On partial regularity of solutions to the quasilinear parabolic system up to the boundary

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Abstract

We study questions of partial regularity up to the boundary for solutions of quasilinear parabolic systems of the form

$$\begin{aligned} \tfrac{\partial u}{\partial t} - \operatorname{div}\bigl(A(z,u)Du\bigr) &= 0 \quad \text{in } (-T,0)\times\Omega, \\ u &= 0 \quad \text{on } (-T,0)\times\partial\Omega, \end{aligned}$$

in Hilbert space setting. Our approach is based on A-caloric approximation lemma.

Introduction

Despite the fact that the heat equation has a smoothing property, the solutions of quasilinear parabolic systems are not smooth in general. Using standard elliptic counterexamples, it is easy to construct a quasilinear parabolic system with smooth coefficients with bounded weak solution which is not Hölder continuous. Namely, one can consider a discontinuous solution $u = u(x)$ of the elliptic system as a stationary solution of a corresponding parabolic system. Moreover, it is possible to construct a quasilinear parabolic system which has a solution which starts as a smooth one at $t = 0$ and develops a discontinuity in some moment $t > 0$, see [8].

In general, it is possible to prove *partial regularity* results only. By the notion "partial regularity" we indicate that the solution need not to be Hölder continuous on the whole domain but it is Hölder continuous on an open subset, whose complement is small.

In this paper we are concerned with the study of regularity properties of parabolic systems of the type

$$\begin{aligned} u_t - \operatorname{div}\bigl(A(z,u)Du\bigr) &= 0 \qquad \text{in } (-T,0)\times\Omega, \\ u &= 0 \qquad \text{on } (-T,0)\times\Gamma, \end{aligned} \tag{1}$$

in Hilbert space setting.

The partial regularity theory of such systems is based on a linearisation argument: one compares a given weak solution locally to the solution of a linear parabolic system with *constant coefficients*.

There are several ways to construct suitable linearisations. One possibility leads to the *blow-up technique* see e.g. Giaquinta and Giusti [12]. Another approach has been introduced in Campanato [13]. Therein, the solution u is compared on a small cylinder with a solution h of initial-Dirichlet problem with "frozen coefficients" where $h = u$ on the parabolic boundary of the cylinder. In this case, the nonlinear parabolic system solved by u must be exploited to obtain further information about u: to be precise, higher integrability of Du is needed. This requires usage of the (parabolic) Gehring type lemma and reverse Hölder inequality for u. See also e.g. [14].

In this paper we apply still another way of linearisation. Our approach is based on the so called A-caloric approximation lemma (lemma 8 below). For harmonic functions, this lemma was originally introduced by De Giorgi [10]. Its elliptic variant, A-harmonic approximation lemma, was used by Duzaar and Grotowski [1] to prove interior partial regularity for nonlinear elliptic systems. The questions of regularity up to the boundary for nonlinear elliptic systems were examined by Grotowski in [3]. Optimal interior partial regularity result for nonlinear parabolic systems was obtained by Duzaar and Mingione [2].

The advantage of the approach based on A-caloric approximation lemma consists in the fact, that we can avoid the necessity of getting higher integrability of Du and using Gehring-type lemma.

The main result in this paper is an alternate proof of the following theorem, which was originally proved in more general framework by Arkhipova [15], cf., theorem 2 therein. The result is not new, but it is proved by different technique, based on A-caloric lemma.

Theorem 1 *Let us suppose structure conditions (H0) — (H4) which are stated below and let u be a weak solution of (1). Furthermore, let the condition*

$$\liminf_{\rho \to 0+} \fint_{A_\rho(y_0)} |u|^2 \, \mathrm{d}z = 0$$

be satisfied at the point $y_0 = (t_0, x_0) \in (-T, 0) \times \Gamma$ *where* $A_\rho(y_0)$ *denotes the intersection of* $(-T, 0) \times \Omega$ *with* $Q_\rho(y_0) = \{(t, x) \in \mathbb{R}^{n+1}; |x - x_0| < \rho, t \in (t_0 - \rho^2, t_0)\}$.

Then there exists a neighborhood $U(y_0)$ *of the point* y_0 *such that u is Hölder continuous on the closure of* $U(y_0) \cap (-T, 0) \times \Omega$.

Formulation of the problem. Notations.

Let $n \geq 2$, Ω be a bounded domain in $\mathbb{R}^n$. Let Γ be a nonempty open subset of $\partial\Omega$ and let $T \in \mathbb{R}, 0 < T < +\infty$, $N \in \mathbb{N}$. We denote $U = (-T, 0) \times \Omega$, $z = (t, x) \in U$ where $x = (x_1, \dots, x_n)$. Let $N \in \mathbb{N}$. For the quasilinear parabolic system with the homogeneous boundary conditions

$$\begin{aligned} \frac{\partial u^i(z)}{\partial t} - D_\alpha\big(A_{ij}^{\alpha\beta}(z, u(z)) D_\beta u^j(z)\big) &= 0 \quad \text{in } U, \quad i = 1, \dots, N, \\ u^i(z) &= 0 \quad \text{on } (-T, 0) \times \Gamma, \quad i = 1, \dots, N \end{aligned} \tag{2}$$

we shall use the abbreviation (1). For the function $u = (u^1, \dots, u^N) : U \to \mathbb{R}^N$, $Du(z) \equiv \{D_\alpha u^i(z)\} \in \mathbb{R}^{N\times n}$ denotes the *spatial* gradient of u and $A(z,u) \equiv A_{ij}^{\alpha\beta}(z,u)$ stands for the coefficient matrix. In what follows, we use summation convention both over the Latin indices $i, j = 1, \dots, N$ and the Greek ones $\alpha, \beta = 1, \dots, n$.

The domain Ω is supposed to be sufficiently smooth, the operator is supposed to be uniformly elliptic. More precisely, it is required

(H0) The boundary $\partial\Omega$ of Ω belongs to the class C^2.

(H1) The mapping $(z,\xi) \mapsto A(z,\xi)$ is uniformly continuous on U.

(H2) There exists some $\lambda > 0$ such that for each $z \in U$, $\xi \in \mathbb{R}^N$ and $v \in \mathbb{R}^{N\times n}$

$$A(z,\xi)\, v\, v \equiv A_{ij}^{\alpha\beta}(z,\xi) v_\alpha^i v_\beta^j \geq \lambda |v|^2. \tag{3}$$

(H3) There exists $L > 0$ such that for any $z \in U$, $\xi \in \mathbb{R}^N$, and $u, v \in \mathbb{R}^{N\times n}$

$$A(z,\xi)\, u\, v \equiv A_{ij}^{\alpha\beta}(z,\xi) u_\alpha^i v_\beta^j \leq L|u||v|. \tag{4}$$

(H4) For each $z \in U$ and $\xi \in \mathbb{R}^N$ $A_{ij}^{\alpha\beta}(z,\xi) = A_{ji}^{\beta\alpha}(z,\xi)$.

The assumptions (H1) and (H3) imply existence of a continuous function $\omega : [0,+\infty) \to [0,+\infty)$ which is bounded, concave, nondecreasing, $\omega(0) = 0$ and

$$\begin{aligned} &\forall x, \bar{x} \in \Omega\ \forall t, \bar{t} \in (-T,0)\ \forall u, \bar{u} \in \mathbb{R}^N \\ &\qquad |A(t,x,u) - A(\bar{t},\bar{x},\bar{u})| \leq \omega(|x-\bar{x}|^2 + |t-\bar{t}| + |u-\bar{u}|^2). \end{aligned} \tag{5}$$

We note that (H3) implies that $A_{ij}^{\alpha\beta}(\cdot, u(\cdot))$ belongs to $\mathrm{L}^\infty(U)$ for any integrable u.

Definition 1 (Weak solution) *Under the weak solution to (2) we understand function $u \in \mathrm{L}^2(-T,0;\mathrm{W}^{1,2}(\Omega,\mathbb{R}^N))$ such that $u(\cdot,t) = 0$ on Γ in the sense of traces for almost every $t \in (-T,0)$ and*

$$\forall \varphi \in C_c^\infty(U,\mathbb{R}^N) \qquad \int_U [u\varphi_t - A(z,u)DuD\varphi]\, \mathrm{d}z = 0. \tag{6}$$

In the conclusion of this section we introduce some notation that will be used throughout the paper. Euclidean norm in $\mathbb{R}^k$ will be denoted by $|\cdot|$, whereas the parabolic metric in $\mathbb{R}\times\mathbb{R}^N$ will be denoted by δ, i.e., for $z^1 = (t^1, x^1), z^2 = (t^2, x^2)$ we have

$$\delta(z^1, z^2) := \max(|x^1 - x^2|, |t^1 - t^2|^{\frac{1}{2}}).$$

For $x_0 = (x_{01}, x_{02}, \dots, x_{0n-1}, 0)$, we write $B_\rho^+(x_0) = \{x \in \mathbb{R}^n; |x - x_0| < \rho, x_n > 0\}$ and $Q_\rho^+(t_0, x_0) = (t_0 - \rho^2, t_0) \times B_\rho^+(x_0)$ and further $Q_\rho^+ = Q_\rho^+(t_0,x_0), Q^+ = Q_1^+$. $D_\rho(x_0)$ denotes the set $\{x \in \mathbb{R}^n; |x - x_0| < \rho, x_n = 0\}$.

For a set $X \subset \mathbb{R} \times \mathbb{R}^n$ with positive finite Lebesgue measure, we denote the average of a given $f \in \mathrm{L}^1(X)$ by $(f)_X \equiv \fint_X f \,\mathrm{d}z := \frac{1}{\operatorname{meas} X} \int_X f \,\mathrm{d}z$, where $\operatorname{meas} X$ denotes the Lebesgue measure of the set X. We also abbreviate $f_{z_0,\rho} := (f)_{Q^+_\rho(z_0)}$.

