1] \to \mathbb{R}^{n \times n}, \mathbb{R}^{m \times m}$ *be continuous. Then a control* $\widetilde{u}$ *in the set of piecewise continuous and left-continuous functions* $[t_0, t_1] \to G_2$ *minimizes*

$$J(u) := \int_{t_0}^{t_1} L(s, x(s), u(s)) ds + G(x(t_1)),$$

where x *is a solution to*

$$\dot{x} = A(t)x(t) + B(t)u(t)$$
$$x(t_0) = x_0,$$

if and only if

$$< \widetilde{w}(t), B(t)v > \leq L_3'(t, \widetilde{x(t)}, \widetilde{u}(t), \widetilde{u}(t)v$$

for all $v \in G_2$ *such that* $\widetilde{u}(t) + v \in G_2$. *Here* $\widetilde{w}$ *is a solution to*

$$\dot{\widetilde{w}}(t) = -A^*(t)\widetilde{w}(t) + L_2'(t, \widetilde{x}(t), \widetilde{u}(t))$$
$$\widetilde{w}(t_1) = -G'(\widetilde{x}(t_1)).$$

Proof. It follows directly from Theorem 2,17 with adding the equation (2,3,6). The sufficiency part follows from Proposition 2,16. □

We recommend to the reader to deduce the Riccati differential equation from Theorem 2,18.

Theorem 2,17 cannot be used for solving the moon landing problem since there are constraints at the end point t_1, namely $x(t_1) = v(t_1) = 0$. The end point is not given and and there is no fix time interval. We will not go into details and refer the reader to e.g. [P-B-G],[A-F],[F-R].

2.4 HAMILTON-JACOBI-BELLMAN EQUATION

In this section we will present an approach giving a sufficient condition for minimization of a functional (2,3,5) in which x is a solution to (2,3,1). We denote by $\mathcal{K}(\xi, \sigma, \tau)$ the set of all $u \in L^\infty(\sigma, \tau, M)$, ($M$ convex $\subset \mathbb{R}^m$) such that a solution to

$$
\begin{aligned}
\dot{x}(t) &= f(t, x(t), u(t)) \\
x(\sigma) &= \xi,
\end{aligned}
$$

exists (in the generalized sense) on the interval $[\sigma, \tau]$. Such a (unique) solution is denoted by $\varphi(t; \sigma, \xi, u)$. Assumptions on f are those which guarantee the local existence and uniqueness of a solution.

Actually, there are two different problems. The firs one concerns the existence of an optimal control. If $\mathcal{K}(x_0, t_1, t_2)$ is compact in certain topology (usually in w^*- topology) and the nonlinear functional $u \to J(u)$ is continuous in this topology (a weaker assumption is sufficient - see [D-M], Ch.6) then there exists an optimal control. The second problem regards the synthesis of this control.

The following procedure is due to R. Bellman and it is called the dynamic programming. It is based on the study of propertied of the **value function** V. This function is defined by

$$
V(t, x) := \inf_{u \in \mathcal{K}(x, t, t_1)} \left[\int_t^{t_1} L(s, \varphi(s; t, x, u) u(s)) ds + G(\varphi(t_1; t, x, u)) \right], \qquad (2,3,7)
$$

where we suppose that $L : [t_0, t_1] \times \mathbb{R}^n \times M \to \mathbb{R}$ is continuous. The notation L comes from variational formulation of the Newton law in classical mechanics, where L is the Lagrange function.

Our aim is to find a differential equation that is satisfied by a value function.

Lemma 2.19. *For any $t_0 \le t \le \tau \le t_1$ and $u \in \mathcal{K}(x, t, t_1)$ the inequality*

$$
V(t, x) \le \int_t^\tau L(s, \varphi(s; t, x, u), u(s)) d\tau + V(\tau, \varphi(\tau; t, x, u)) \qquad (2,3,8)
$$

holds. Moreover, if $\widehat{u}$ is optimal (i.e. $V(t, x) = \int_t^{t_1} L(s, \varphi(s, t, x, \hat{u}), \hat{u}(s)) ds + G(\varphi(t_1; t, x, \hat{u})$ the equality in (2,3,8) is true.

Proof. Choose $u_1 \in \mathcal{K}(\varphi(\tau; t, x, u), s, t_1))$ for which V in the right-hand side is almost achieved and concatenate $u\mid_{[t,s]}$ and u_1. The equality for an optimal $\hat{u}$ follows from the observation that the restriction of $\hat{u}$ is again optimal. $\qquad \square$

Corollary 2.20. *Let the greatest lower bound in (2,3,7) is achieved at a continuous control $\hat{u}$. Assume that L is continuous and the value function V is continuously differentiable. Then*

$$
\frac{\partial V}{\partial t}(t, x) = -L(t, x, \hat{u}(t)) - \langle \nabla_x V(t, x), f(t, x, \hat{u}(t) \rangle \qquad (2,3,9)
$$

and

$$
\frac{\partial V}{\partial t}(t, x) = -\min_{v \in M} [L(t, x, v) + \langle \nabla_x V(t, x), f(t, x, v \rangle]. \qquad (2,3,10)
$$

Proof. By differentiating the right-hand side in (2,3,8) with respect to τ we obtain

$$L(\tau,\varphi(\tau,t,x,\hat{u}(\tau)))+\frac{\partial V}{\partial t}(\tau,\varphi(\tau,t,x,\hat{u}))+\nabla_x V(\tau,\varphi(\tau,t,x,u)),f(\tau,\varphi(\tau,t,x,\hat{u},\hat{u}(\tau))>.$$

Since $\hat{u}$ is optimal, the equality in (2,3,8) holds and the derivative of the left-hand side with respect to τ is zero. Putting $\tau = t$ we obtain (2,3,9). If u is any control in $\mathcal{K}(x,t,t_1)$ and $w(\tau,x,u)$ denotes the right-hand side in(2,3,8), we have

$$w(\tau,\hat{x}(\tau),\hat{u}) - w(\tau,x(\tau),u) \le 0. \tag{2,3,11}$$

Here $x(\cdot),\hat{x}(\cdot)$ are solutions starting at x at time t. From that we can conclude that the function in (2,3,11) is a non-increasing function of τ and hence

$$-\frac{\partial V}{\partial t}(t,x) = L(t,x,\hat{u}(t)) + <\nabla_x V(t,x),f(t,x,\hat{u}(t)>\le$$
$$\le L(t,x,u(t)) + <\nabla_x V(t,x),f(t,x,u(t)>,$$

i.e. (2,3,10) is true. $\qquad\square$

It is a quite delicate matter to prove the differentiability of value function for a given cost functional. The interested reader can find more details e.g. in Chapter 4 of [F-R].

The partial differential equation (2,3,10) is called the Hamilton-Jacobi-Bellman equation. As quite often in partial differential equations,a an appropriate notion of solution is not obvious. The concept of viscosity solutions (see e.g. [C-I-L] or [B-CD]) seems to be a good one. It is far from the scope of these lectures to explain details. Instead assume that W is classical solution of (2,3,10) satisfying the condition

$$W(t_1,x) = G(x). \tag{2,3,12}$$

In order to prove that W is actually the value function V we need the following result which is often called a verification lemma.

Lemma 2.21. *Let $\widetilde{V} : [t_0,t_1] \times \mathbb{R}^n \to \mathbb{R}$ have the following properties:*
(i) For any $t_0 \le t < \tau \le t_1$, $x \in \mathbb{R}^n$ and $u \in \mathcal{K}(x,t,t_1)$ the function $\widetilde{V}$ satisfies (2,3,8).
(ii) For any $t_0 \le t < \tau \le t_1$, $x \in \mathbb{R}^n$, there is $\widetilde{u} \in \mathcal{K}(x,t,t_1)$ for which the equality in (2,3,8) takes place.
(iii) $\widetilde{V}(t_1,x) = G(x)$ for $x \in \mathbb{R}^n$.
Then $V = \widetilde{V}$.