We will often make use of the following elementary observation: if X has positive finite Lebesgue measure and the function f lies in $\mathrm{L}^2(X)$, then $\inf_{\nu\in\mathbb{R}} \int_X |f-\nu|^2 \,\mathrm{d}z$ is achieved for $\nu = \fint_X f \,\mathrm{d}z$.

By $V_\rho(z_0)$ we denote the Bochner space $\mathrm{L}^2\big(t_0-\rho^2, t_0; \mathrm{W}^{1,2}(B^+_\rho(x_0), \mathbb{R}^N)\big)$ equipped with the norm

$$\|f\|^2_{V_\rho(z_0)} := \int_{Q^+_\rho(z_0)} \big(\frac{1}{\rho^2}|f|^2 + |Df|^2\big) \,\mathrm{d}z.$$

We abbreviate $V_\rho := V_\rho(t_0, x_0), V := V_1$. We also make use of the space

$$Z := \{f \in \mathrm{W}^{1,2}(B^+, \mathbb{R}^N);\ \operatorname{trace} f = 0 \text{ on } D\}.$$

The space of $\alpha-$Hölder continuous functions with respect to the parabolic metric δ on a set $\bar{Q}^+_\rho(z_0)$, is denoted by $C^{0,\alpha}(\bar{Q}^+_\rho(z_0), \delta)$. It is equipped with the seminorm

$$[f]_{C^{0,\alpha}(\bar{Q}^+_\rho(z_0),\delta)} := \sup\Big\{\frac{|f(z^1)-f(z^2)|}{\delta^\alpha(z^1,z^2)};\ z^1, z^2 \in \bar{Q}^+_\rho(z_0),\ z^1 \neq z^2\Big\}.$$

By $\mathcal{L}^{p,\lambda}(Q^+_\rho(z_0), \delta)$ we will denote the appropriate Campanato space with the seminorm

$$[f]^p_{\mathcal{L}^{p,\lambda}(Q^+_\rho(z_0),\delta)} := \sup\Big\{\rho^{-\lambda} \int_{U(z,\rho)} |f(t,x) - f_{U(z,\rho)}|^p \,\mathrm{d}x\,\mathrm{d}t;\ z \in \bar{Q}^+_\rho(z_0),\ \rho > 0\Big\},$$

where $U(z,\rho) := \{y \in Q^+_\rho;\ \delta(y,z) < \rho\}$. The spaces $\mathcal{L}^{p,\lambda}(Q^+_\rho(z_0), \delta)$ and $C^{0,\alpha}(\bar{Q}^+_\rho(z_0), \delta)$ are isomorphic for $\alpha = \frac{\lambda-(n+2)}{p}$, for proof see [4].

By c we denote a generic constant which depends only on the data of the problem (1).

Statement of the result

The main result of this paper is the following local boundary partial regularity theorem:

Theorem 2 *Let $u \in \mathrm{L}^2(-T, 0; \mathrm{W}^{1,2}(\Omega, \mathbb{R}^N))$ be a weak solution of (2) under the assumptions (H0), (H1), (H2), (H3) and (H4). Let $\alpha \in (0,1)$ and $y_0 \in (-T, 0) \times \Gamma$ be a point where the condition*

$$\liminf_{\rho\to 0+} \fint_{Q_\rho(y_0)\cap U} |u|^2 \,\mathrm{d}z = 0 \tag{7}$$

is satisfied.

Then there exists some neighborhood $U(y_0)$ of the point y_0 such that $u \in C^{0,\alpha}(\overline{U(y_0) \cap U}, \mathbb{R}^N, \delta)$.

We illustrate the main step in the proof of the boundary regularity in a model situation — on a half cylinder. The general result then follows by a transformation argument.

Theorem 3 *Let $u \in \mathrm{L}^2(-1,0;\mathrm{W}^{1,2}(B^+,\mathbb{R}^N))$ be a weak solution of (2) on $U = Q^+$ with $\Gamma = D$. We assume the structure conditions (H1), (H2), (H3) and (H4) with $\Omega = B^+$, $T = 1$. Let $\alpha \in (0,1)$ and $y_0 \in (-1,0) \times D$ be a point where the condition*

$$\liminf_{\rho \to 0+} \fint_{Q^+_\rho(y_0)} |u|^2 \,\mathrm{d}z = 0 \tag{8}$$

is satisfied.

Then there exists some neighborhood $U(y_0)$ of the point y_0 such that $u \in C^{0,\alpha}(\overline{U(y_0) \cap Q^+}, \mathbb{R}^N, \delta)$.

Preliminary results

In this section, we summarize auxiliary lemmas that will be used in the main section. These include a-priory estimates for parabolic systems, Campanato lemma and A-caloric lemma.

In the whole section we put $z_0 = (t_0, x_0)$ where $x_0 = (x_{01}, \ldots, x_{0n-1}, 0)$.

Lemma 4 (Caccioppoli lemma) *Let $\rho > 0$ and $Q^+_\rho(z_0) \subset\subset Q^+$. Let $u \in V$ be a weak solution of (2) on $U = Q^+$ with $\Gamma = D$ under the conditions (H2) and (H3). Then there holds*

$$\int_{Q^+_{\frac{\rho}{2}}(z_0)} |Du|^2 \,\mathrm{d}z \leq \frac{c}{\rho^2} \int_{Q^+_\rho(z_0)} |u|^2 \,\mathrm{d}z,$$

where c does not depend on u, ρ and z_0.

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Now, we formulate properties of solutions to the parabolic systems with constant coefficients which we call $\mathcal{A}$-caloric functions:

Definition 2 *Let $h \in V_\rho(z_0), \lambda > 0$ and $\mathcal{A}$ be a bilinear form defined on $\mathbb{R}^{N\times n}$ (i.e. $\mathcal{A} \equiv \mathcal{A}^{\alpha\beta}_{ij}$ is constant) which satisfies*

$$\mathcal{A}\, v\, v \geq \lambda |v|^2 \quad \textit{for all } v \in \mathbb{R}^{N\times n}. \tag{9}$$

We say that the function h is $\mathcal{A}$-caloric *on $Q^+_\rho(z_0)$ if*

$$\int_{Q^+_\rho} \left(h\varphi_t - \mathcal{A}\, Dh\, D\varphi\right) \mathrm{d}z = 0 \tag{10}$$

holds for all $\varphi \in C^\infty_c(Q^+_\rho(z_0), \mathbb{R}^N)$.

We will often use homogeneous Dirichlet boundary conditions on the "flat boundary" $(t_0 - \rho^2, t_0) \times D_\rho(x_0)$ which are interpreted in the sense of traces on almost all time levels.

Lemma 5 (Infinite differentiability) *Let $\mathcal{A}$ be a symmetric ($\mathcal{A}_{ij}^{\alpha\beta} = \mathcal{A}_{ji}^{\beta\alpha}$) bilinear form on $\mathbb{R}^{N\times n}$ which satisfies (9). Let $h \in \mathrm{L}^2(t_0-\rho^2, t_0; \mathrm{W}^{1,2}(B_\rho^+(x_0), \mathbb{R}^N))$ be an $\mathcal{A}$-harmonic function with $h = 0$ on $(t_0-\rho^2, t_0) \times D_\rho(x_0)$.*

Then, for any $\sigma < \rho$, the function h belongs to $C^\infty(\bar{Q}_\sigma^+(z_0), \mathbb{R}^N)$ and there holds

$$\int_{Q_\sigma^+(z_0)} |D^\alpha \frac{\partial^k h}{\partial t^k}|^2 \,\mathrm{d}z < c(\alpha, k, \sigma, \rho) \int_{Q_\rho^+(z_0)} |h|^2 \,\mathrm{d}z \tag{11}$$

for any multiindex α and any $k \in \mathbb{N}_0$.

Lemma 6 (Campanato) *Let $\mathcal{A}$ be a symmetric ($\mathcal{A}_{ij}^{\alpha\beta} = \mathcal{A}_{ji}^{\beta\alpha}$) bilinear form on $\mathbb{R}^{N\times n}$ which satisfies (9). Let $h \in \mathrm{L}^2(t_0-\rho^2, t_0; \mathrm{W}^{1,2}(B_\rho^+(x_0), \mathbb{R}^N))$ be an $\mathcal{A}$-caloric function fulfilling $h = 0$ on $(t_0-\rho^2, t_0) \times D_\rho(x_0)$. Then there exists a constant C_{camp} such that for any $\sigma \in (0, \rho)$ there holds*

$$⨍_{Q_\sigma(z_0)} |h|^2 \,\mathrm{d}z \le C_{camp} \left(\frac{\sigma}{\rho}\right)^2 ⨍_{Q_\rho(z_0)} |h|^2 \,\mathrm{d}z. \tag{12}$$

The proof is originally due to Campanato [4].

Proof: Without loss of generality, we suppose $\sigma \in (0, \frac{\rho}{2})$. (For $\sigma \in [\frac{\rho}{2}, \rho)$ we choose $C_{camp} = 2^{n+2}$.) Let $z_0 = (t_0, x_0)$. By the previous lemma, for any $k = 1, 2, \ldots$ there holds

$$\|h\|^2_{\mathrm{W}^{k,2}(Q^+_{\frac{\rho}{2}}(z_0), \mathbb{R}^N)} \le c(\rho, k) \int_{Q_\rho^+(z_0)} |h|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{13}$$

Thanks to Sobolev embedding theorem we have $h \in C^1(\bar{Q}^+_{\frac{\rho}{2}}(z_0), \mathbb{R}^n)$, and for sufficiently big k

$$\sup_{\bar{Q}^+_{\frac{\rho}{2}}(z_0)} \left(|Dh|^2 + |\frac{\partial h}{\partial t}|^2\right) \le c(\rho, k) \|h\|^2_{\mathrm{W}^{k,2}(Q^+_{\frac{\rho}{2}}(z_0), \mathbb{R}^N)}. \tag{14}$$

Let $z = (t, x_1, \ldots, x_n) \in Q_\sigma^+(z_0)$ be arbitrary. We denote the orthogonal projection of z on $(t_0-\sigma^2, t_0) \times D_\sigma(z_0)$ by z', i.e. $z' = (t, x_1, \ldots, x_{n-1}, 0)$. As $h \in C^1(\bar{Q}^+_{\frac{\rho}{2}}(z_0), \mathbb{R}^n)$ and $h = 0$ on $(t_0-\sigma^2, t_0) \times D_\sigma(x_0)$, we can estimate

$$|h(z) - \underbrace{h(z')}_{=0}|^2 \le \sigma^2 \sup_{\bar{Q}^+_{\frac{\rho}{2}}(z_0)} \sum_{i=1}^N |\frac{\partial h^i}{\partial x_n}|^2.$$

This yields,

$$\int_{Q_\sigma^+(z_0)} |h|^2 \,\mathrm{d}x\,\mathrm{d}t \le \mathrm{meas}\left(Q_\sigma^+(z_0)\right) \sigma^2 \sup_{\bar{Q}^+_{\frac{\rho}{2}}(z_0)} \left(|Dh|^2 + |\frac{\partial h}{\partial t}|^2\right),$$

and thus, cf. (14), (13):

$$\int_{Q_\sigma^+(z_0)} |h|^2 \,\mathrm{d}x\,\mathrm{d}t \le c(\rho)\, \mathrm{meas}\left(Q_\sigma^+(z_0)\right) \sigma^2 \int_{Q_\rho^+(z_0)} |h|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

It now remains to precise the dependence of the constant $c(\rho)$ on ρ. Without loss of generality, we suppose $z_0 = 0$. We consider $v(\tau, y) := h(\lambda^2\tau, \lambda y)$ with $\lambda > 0$, which is also $\mathcal{A}$-caloric on the half-cylinder $Q^+_{\frac{\rho}{\lambda}}(0)$ and fulfils $v = 0$ on $(-\frac{\rho^2}{\lambda^2}, 0) \times D_{\frac{\rho}{\lambda}}(0)$. Setting $\lambda := \rho$, we thus have

$$\int_{Q^+_{\frac{\sigma}{\rho}}} |v(\tau, y)|^2 \,\mathrm{d}y \,\mathrm{d}\tau \leq c(1) \,\mathrm{meas}\,\big(Q^+_{\frac{\sigma}{\rho}}\big) \,\Big(\frac{\sigma}{\rho}\Big)^2 \int_{Q^+_1} |v(\tau, y)|^2 \,\mathrm{d}y \,\mathrm{d}\tau.$$

Transformation of variables $y = \frac{x}{\rho}, \tau = \frac{t}{\rho^2}$ yields

$$\int_{Q^+_\sigma} |h(t, x)|^2 \,\mathrm{d}x \,\mathrm{d}t \leq c(1) \,\mathrm{meas}\,\big(Q^+_{\frac{\sigma}{\rho}}\big) \,\Big(\frac{\sigma}{\rho}\Big)^2 \int_{Q^+_\rho} |h(t, x)|^2 \,\mathrm{d}x \,\mathrm{d}t,$$

thus $c(\rho) = \frac{c(1)}{\rho^{n+4}}$. We have proved the inequality (12). □

We now state the $\mathcal{A}$-caloric approximation lemma which is the main tool in this paper. The interior version of this lemma can be found in [2]. The elliptic boundary variant is presented in [3]. For simplicity, we formulate the estimates on Q^+ but they can be easily transformed on any $Q^+_\rho(z)$.

In the proof, we use the following proposition which is due to Simon, cf. [5], theorem 3 there:

Proposition 7 *Assume X and B are Banach spaces, $X \subset\subset B$, $F \subset \mathrm{L}^p(0, T; B)$ where $1 \leq p \leq \infty$, F is bounded in $\mathrm{L}^1_{\mathrm{loc}}(0, T; X)$ and*

$$\|f(\cdot + h) - f(\cdot)\|_{\mathrm{L}^p(0,T-h;B)} \to 0 \text{ as } h \to 0, \text{ uniformly for } f \in F. \tag{15}$$

Then F is relatively compact in $\mathrm{L}^p(0, T; B)$.

Lemma 8 ($\mathcal{A}$-caloric lemma) *Consider fixed positive $\lambda, L \in \mathbb{R}$ and $n, N \in \mathbb{N}$ with $n \geq 2$. Then for any given $\epsilon > 0$ there exists $\delta = \delta(n, N, \lambda, L, \epsilon) \in (0, 1]$ with the following property: for any bilinear form $\mathcal{A}$ on $\mathbb{R}^{N\times n}$ which fulfils*

$$\mathcal{A}\, v\, v \geq \lambda|v|^2 \quad \text{for all } v \in \mathbb{R}^{N\times n}, \tag{16}$$

$$|\mathcal{A}\, u\, v| \leq L|u||v| \quad \text{for all } u, v \in \mathbb{R}^{N\times n}, \tag{17}$$

and for any $u \in V_1$ satisfying

- $$\int_{Q^+} \big(|u|^2 + |Du|^2\big) \,\mathrm{d}z \leq 1, \tag{18}$$
- $$\Big|\int_{Q^+} \big(u\varphi_t - \mathcal{A}\, Du\, D\varphi\big) \,\mathrm{d}z\Big| \leq \delta \sup_{Q^+} |D\varphi|, \quad \text{for all } \varphi \in C^1_c(Q^+, \mathbb{R}^N), \tag{19}$$
- $$u = 0 \quad \text{on } (-1, 0) \times D. \tag{20}$$

there exists an $\mathcal{A}$-caloric function $h \in V_1$ satisfying

- $$\int_{Q^+} \big(|h|^2 + |Dh|^2\big) \,\mathrm{d}z \leq 1, \tag{21}$$
- $$\int_{Q^+} |h - u|^2 \,\mathrm{d}z \leq \epsilon, \tag{22}$$
- $$h = 0 \quad \text{on } (-1, 0) \times D. \tag{23}$$

The proof follows ideas of Duzaar and Mingione [2] and we give it here for the reader's convenience.

Proof. We proceed by a contradiction argument. Let us suppose that the assertion is false. We can thus find $\epsilon > 0$, a sequence $\{\mathcal{A}_k\}$ of bilinear forms on $\mathbb{R}^{N\times n}$, with uniform ellipticity (16) and upper bounds (17), and a sequence of functions $\{v_k\}$ with $v_k \in \mathrm{L}^2(-1,0;\mathrm{W}^{1,2}(B^+,\mathbb{R}^N))$ such that

$$\int_{Q^+} \big(|v_k|^2 + |Dv_k|^2\big)\,\mathrm{d}z \le 1, \tag{24}$$

$$v_k = 0 \quad \text{on } (-1,0)\times D, \tag{25}$$

and

$$\Big|\int_{Q^+} \big(v_k\varphi_t - \mathcal{A}_k\, Dv_k\, D\varphi\big)\,\mathrm{d}z\Big| \le \frac{1}{k}\sup_{Q^+}|D\varphi| \tag{26}$$

for all $\varphi \in C^1_c(Q^+,\mathbb{R}^N)$ and $k\in\mathbb{N}$, but

$$\int_{Q^+} |v_k - h|^2\,\mathrm{d}z > \epsilon \tag{27}$$

for all $h \in \mathrm{L}^2(-1,0;\mathrm{W}^{1,2}(B^+,\mathbb{R}^N))$ satisfying

- h is $\mathcal{A}_k$-caloric function on Q^+, (28)
- $\displaystyle\int_{Q^+} \big(|h|^2 + |Dh|^2\big)\,\mathrm{d}z \le 1,$ (29)
- $h = 0 \quad \text{on } (-1,0)\times D.$ (30)

Passing to a subsequence (also labeled with k) we obtain the existence of $v \in \mathrm{L}^2(-1,0;\mathrm{W}^{1,2}(B^+,\mathbb{R}^N))$ and $\mathcal{A}$ such that there holds

$$\begin{aligned} v_k &\rightharpoonup v && \text{weakly in } \mathrm{L}^2(Q^+,\mathbb{R}^N),\\ Dv_k &\rightharpoonup Dv && \text{weakly in } \mathrm{L}^2(Q^+,\mathbb{R}^{N\times n}),\\ \mathcal{A}_k &\to \mathcal{A}. \end{aligned} \tag{31}$$