Proof. It is straightforward. $\qquad\square$

The simplest situation in the Hamilton-Jacobi-Bellman equations is that there is a unique minimizer in (2,3,10). Define the Hamiltonian

$$H(t,x,v,\lambda) = L(t,x,v) + <\lambda,f(t,x,v)>$$

and suppose that $\alpha(t,x,\lambda)$ is unique $v \in M$ which minimizes H, i.e.

$$H(t,x,\alpha(t,x,\lambda),\lambda) = \min_{v\in M} H(t,x,v,\lambda) \tag{2,3,13}$$

and, moreover α is continuous on $[t_0,t_1] \times \mathbb{R}^n \times \mathbb{R}^n$. Notice that the continuity is also a strong assumption. This α generally does not belong to $\mathcal{K}(x,t_0,t_1)$. We need to assume that α yields an admissible feedback in the following sense.

Definition 2.22. A function $F : [t_0, t_1] \times \mathbb{R}^n \to M$ is an admissible feedback if for any $\tau \in [t_0, t_1]$ and each $x \in \mathbb{R}^n$ there is a unique $u \in \mathcal{K}(x, \tau, t_1)$ such that a corresponding solution $\varphi(\cdot, \tau, x, u)$ satisfies

$$\begin{aligned}\varphi(\tau) &= x \\ u(t) &= F(t, \varphi(t)), \qquad t \in [\tau, t_1].\end{aligned}$$

Theorem 2.23. *Let there be a continuous function α satisfying (2,3,11) and let V be a classical solution to (2,3,10) which satisfies (2,3,12), such that the funcftion*

$$F(t, x) = \alpha(t, x, \nabla_x V(t, x))$$

is an admissible feedback. Then V is the value function and $F(t, x_0)$ is an optimal control.

Proof. is done by verification of properties in Lemma 2,21. $\qquad\square$

Perhaps, the reader has obtain an opinion that Theorem 2,23 can be scarcely applied. She/he is mostly right but there is a standard example.

Example 2.24. Assume that $f(t, x, u) = A(t)x + B(t)u$, where A, B are continuous matrices of corresponding orders, and

$$L(t, x, u) = < Qx, x > + < Ru, u >$$

with a non-negative symmetric matrix Q and a positive definite matrix R. Moreover, suppose that $G(x) = < Q_0 x, x >$ with a non-negative symmetric matrix Q_0 (cf. Section 1,6). An easy computation shows that

$$\alpha(t, x, \lambda) = -\frac{1}{2}R^{-1}B^*(t)\lambda \qquad (2,3,14)$$

is a unique minimum in (2,3,13). We will try to find a solution to (2,3,10) for which $x \to \nabla_x V(t, x)$ is linear in x, say

$$\nabla_x V(t, x) = < P(t)x, x >, \qquad (2,3,15)$$

i.e. $\nabla_x V(t, x) = 2P(t)x$. Assuming (2,3,15) we get a feedback in the form

$$F(t, x) = -R^{-1}B^*(t)P(t)x.$$

If P is continuous on $[t_0, t_1]$ then this feedback is admissible, since φ is a solution to

$$\begin{aligned}\dot{y} &= [A(t) - B(t)B^*P(t)]y \\ y(\tau) &= x.\end{aligned}$$

If we substitute φ for x and F for u in (2,3,8) we find that $\widetilde{P}(t) := P(t_1 - t)$ satisfies the Riccati differential equation (1,6,5) and the value function V is continuously differentiable.

Bibliography

[A-B-H-N] Arendt W., C.Batty, M.Hieber, F.Neubrander,*Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser 2001.

[A-I] Avdonin S.A., S.A.Ivanov, *Families of Exponentials*, Cambridge Univ.Press 1995.

[A-F] Athans M., P.L.Falb, *Optimal Control*, Springer 1975.

[B] Byrnes C.I., *Pole asignement by output feedback*, pp.31-78 in Lecture Notes in Control and Information Sciences 135, Springer 1989.

[B-CD] Bardi M., I.Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser 1997.

[B-DP-D-M] Bensoussan A., G.Da Prato, M.C.Delfour, S.K.Mitter, *Representation and Control of Infinite Dimensional Systems I,II*, Birkhäuser 1993.