We thus have $v_k \rightharpoonup v$ in $\mathrm{L}^2(-1,0;Z)$ and so $v \in \mathrm{L}^2(-1,0;Z)$. Thus, $v = 0$ on $(-1,0)\times D$. Using the lower semicontinuity of $v \mapsto \int_{Q^+}\big(|v|^2 + |Dv|^2\big)\,\mathrm{d}z$ with respect to weak convergence in $\mathrm{L}^2(-1,0;\mathrm{W}^{1,2}(B^+,\mathbb{R}^N))$ we obtain

$$\int_{Q^+} \big(|v|^2 + |Dv|^2\big)\,\mathrm{d}z \le 1. \tag{32}$$

Moreover, for $\varphi \in \mathcal{C}^\infty_0(Q^+,\mathbb{R}^N)$ we have

$$\begin{aligned} \int_{Q^+} \big(v\varphi_t - \mathcal{A}\,Dv\,D\varphi\big)\,\mathrm{d}z &= \int_{Q^+} \big((v-v_k)\varphi_t - \mathcal{A}\,(Dv - Dv_k)\,D\varphi\big)\,\mathrm{d}z\\ &\quad - \int_{Q^+} (\mathcal{A}-\mathcal{A}_k)\,Dv_k\,D\varphi\,\mathrm{d}z\\ &\quad + \int_{Q^+} \big(v_k\varphi_t - \mathcal{A}_k\,Dv_k\,D\varphi\big)\,\mathrm{d}z. \end{aligned} \tag{33}$$

Passing to the limit $k \to \infty$ we see that the first term of the right-hand side converges to 0 due to (31), the same holds for the second term in view of the uniform bound of Dv_k in $\mathrm{L}^2(Q^+, \mathbb{R}^{N\times n})$ – see (24) and the convergence of $\{\mathcal{A}_k\}$, the third term vanishes in the limit $k \to \infty$ via (26). This shows that the weak limit v is an $\mathcal{A}$-caloric function on Q^+, i.e.,

$$\forall \varphi \in \mathcal{C}_0^\infty(Q^+, \mathbb{R}^N) \qquad \int_{Q^+} \big(v\varphi_t - \mathcal{A}\, Dv\, D\varphi\big)\, \mathrm{d}z = 0. \tag{34}$$

We want to show that (up to a subsequence) $v_k \to v$ in $\mathrm{L}^2(Q^+, \mathbb{R}^N)$. To get this convergence, we now check the assumptions of the proposition 7, where we put $X = \mathrm{W}^{1,2}(B^+, \mathbb{R}^N)$, $B = \mathrm{L}^2(B^+, \mathbb{R}^N)$, $F = \{v_k\}_{k\in\mathbb{N}}$.

For $\varphi \in \mathcal{C}_0^\infty(Q^+, \mathbb{R}^N)$ we have

$$\begin{aligned}
\Big| \int_{Q^+} v_k \varphi_t \, \mathrm{d}z \Big| &\le \Big| \int_{Q^+} \mathcal{A}_k\, Dv_k\, D\varphi \, \mathrm{d}z \Big| + \frac{1}{k} \sup_{Q^+} |D\varphi| \\
&\le |\mathcal{A}_k| \Big(\int_{Q^+} |Dv_k|^2 \, \mathrm{d}z \Big)^{\frac{1}{2}} \Big(\int_{Q^+} |D\varphi|^2 \, \mathrm{d}z \Big)^{\frac{1}{2}} + \frac{1}{k} \sup_{Q^+} |D\varphi| \\
&\le |\mathcal{A}_k| \Big(\int_{Q^+} |D\varphi|^2 \, \mathrm{d}z \Big)^{\frac{1}{2}} + \frac{1}{k} \sup_{Q^+} |D\varphi|.
\end{aligned} \tag{35}$$

Here, we have used (24) and (26). Now, for $-1 < s_1 < s_2 < 0$ and $\epsilon > 0$ small enough we choose

$$\zeta_\epsilon(t) = \begin{cases} 0 & \text{for} \quad -1 \le t \le s_1 - \epsilon, \\ \frac{1}{\epsilon}(t - s_1 + \epsilon) & \text{for} \quad s_1 - \epsilon \le t \le s_1, \\ 1 & \text{for} \quad s_1 \le t \le s_2, \\ -\frac{1}{\epsilon}(t - s_2 - \epsilon) & \text{for} \quad s_2 \le t \le s_2 + \epsilon, \\ 0 & \text{for} \quad s_2 + \epsilon \le t \le 0, \end{cases} \tag{36}$$

and let $\varphi(t,x) = \zeta_\epsilon(t)\psi(x)$ for $\psi \in \mathcal{C}_0^\infty(B^+, \mathbb{R}^N)$. Testing (35) with (mollified) φ we obtain

$$\begin{aligned}
&\Big| \int_{B^+} \Big(\frac{1}{\epsilon} \int_{s_1-\epsilon}^{s_1} v_k(t,x)\, \mathrm{d}t - \frac{1}{\epsilon} \int_{s_2}^{s_2+\epsilon} v_k(t,x)\, \mathrm{d}t \Big) \psi(x)\, \mathrm{d}x \Big| \\
&\quad \le |\mathcal{A}_k| \Big(\int_{-1}^{0} \zeta_\epsilon(t)^2 \, \mathrm{d}t \Big)^{\frac{1}{2}} \|D\psi\|_{\mathrm{L}^2(B^+, \mathbb{R}^N)} + \frac{1}{k} \sup_{B^+} |D\psi| \cdot \sup_{-1\le t\le 0} |\zeta_\epsilon(t)| \\
&\quad \le \Big(|\mathcal{A}_k| \sqrt{s_2 - s_1 + 2\epsilon} + \frac{1}{k} \Big) \sup_{B^+} |D\psi|.
\end{aligned} \tag{37}$$

By Sobolev-embedding

$$\sup_{B^+} |D\psi| \le c(n,l) \|\psi\|_{W_0^{l,2}(B^+)}, \quad l > \frac{n+2}{2}, \tag{38}$$

we see that

$$\Big|\int_{B^+}\Big(\frac{1}{\epsilon}\int_{s_1-\epsilon}^{s_1} v_k(t,x)\,\mathrm{d}t-\frac{1}{\epsilon}\int_{s_2}^{s_2+\epsilon} v_k(t,x)\,\mathrm{d}t\Big)\psi(x)\,\mathrm{d}x\Big|$$
$$\le c(n,l)\big(|\mathcal{A}_k|\sqrt{s_2-s_1+2\epsilon}+\frac{1}{k}\big)\|\psi\|_{\mathrm{W}_0^{l,2}(B^+)}. \tag{39}$$

Passing to the limit $\epsilon\to 0$ we obtain for a.e. $-1<s_1<s_2<0$

$$\Big|\int_{B^+}\big(v_k(s_2,\cdot)-v_k(s_1,\cdot)\big)\psi\,\mathrm{d}x\Big|\le c(n,l)\big(|\mathcal{A}_k|\sqrt{s_2-s_1}+\frac{1}{k}\big)\|\psi\|_{\mathrm{W}_0^{l,2}(B^+)} \tag{40}$$

for any $\psi\in\mathcal{C}_0^\infty(B^+,\mathbb{R}^N)$. By density of $\mathcal{C}_0^\infty(B^+,\mathbb{R}^N)$ in $\mathrm{W}_0^{l,2}(B^+,\mathbb{R}^N)$ the last inequality is also valid for any $\psi\in\mathrm{W}_0^{l,2}(B^+,\mathbb{R}^N)$. Taking the supremum over all $\psi\in\mathrm{W}_0^{l,2}(B^+,\mathbb{R}^N)$ with $\|\psi\|_{\mathrm{W}_0^{l,2}(B^+)}\le 1$ we infer

$$\|v_k(s_2,\cdot)-v_k(s_1,\cdot)\|_{\mathrm{W}^{-l,2}(B^+,\mathbb{R}^N)}\le c(l,n)\big(|\mathcal{A}_k|\sqrt{s_2-s_1}+\frac{1}{k}\big). \tag{41}$$

Interpolating $\mathrm{L}^2(B^+,\mathbb{R}^N)$ between $\mathrm{W}^{1,2}(B^+,\mathbb{R}^N)$ and $\mathrm{W}^{-l,2}(B^+,\mathbb{R}^N)$ it follows for $\mu>0$ that

$$\begin{aligned}
\int_{-1}^{-h}&\|v_k(t+h,\cdot)-v_k(t,\cdot)\|^2_{\mathrm{L}^2(B^+)}\,\mathrm{d}t\\
&\le\mu\int_{-1}^{-h}\|v_k(t+h,\cdot)-v_k(t,\cdot)\|^2_{\mathrm{W}^{1,2}(B^+)}\,\mathrm{d}t\\
&\quad+c(\mu)\int_{-1}^{-h}\|v_k(t+h,\cdot)-v_k(t,\cdot)\|^2_{\mathrm{W}^{-l,2}(B^+)}\,\mathrm{d}t\\
&\le 4\mu\int_{-1}^{0}\|v_k(t,\cdot)\|^2_{\mathrm{W}^{1,2}(B^+)}\,\mathrm{d}t+c(\mu)c^2\big(|\mathcal{A}_k|\sqrt{h}+\frac{1}{k}\big)^2\\
&\le 4\mu+2c(\mu)c^2\big(|A_k|^2h+\frac{1}{k^2}\big).
\end{aligned} \tag{42}$$

Here, we have used in the first line the interpolation inequality

$$\|w\|^2_{\mathrm{L}^2(B^+)}\le\mu\|w\|^2_{W^{1,2}(B^+)}+c(\mu)\|w\|^2_{W^{-l,2}(B^+)} \tag{43}$$

which is valid for $w\in\mathrm{W}^{1,2}(B^+,\mathbb{R}^N)$. Moreover, in the second-last line we have used the bound (41) for $\|v_k(t+h,\cdot)-v_k(t,\cdot)\|_{\mathrm{W}^{-l,2}(B^+)}$ from above and the bound (24).

We are now in the position to show that

$$\lim_{h\to 0}\int_{-1}^{-h}\|v_k(t+h,\cdot)-v_k(t,\cdot)\|^2_{\mathrm{L}^2(B^+)}\,\mathrm{d}t=0 \text{ uniformly in } k. \tag{44}$$

In order to do this we recall that $\mathcal{A}_k\to\mathcal{A}$ as $k\to\infty$ so that $\sup_{k\in\mathbb{N}}|\mathcal{A}_k|\le a<\infty$. Using this in (42) we obtain

$$\int_{-1}^{-h}\|v_k(t+h,\cdot)-v_k(t,\cdot)\|^2_{\mathrm{L}^2(B^+)}\,\mathrm{d}t\le 4\mu+2c(\mu)c^2\big(a^2h+\frac{1}{k^2}\big). \tag{45}$$

For given $\theta > 0$ we choose $\mu = \frac{\theta}{12}$. This fixes μ and also $c(\mu) = c(\frac{\theta}{12})$. Next we choose $k_0 \in \mathbb{N}$ such that $\frac{2c(\mu)c^2}{k^2} < \frac{\theta}{3}$ for any $k \geq k_0$. Then, for $k = 1, \ldots, k_0 - 1$ we choose $h_1 > 0$ such that

$$\forall 0 < h < h_1,\ k = 1, \ldots, k_0 - 1 \quad \int_{-1}^{-h} \|v_k(t+h, \cdot) - v_k(t, \cdot)\|^2_{\mathrm{L}^2(B^+)} \, \mathrm{d}t < \theta. \tag{46}$$

Finally, we choose $h_2 > 0$ such that $2c(\mu)c^2a^2h < \frac{\theta}{3}$ for any $0 < h < h_2$. Then, for any $k \in \mathbb{N}$ and $0 < h < h_0 := \min(h_1, h_2)$ we have

$$\int_{-1}^{-h} \|v_k(t+h, \cdot) - v_k(t, \cdot)\|^2_{\mathrm{L}^2(B^+)} \, \mathrm{d}t < \theta \tag{47}$$

which proves (44).

Since the sequence $\{v_k\}$ is also bounded in $\mathrm{L}^2(-1, 0; \mathrm{W}^{1,2}(B^+, \mathbb{R}^N))$ we are able to apply the Simon's theorem 7 to obtain a subsequence $\{v_k\}$ (again labelled by k) such that

$$v_k \to v \quad \text{strongly in } \mathrm{L}^2(Q^+, \mathbb{R}^N). \tag{48}$$

To obtain the desired contradiction we denote by $w_k : Q^+ \to \mathbb{R}^N$ a solution to the following initial-Dirichlet problem; its existence can be deduced from standard existence arguments, cf. [6], [7] for instance.

$$\begin{cases} w_k \in \mathcal{C}([-1, 0]; \mathrm{L}^2(B^+, \mathbb{R}^N)) \cap \mathrm{L}^2(-1, 0; \mathrm{W}_0^{1,2}(B^+, \mathbb{R}^N)), \\ \frac{\partial w_k}{\partial t} \in \mathrm{L}^2(-1, 0; \mathrm{W}^{-1,2}(B^+, \mathbb{R}^N)), \\ w_k(\cdot, -1) = 0, \end{cases} \tag{49}$$

$$\forall \varphi \in \mathcal{C}_0^\infty(Q^+, \mathbb{R}^N) \int_{Q^+} \big(w_k \varphi_t - \mathcal{A}_k \, Dw_k \, D\varphi\big) \, \mathrm{d}z = \int_{Q^+} (\mathcal{A} - \mathcal{A}_k) \, Dv \, D\varphi \, \mathrm{d}z. \tag{50}$$

A standard a-priori estimate holds for the solution w_k:

$$\begin{aligned} \frac{1}{2}\|w_k(\cdot, t)\|^2_{\mathrm{L}^2(B^+)} &+ \int_{(-1,t)\times B^+} \mathcal{A}_k \, Dw_k \, Dw_k \, \mathrm{d}z \\ &= \int_{(-1,t)\times B^+} (\mathcal{A}_k - A) \, Dv \, Dw_k \, \mathrm{d}z \qquad \text{for a.e. } t \in (-1, 0). \end{aligned} \tag{51}$$

Using the ellipticity of the bilinear forms $\mathcal{A}_k$ we see that the second term of the left-hand side of (51) is bounded from below by $\lambda \int_{(-1,t)\times B^+} |Dw_k|^2 \, \mathrm{d}z$. Moreover the right-hand side of (51) is estimated easily by Cauchy-Schwarz inequality, the bound $\int_{Q^+} |Dv|^2 \, \mathrm{d}z \leq 1$ from (32), and Young's inequality

$$\begin{aligned} \int_{(-1,t)\times B^+} &(\mathcal{A}_k - \mathcal{A}) \, Dv \, Dw_k \, \mathrm{d}z \\ &\leq |\mathcal{A} - \mathcal{A}_k| \Big(\int_{Q^+} |Dv|^2 \, \mathrm{d}z \Big)^{\frac{1}{2}} \Big(\int_{(-1,t)\times B^+} |Dw_k|^2 \, \mathrm{d}z \Big)^{\frac{1}{2}} \\ &\leq \frac{1}{2\lambda} |\mathcal{A} - \mathcal{A}_k|^2 + \frac{\lambda}{2} \int_{(-1,t)\times B^+} |Dw_k|^2 \, \mathrm{d}z. \end{aligned} \tag{52}$$

This implies in particular

$$\frac{1}{2}\int_{B^+}|w_k(t,\cdot)|^2\,\mathrm{d}x+\frac{\lambda}{2}\int_{(-1,t)\times B^+}|Dw_k|^2\,\mathrm{d}z\le\frac{1}{2\lambda}|\mathcal{A}_k-\mathcal{A}|^2 \tag{53}$$

for a.e. $t\in[-1,0]$ and $k\in\mathbb{N}$.

Taking the supremum over $t\in(-1,0)$ we arrive at

$$\sup_{t\in(-1,0)}\frac{1}{2}\int_{B^+}|w_k(t,\cdot)|^2\,\mathrm{d}x+\frac{\lambda}{2}\int_{Q^+}|Dw_k|^2\,\mathrm{d}z\to 0\quad\text{as } k\to\infty. \tag{54}$$

Letting $g_k:=v-w_k\in\mathrm{L}(-1,0;\mathrm{W}^{1,2}(B^+,\mathbb{R}^N))$ we easily see that g_k agrees with v on the parabolic boundary $\partial_p Q^+$ of Q^+ and satisfies

$$\forall\varphi\in\mathcal{C}_0^\infty(Q^+,\mathbb{R}^N)\qquad\int_{Q^+}\big(g_k\varphi_t-\mathcal{A}_k\,Dg_k\,D\varphi\big)\,\mathrm{d}z=0. \tag{55}$$

From (54) and the definition of g_k we see that

$$\int_{Q^+}\big(|g_k-v|^2+|Dg_k-Dv|^2\big)\,\mathrm{d}z\to 0\quad\text{as } k\to\infty, \tag{56}$$

which implies in particular that

$$\int_{Q^+}\big(|g_k|^2+|Dg_k|^2\big)\,\mathrm{d}z\to\int_{Q^+}\big(|v|^2+|Dv|^2\big)\,\mathrm{d}z\le 1\quad\text{as } k\to\infty. \tag{57}$$

Letting

$$b_k:=\max\Big\{1,\int_{Q^+}\big(|g_k|^2+|Dg_k|^2\big)\,\mathrm{d}z\Big\},\quad \tilde{g}_k:=\frac{g_k}{b_k} \tag{58}$$

we see that $b_k\to 1$ and $\int_{Q^+}(|\tilde{g}_k|^2+|D\tilde{g}_k|^2)\,\mathrm{d}z\le 1$ for any $k\in\mathbb{N}$. Note that $\tilde{g}_k\in\mathrm{L}^2(-1,0;\mathrm{W}^{1,2}(B^+,\mathbb{R}^N))$ and

- $\tilde{g}_k$ is $\mathcal{A}_k$-caloric function on Q^+, (59)
- $\displaystyle\int_{Q^+}\big(|\tilde{g}_k|^2+|D\tilde{g}_k|^2\big)\,\mathrm{d}z\le 1,$ (60)
- $\tilde{g}_k=0\quad\text{on } (-1,0)\times D.$ (61)

Furthermore,

$$\Big(\int_{Q^+}|\tilde{g}_k-v|^2\,\mathrm{d}z\Big)^{\frac{1}{2}}\le\Big(\int_{Q^+}|g_k-v|^2\,\mathrm{d}z\Big)^{\frac{1}{2}}+(1-\frac{1}{b_k})\Big(\int_{Q^+}|g_k|^2\,\mathrm{d}z\Big)^{\frac{1}{2}}\to 0 \tag{62}$$

as $k\to\infty$, which yields the desired contradiction to (27). $\square$

We now formulate another variant of the previous lemma which fits better for our purposes, cf. [1].