[C-I-L] Crandall M.G., H.Ishii, P.L.Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull.Amer.Math.Soc. 27(1992),1-67.

[C-Z] Curtain R.F., H.J.Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer 1995.

[D-F-M] Desch W., E.Fašanga, J.Milota, *Unbounded observers and Riccati operators in nonreflexive spaces*, pp.

[DFMS] Desch W.,E.Fašangová, J.Milota, W.Schappacher, *Riccati Operators in Nonreflexive Spaces*, Differential and Integral Eqs. 15 (2002), 1493-1510.

[D-F-T] Doyle J.C., B.A.Francis, A.R.Tannenbaum, *Feedback Control Theory*, Macmillan 1992.

[D-M] Drábek P., J.Milota, *Methods of Nonlinear Analysis*, Birkhäuser 2007.

[D-M-S] Desch W., J.Milota, W.Schappacher, *Least Square Control Problems in Nonreflexive Spaces*, Semigroup Forum 62 (2001), 337-357.

[D-S] Desch W., W.Schappacher, *Spectral properies of finite-dimensional perturbed linear semigroups*, J.Diff.Eqs. 59(1985),80-102.

[E-N] Engel K.J., R.Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer 1999.

[F-F] Foias C., A.E.Frazho, *The Commutant Lifting Approach to Interpolation*, Birkhäuser 1990.

[F-R] Fleming W.H., R.W.Rishel, *Deterministic and Stochastic Optimal Control*, Springer 1975.

[I] Isidori A., *Nonlinear Control Systems*, Springer 1995.

[L-M] Lee E.E., L.Markus, *Foundations of Optimal Control Theory*, J.Wiley 1968.

[L-T1] Lasiecka I., R.Triggiani, Lecture Notes in Control and Information Sciences 164, Springer 1991.

[L-T2] Lasiecka I., R.Triggiani, *Control Theory for Partial Differential Equatiuons: Continuous and Approximation Theory I,II*, Cambridge Univ.Press 2000.

[L-Y] Li X., J.Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birkhäuser, 1995.

[M] Milota J., *Continuous semigroup and evolution equations*, pp.7-66 in Proc.Sem. Diff.Eqs., ed.P.Drábek, Univ.West Bohemia, Plzeň 2001.

[M-S] Macki J., A.Strauss, *Introduction to Optimal Control Theory*, Springer 1982.

[N] Nikolski N.K., *Operators, Functions and Systems: An Easy Reading I,II*, Amer. Math.Soc. 2002.

[Pa] Partington J.R.,*Linear Operators and Linear Systems*, Cambridge Univ.Press 2004.

[P-B-G-M] Pontryagin L.S., V.G.Boltyanski, R.V.Gamkrelidze, E.F.Mischenko, *The Mathematical Theory of Optimal Processes*, Interscience 1962.

[Pe] Peller V.V., *Hankel Operators and Their Applications*, Springer 2003.

[Sa1] Salamon D., *Infinite dimensional linear systems with unbounded control and observation*, Trans.Math.Amer.Math.Soc. 300 (1987),383-415.

[Sa2] Salamon D., *Realization theory in Hilbert spaces*, Math. Systems Theory 21(1989),147-164.

[St1] Staffans O.J., *Quadratic optimal control through spectral and coprime factorization*, European J.Control 5(1999),157-179.

[St2] Staffans O.J., *Well Posed Linear Systems, Part I: General Theory*, Cambridge Univ.Press 2005.

[V] Vidyasagar M., *Control System Synthesis: A Factorization Approach*, MIT Press 1985.

[W1] Weiss G., *The representation of regular linear systems in Hilbert spaces*, pp.401-416 in Control and Estimations of Distributed Parameter Systems, Birkhäuser 1989.

[W2] Weiss G., *Representation of shift invariant operators on L^2 by H^∞ transfer functions*, Math.Control,Signals,and Systems 4(1991),193-203.

[Z] Zabczyk J., *Mathematical Control Theory: An Introduction*, Birkhäuser 1992.