Lemma 9 *Let $\lambda > 0$, $L > 0$, and $n, N \in \mathbb{N}$ with $n \geq 2$. Then for any given $\epsilon > 0$ there exists a constant $C_\epsilon = C(n, N, \lambda, L, \epsilon)$ with the following property: for any bilinear form $\mathcal{A}$ on $\mathbb{R}^{N\times n}$ which fulfils (16) and (17) and for any $u \in V_\rho(z_0)$ with $u = 0$ on $(t_0-\rho^2, t_0)\times D_\rho(x_0)$ there exists an $\mathcal{A}$-caloric function $h \in V_\rho(z_0)$ which satisfies:*

- $h = 0$ *on* $(t_0 - \rho^2, t_0) \times D_\rho(x_0)$, (63)
- $$\frac{1}{\rho^2}\int_{Q_\rho^+(z_0)} |h|^2 \,\mathrm{d}z + \int_{Q_\rho^+(z_0)} |Dh|^2 \,\mathrm{d}z \leq \frac{1}{\rho^2}\int_{Q_\rho^+(z_0)} |u|^2 \,\mathrm{d}z + \int_{Q_\rho^+(z_0)} |Du|^2 \,\mathrm{d}z \tag{64}$$
- *there exists a function* $\varphi \in C_c^1(Q_\rho^+(z_0), \mathbb{R}^N)$ *such that* $\sup\limits_{Q_\rho^+(z_0)} |D\varphi| \leq \dfrac{1}{\rho}$ *and*

$$\fint_{Q_\rho^+(z_0)} |u-h|^2 \,\mathrm{d}z \leq C_\epsilon\Big(\rho^2 \fint_{Q_\rho^+(z_0)} (u\varphi_t - \mathcal{A}\, Du\, D\varphi)\,\mathrm{d}z\Big)^2 + \epsilon \fint_{Q_\rho^+(z_0)} \big(|u|^2 + \rho^2 |Du|^2\big)\,\mathrm{d}z. \tag{65}$$

Proof: It is sufficient to consider the case $z_0 = 0, \rho = 1$. The general result can be obtained by the transformation from $Q_\rho^+(z_0)$ to Q^+ and back, i.e., $\tilde{u}(T,X) := u(t_0 + \rho^2 T, x_0 + \rho X)$ and $h(t,x) = \tilde{h}(\frac{1}{\rho^2}(t-t_0), \frac{1}{\rho}(x-x_0))$.

We may assume $\|u\|_{V_1} \neq 0$. Let us put $\bar{u} := \frac{u}{\|u\|_{V_1}}$ and consider two different cases.

First case: Let us assume

$$\forall \varphi \in C_c^1(Q^+, \mathbb{R}^N) \quad \Big|\int_{Q^+} (\bar{u}\varphi_t - \mathcal{A}\, D\bar{u}\, D\varphi)\,\mathrm{d}z\Big| \leq \delta \sup_{Q^+} |D\varphi|, \tag{66}$$

where δ corresponds to ϵ from lemma 8. By lemma 8 we obtain an $\mathcal{A}$-caloric function $\bar{h}$ satisfying $\bar{h} = 0$ on $(-1,0)\times D$ and

$$\int_{Q^+} \big(|\bar{h}|^2 + |D\bar{h}|^2\big)\,\mathrm{d}z \leq 1, \quad \int_{Q^+} |\bar{u} - \bar{h}|^2 \,\mathrm{d}z \leq \epsilon. \tag{67}$$

We now set $h := \bar{h}\|u\|_{V_1}$. This yields

$$\int_{Q^+} \big(|h|^2 + |Dh|^2\big)\,\mathrm{d}z \leq \|u\|_{V_1}^2 \int_{Q^+} \big(|\bar{h}|^2 + |D\bar{h}|^2\big)\,\mathrm{d}z \leq \|u\|_{V_1}^2 = \int_{Q^+} \big(|u|^2 + |Du|^2\big)\,\mathrm{d}z, \tag{68}$$

$$\int_{Q^+} |u-h|^2 \,\mathrm{d}z = \|u\|_{V_1}^2 \int_{Q^+} |\bar{u}-\bar{h}|^2 \,\mathrm{d}z \leq \|u\|_{V_1}^2 \epsilon = \epsilon \int_{Q^+} \big(|u|^2 + |Du|^2\big)\,\mathrm{d}z, \tag{69}$$

Thus, we have proved (63), (64) and (65) (with $\rho = 1$) provided (66) holds.

Second case: If (66) does not hold then there is a function $\varphi \in C^1_c(Q^+, \mathbb{R}^N)$ (which is not identically zero) which satisfies

$$\Big| \int_{Q^+} (\bar{u}\varphi_t - \mathcal{A}\, D\bar{u}\, D\varphi)\, \mathrm{d}z \Big| > \delta \sup_{Q^+} |D\varphi|. \tag{70}$$

Writing $\bar{\varphi} := \frac{\varphi}{\sup_{Q^+}|D\varphi|}$, this implies

$$\Big| \int_{Q^+} (u\bar{\varphi}_t - \mathcal{A}\, Du\, D\bar{\varphi})\, \mathrm{d}z \Big| > \delta \Big(\int_{Q^+} \big(|u|^2 + |Du|^2\big)\, \mathrm{d}z \Big)^{\frac{1}{2}}. \tag{71}$$

We now set $h = 0$ and verify that

$$\begin{aligned} \int_{Q^+} |u-h|^2\, \mathrm{d}z = \int_{Q^+} |u|^2\, \mathrm{d}z &\le \int_{Q^+} \big(|u|^2 + |Du|^2\big)\, \mathrm{d}z \\ &< \frac{1}{\delta^2} \Big(\int_{Q^+} (u\bar{\varphi}_t - \mathcal{A}\, Du\, , D\bar{\varphi})\, \mathrm{d}z \Big)^2. \end{aligned} \tag{72}$$

Thus, (65) holds with $C_\epsilon = \frac{1}{\delta^2}$. $\square$

Proof of the main theorem

We will now prove the theorem 3. Let us define

$$I(z_0, \rho) := \fint_{Q^+_\rho(z_0)} |u|^2\, \mathrm{d}z, \qquad I(\rho) := I(z_0, \rho). \tag{73}$$

The most important part of the proof including application of the A-caloric lemma is contained in the following lemma.

Lemma 10 *Let u be a weak solution of (2) where $\Omega = Q^+, \Gamma = D$. Then there is a constant $c > 0$ with the following property: for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for any $z_0 \in (-1,0) \times D$ and $\rho > 0$ which fulfil $Q^+_{2\rho}(z_0) \subset\subset Q^+$ and for any $\sigma \in (0, \rho)$ holds*

$$I(\sigma) \le c \Big[\big(\frac{\rho}{\sigma}\big)^{n+2} \Big(C_\epsilon \omega\big(I(\rho) + 2\rho^2\big) + \epsilon \Big) + \big(\frac{\sigma}{\rho}\big)^2 \Big] I(2\rho). \tag{74}$$

Proof. Let $u \in \mathrm{L}^2(-1,0; \mathrm{W}^{1,2}(B^+, \mathbb{R}^N))$ be a weak solution to the system (2) on the domain $Q^+ = (-1,0) \times B^+$, with the homogeneous boundary condition $u = 0$ on the flat part $(-1,0) \times D$ of the boundary. Let $z_0 = (t_0, x_0) \in (-1,0) \times D$, $Q^+_{2\rho}(z_0) \subset Q^+$ and let $\varphi \in C^1_c(Q^+, \mathbb{R}^N)$ be a test function with $\operatorname{supp} \varphi \subset Q^+_\rho(z_0)$ and $\sup_{Q^+_\rho(z_0)} |D\varphi| \le \frac{1}{\rho}$.

Using (6) and Hölder inequality, we obtain

$$
\begin{aligned}
\Big|\int_{Q_\rho^+(z_0)} \big[u\varphi_t - A(z_0,0)DuD\varphi\big]\,\mathrm{d}z\Big| &= \Big|\int_{Q_\rho^+(z_0)} \big[\big(A(z,u)-A(z_0,0)\big)DuD\varphi\big]\,\mathrm{d}z\Big| \\
&\leq \underbrace{\Big(\int_{Q_\rho^+(z_0)} |A(z,u)-A(z_0,0)|^2\,\mathrm{d}z\Big)^{\frac{1}{2}}}_{I_1} \underbrace{\Big(\int_{Q_\rho^+(z_0)} |Du|^2\,\mathrm{d}z\Big)^{\frac{1}{2}}}_{I_2} \sup_{Q_\rho^+(z_0)} |D\varphi|.
\end{aligned}
\tag{75}
$$

The Caccioppoli inequality provides

$$
I_2 \leq \frac{c_{cacc}}{4\rho^2} \int_{Q_{2\rho}^+(z_0)} |u|^2\,\mathrm{d}z. \tag{76}
$$

Using the continuity of A, cf. (5) and concavity of ω we get

$$
\begin{aligned}
\frac{1}{\operatorname{meas} Q_\rho^+} I_1 &\leq ⨍_{Q_\rho^+(z_0)} 2L|A(z,u)-A(z_0,0)|\,\mathrm{d}z \\
&\leq 2L ⨍_{Q_\rho^+(z_0)} \omega\big(|x-x_0|^2+|t-t_0|+|u(t,x)|^2\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\leq 2L\omega\Big(⨍_{Q_\rho^+(z_0)} \big(\underbrace{|x-x_0|^2+|t-t_0|}_{\leq 2\rho^2}+|u(t,x)|^2\big)\,\mathrm{d}x\,\mathrm{d}t\Big) \\
&= 2L\omega\Big(2\rho^2 + ⨍_{Q_\rho^+(z_0)} |u|^2\,\mathrm{d}z\Big).
\end{aligned}
\tag{77}
$$

We use the estimates for I_1 and I_2 and recall $\sup_{Q_\rho^+(z_0)} |D\varphi| \leq \frac{1}{\rho}$ to obtain

$$
\begin{aligned}
&\Big|\int_{Q_\rho^+(z_0)} \big[u\varphi_t - A(z_0,0)DuD\varphi\big]\,\mathrm{d}z\Big| \\
&\qquad \leq c\sqrt{\operatorname{meas}(Q_\rho^+)\,\omega\big(I(\rho)+2\rho^2\big)}\,\sqrt{\frac{1}{\rho^2}\operatorname{meas}(Q_\rho^+)\,I(2\rho)}\,\frac{1}{\rho} \\
&\qquad \leq c\operatorname{meas}(Q_\rho^+)\frac{1}{\rho^2}\sqrt{I(2\rho)\omega\big(I(\rho)+2\rho^2\big)}.
\end{aligned}
\tag{78}
$$

We now take fixed $\epsilon > 0$ which we will specify later and apply lemma 9. It ensures existence of an $\mathcal{A} := A(z_0,0)$-caloric function h which approximates u. More precisely, there holds (63), (64) and (65). We utilize (65), (78) and the

Caccioppoli inequality to get:

$$
\begin{aligned}
\fint_{Q_\rho^+(z_0)} |u-h|^2 \,\mathrm{d}z &\le C_\epsilon \Big(\rho^2 \fint_{Q_\rho^+(z_0)} [u\varphi_t - A(z_0,0) Du D\varphi] \,\mathrm{d}z \Big)^2 \\
&\quad + \epsilon \fint_{Q_\rho^+(z_0)} \big(|u|^2 + \rho^2 |Du|^2 \big) \,\mathrm{d}z \\
&\le C_\epsilon \Big(\rho^2 \frac{c}{\rho^2} \sqrt{I(2\rho) \omega\big(I(\rho) + 2\rho^2\big)} \Big)^2 \\
&\quad + \epsilon \fint_{Q_\rho^+(z_0)} |u|^2 \,\mathrm{d}z + \epsilon c_{cacc} \fint_{Q_{2\rho}^+(z_0)} |u|^2 \,\mathrm{d}z \\
&\le I(2\rho) \Big(C_\epsilon \omega\big(I(\rho) + 2\rho^2\big) + c\epsilon \Big).
\end{aligned} \tag{79}
$$

Let σ be any positive number satisfying $\sigma < \rho$. We use the Campanato inequality (12) and (63) on the second line

$$
\begin{aligned}
\int_{Q_\sigma^+(z_0)} |u|^2 \,\mathrm{d}z &\le 2 \int_{Q_\sigma^+(z_0)} |u-h|^2 \,\mathrm{d}z + 2 \int_{Q_\sigma^+(z_0)} |h|^2 \,\mathrm{d}z \\
&\le 2 \int_{Q_\rho^+(z_0)} |u-h|^2 \,\mathrm{d}z + c \big(\frac{\sigma}{\rho} \big)^{n+4} \int_{Q_\rho^+(z_0)} |h|^2 \,\mathrm{d}z.
\end{aligned} \tag{80}
$$

The relation (64) and the Caccioppoli inequality imply

$$
\begin{aligned}
\int_{Q_\rho^+(z_0)} |h|^2 \,\mathrm{d}z &\le \int_{Q_\rho^+(z_0)} \big(|h|^2 + \rho^2 |Dh|^2 \big) \,\mathrm{d}z \\
&\le \int_{Q_\rho^+(z_0)} \big(|u|^2 + \rho^2 |Du|^2 \big) \,\mathrm{d}z \le c \int_{Q_{2\rho}^+(z_0)} |u|^2 \,\mathrm{d}z.
\end{aligned} \tag{81}
$$

Using (80), (81) and (79) we obtain

$$
\begin{aligned}
I(\sigma) &\le 2 \big(\frac{\rho}{\sigma} \big)^{n+2} \fint_{Q_\rho^+(z_0)} |u-h|^2 \,\mathrm{d}z + 2 \big(\frac{\sigma}{\rho} \big)^2 \fint_{Q_\rho^+(z_0)} |h|^2 \,\mathrm{d}z \\
&\le 2 \big(\frac{\rho}{\sigma} \big)^{n+2} \fint_{Q_\rho^+(z_0)} |u-h|^2 \,\mathrm{d}z + c \big(\frac{\sigma}{\rho} \big)^2 \fint_{Q_{2\rho}^+(z_0)} |u|^2 \,\mathrm{d}z \\
&\le 2 \big(\frac{\rho}{\sigma} \big)^{n+2} I(2\rho) \Big(C_\epsilon \omega\big(I(\rho) + 2\rho^2\big) + c\epsilon \Big) + c \big(\frac{\sigma}{\rho} \big)^2 \fint_{Q_{2\rho}^+(z_0)} |u|^2 \,\mathrm{d}z. \\
&\le c \Big[\big(\frac{\rho}{\sigma} \big)^{n+2} \Big(C_\epsilon \omega\big(I(\rho) + 2\rho^2\big) + \epsilon \Big) + \big(\frac{\sigma}{\rho} \big)^2 \Big] I(2\rho).
\end{aligned} \tag{82}
$$

□

The rest of the proof of the theorem 3 consists in a standard iterative procedure for estimates on cylinders and half-cylinders. At first, we deduce the estimate on half-cylinders.

Lemma 11 *Let $R_0 > 0$ be arbitrarily small. Under the assumptions of theorem 3 there exist $\rho_0 \in (0, R_0)$, $c_1 > 0$ and a "flat neighborhood" $V(y_0) \subset (-1,0) \times D$ of the point y_0 such that*

$$\forall z_0 \in V(y_0)\ \forall \rho \in (0, \rho_0) \quad I(z_0, \rho) \le c_1 \big(\frac{\rho}{\rho_0}\big)^{2\alpha} I(z_0, \rho_0). \tag{83}$$

Proof. Let $z_0 \in (-1,0) \times D$, $\rho > 0$, $Q^+_{2\rho}(z_0) \subset\subset Q^+$, $\sigma \in (0, \rho)$ and $\epsilon > 0$. From the previous lemma, we get the estimate (74). To simplify our reasoning, we rewrite it using $I(z_0, \rho) \le 2^{n+2} I(z_0, 2\rho)$ and the fact that ω is nondecreasing function:

$$I(z_0, \sigma) \le c\Big[\big(\frac{\rho}{\sigma}\big)^{n+2}\Big(C_\epsilon \omega\big(2^{n+2} I(z_0, 2\rho) + 2\rho^2\big) + \epsilon\Big) + \big(\frac{\sigma}{\rho}\big)^2\Big] I(z_0, 2\rho). \tag{84}$$

Let $\sigma = 2\rho\tau$, where $\tau \in (0, \frac{1}{2})$ is to be specified later. We get

$$I(z_0, 2\tau\rho) \le \tau^{2\alpha} \underbrace{\tau^{2-2\alpha} c\Big[\tau^{-n-4}\Big(C_\epsilon \omega\big(2^{n+2} I(z_0, 2\rho) + 2\rho^2\big) + \epsilon\Big) + 1\Big]}_{\text{we want to make this smaller than 1}} I(z_0, 2\rho). \tag{85}$$

Since $\alpha \in (0,1)$ we can choose $\tau \in (0, \frac{1}{2})$ so that $2c\tau^{2-2\alpha} < 1$. Now we set $\epsilon := \frac{1}{2}\tau^{n+4}$ and find $\tilde{\epsilon} > 0$ such that

$$C_\epsilon\, \omega(\tilde{\epsilon}) \le \frac{1}{2}\tau^{n+4}.$$

Recalling (8), we choose $\rho_0 \in (0, \frac{R_0}{2})$ small enough to ensure that

$$2^{n+2} I(y_0, 2\rho_0) + 2\rho_0^2 \le \frac{\tilde{\epsilon}}{2}.$$

The continuity of $I(\cdot, 2\rho)$ implies that there exists some "flat neighborhood" $V(y_0) \subset (-1,0) \times D$ of y_0 such that

$$2^{n+2} I(z_0, 2\rho_0) + 2\rho_0^2 \le \tilde{\epsilon} \quad \text{holds for all } z_0 \in V(y_0). \tag{86}$$

Thus we get

$$C_\epsilon\, \omega\big(2^{n+2} I(z_0, 2\rho_0) + 2\rho_0^2\big) \le \frac{1}{2}\tau^{n+4} \qquad \forall z_0 \in V(y_0).$$

Our choice of $V(y_0)$ and constants τ, ϵ and ρ_0 implies

$$\forall z_0 \in V(y_0) \quad c\tau^{2-2\alpha}\Big[\tau^{-n-4}\Big(C_\epsilon \omega\big(2^{n+2} I(z_0, 2\rho_0) + 2\rho_0^2\big) + \epsilon\Big) + 1\Big] < 1 \tag{87}$$

and thus by (85) we get

$$\forall z_0 \in V(y_0) \quad I(z_0, 2\rho_0\tau) \le \tau^{2\alpha} I(z_0, 2\rho_0).$$

Using stepwisely $2\rho_0\tau^k$ with $k = 1, 2, 3, \ldots$ instead of $2\rho_0$, we see that (87) remains true in all steps and thus we obtain

$$\forall k \in \mathbb{N}_0 \quad I(z_0, 2\rho_0\tau^{k+1}) \leq \tau^{2\alpha} I(z_0, 2\rho_0\tau^k). \tag{88}$$

This yields

$$\forall k \in \mathbb{N} \quad I(z_0, 2\rho_0\tau^k) \leq \tau^{2\alpha k} I(z_0, 2\rho_0). \tag{89}$$

Let us now choose arbitrary $\rho \in (0, 2\rho_0)$ and find $k \in \mathbb{N}_0$ such that $2\rho_0\tau^{k+1} < \rho \leq 2\rho_0\tau^k$. We estimate

$$\begin{aligned}
I(z_0,\rho) \leq \frac{1}{\text{meas } Q_\rho^+} \int_{Q^+_{2\rho_0\tau^k}(z_0)} |u|^2 \,\mathrm{d}z &\leq \frac{1}{\tau^{n+2}} \frac{1}{\text{meas } Q^+_{2\rho_0\tau^k}} \int_{Q^+_{2\rho_0\tau^k}(z_0)} |u|^2 \,\mathrm{d}z \\
&\leq \text{by (89)} \leq \frac{1}{\tau^{n+2}} \tau^{2\alpha k} \frac{1}{\text{meas } Q^+_{2\rho_0}} \int_{Q^+_{2\rho_0}(z_0)} |u|^2 \,\mathrm{d}z \\
&\leq \frac{1}{\tau^{n+2+2\alpha}} \Big(\frac{\rho}{2\rho_0}\Big)^{2\alpha} ⨍_{Q^+_{2\rho_0}(z_0)} |u|^2 \,\mathrm{d}z
\end{aligned} \tag{90}$$

(In the last inequality, we have used $\tau^{k+1} < \frac{\rho}{2\rho_0}$.) This yields

$$\forall \rho \in (0, 2\rho_0) \quad I(z_0,\rho) \leq c(\tau,\alpha)\Big(\frac{\rho}{2\rho_0}\Big)^{2\alpha} I(z_0, 2\rho_0).$$

This inequality implies (83) where we write ρ_0 instead of $2\rho_0$. $\square$

Analogical estimates hold on the whole cylinder:

Proposition 12 *Let u be a weak solution of (2). Let us assume the conditions (H1), (H2), (H3) and let $\alpha \in (0,1)$ be arbitrary. Then there exist $\epsilon > 0$ and $c_2 > 0$ such that for any (interior point) $z_0 \in U$ the condition*

$$⨍_{Q_R(z_0)} |u - u_{Q_R(z_0)}|^2 \,\mathrm{d}z + R^2 < \epsilon \tag{91}$$

(with any $R > 0$ such that $Q_R(z_0) \subset\subset U$) implies

$$\forall r \in (0,R) \quad ⨍_{Q_r(z_0)} |u - u_{Q_r(z_0)}|^2 \,\mathrm{d}z \leq c_2\Big(\frac{r}{R}\Big)^{2\alpha} ⨍_{Q_R(z_0)} |u - u_{Q_R(z_0)}|^2 \,\mathrm{d}z. \tag{92}$$

The estimate (92) can be proved in a similar way as the estimates on half-cylinders. We refer to [2] where the estimates are presented for solutions of nonlinear parabolic systems. Cf. also [1] , theorem A.4 and [3].

For any $z_0 \in Q^+$, let us define

$$J(z_0,\rho) := ⨍_{Q_\rho(z_0)\cap Q^+} |u - (u)_{Q_\rho(z_0)\cap Q^+}|^2 \,\mathrm{d}z.$$

We now prove the general decay estimate for J:

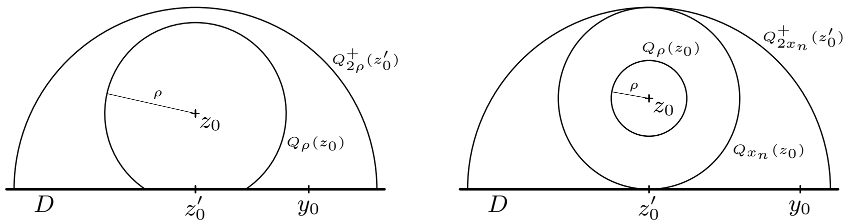

Figure 1: Estimates on two cylinders near the flat boundary. Left-hand side: $\rho > x_n$, right-hand side: $\rho < x_n$.

Lemma 13 *Under the assumptions of theorem 3 there exists $c > 0$, $\rho_0 > 0$ and a neighborhood of the point y_0, whose intersection with Q^+ we denote by $U(y_0)$, such that*

$$\forall z_0 \in U(y_0)\ \forall \rho \in (0, \rho_0) \quad J(z_0, \rho) \le c\Big(\frac{\rho}{\rho_0}\Big)^{2\alpha}. \tag{93}$$

Proof. Let $z_0 = (t, x_1, \ldots, x_n) \in Q^+$ and $\rho > 0$. We realize that dist $(z_0, (-1, 0)\times D) = x_n$ and denote z_0' the orthogonal projection of z_0 onto $(-1, 0) \times D$, i.e., $z_0' = (t, x_1, \ldots, x_{n-1}, 0)$. The idea of the proof is outlined on the figure 1. If the cylinder $Q_\rho(z_0)$ intersects the flat boundary $(-1, 0) \times D$ (left-hand side of the figure) then we initially estimate $J(z_0, \rho)$ by $I(z_0', 2\rho)$ and then use the lemma 11 to estimate $I(z_0', 2\rho)$ by $I(z_0', \rho_0)$. If the cylinder $Q_\rho(z_0)$ does not intersect the flat boundary $(-1, 0)\times D$ (right-hand side of the figure) then we initially use the proposition 12 to estimate $J(z_0, \rho)$ by $J(z_0, x_n)$, then we estimate $J(z_0, x_n)$ by $I(z_0', 2x_n)$ and then use the lemma 11 to estimate $I(z_0', 2x_n)$ by $I(z_0', \rho_0)$.

Apparently, we have to choose $U(y_0)$ and ρ_0 small enough to ensure that the assumptions of lemma 11 and proposition 12 are fulfilled. To do this we consider ϵ from the proposition 12, c_1 from lemma 11 and find $\rho_0 > 0$ and $V(y_0)$ such that the assumptions of the lemma 11 are fulfilled and furthermore

$$\rho_0^2 \le \frac{\epsilon}{2}, \tag{94}$$

$$2^{n+1} c_1\, I(y_0, \rho_0) < \frac{\epsilon}{4} \qquad \text{(using (8))}. \tag{95}$$

Using the continuity of $I(\cdot, \rho_0)$, we choose $\tilde V(y_0) \subset (-1, 0) \times D$ to be a "flat neighborhood" of y_0 such that $\tilde V(y_0) \subset V(y_0)$ and

$$\forall z' \in \tilde V(y_0) \quad 2^{n+1} c_1 I(z', \rho_0) < \frac{\epsilon}{2}. \tag{96}$$

Now we put $U(y_0) = \tilde V(y_0) \times (0, \frac{\rho_0}{2})$. For any $z_0 \in U(y_0)$ we get

$$\begin{aligned} J(z_0, x_n) &= \fint_{Q_{x_n}(z_0)} |u|^2 \,\mathrm{d}z \le 2^{n+1} \fint_{Q^+_{2x_n}(z_0')} |u|^2 \,\mathrm{d}z \le \text{(lemma 11)} \\ &\le 2^{n+1} c_1 \underbrace{\Big(\frac{2x_n}{\rho_0}\Big)^{2\alpha}}_{\le 1} \fint_{Q^+_{\rho_0}(z_0')} |u|^2 \,\mathrm{d}z < \text{due to (96)} < \frac{\epsilon}{2}. \end{aligned} \tag{97}$$

Thus, with this choice of ρ_0 and $U(y_0)$, the condition (91) holds for any $z_0 \in U(y_0)$ and $R = x_n$. We can thus estimate $J(z_0, \rho)$ by $J(z_0, x_n)$ (prop. 12) for any $z_n \in U(y_0)$ a $\rho \in (0, x_n)$.

Let us now assume $z_0 \in U(y_0)$, $\rho \in (0, \frac{\rho_0}{2})$. We distinguish two different cases: $\rho \in \langle x_n, \frac{\rho_0}{2})$ and $\rho \in (0, x_n)$, cf. figure 1.

First case: let $\rho \in \langle x_n, \frac{\rho_0}{2})$. We see that $Q_\rho(z_0) \subset Q^+_{2\rho}(z_0')$. We get

$$\begin{aligned} J(z_0,\rho) &= \fint_{Q_\rho(z_0)\cap Q^+} |u|^2 \,\mathrm{d}z \le 2^{n+1} \fint_{Q^+_{2\rho}(z_0')} |u|^2 \,\mathrm{d}z \le \text{(lemma 11)} \\ &\le 2^{n+1} c_1 \Big(\frac{2\rho}{\rho_0}\Big)^{2\alpha} \fint_{Q^+_{\rho_0}(z_0')} |u|^2 \,\mathrm{d}z \le \text{(due to (96))} \le \epsilon 2^{2\alpha} \Big(\frac{\rho}{\rho_0}\Big)^{2\alpha}. \quad (98) \end{aligned}$$

Second case: let $\rho \in (0, x_n)$. We see that $Q_\rho(z_0) \subset Q_{x_n}(z_0) \subset Q^+_{2x_n}(z_0')$. We get

$$\begin{aligned} J(z_0,\rho) &\le \text{(proposition 12)} \le c_2 \Big(\frac{\rho}{x_n}\Big)^{2\alpha} J(z_0, x_n) \le c_2 2^{n+1} \Big(\frac{\rho}{x_n}\Big)^{2\alpha} I(z_0', 2x_n) \\ &\le \text{(lemma 11)} \le c_2 2^{n+1} \Big(\frac{\rho}{x_n}\Big)^{2\alpha} c_1 \Big(\frac{2x_n}{\rho_0}\Big)^{2\alpha} I(z_0', \rho_0) \\ &\le \text{(due to (96))} \le c_2 \epsilon 2^{2\alpha} \Big(\frac{\rho}{\rho_0}\Big)^{2\alpha}. \quad (99) \end{aligned}$$

We see that (93) holds with $c = \max(\epsilon 2^{2\alpha}, c_2 \epsilon 2^{2\alpha})$. $\square$

In accordance to the previous lemma, the solution u belongs to the Campanato space $\mathcal{L}^{2,n+2+2\alpha}(U(y_0), \delta)$ on some neighborhood of the "regular point" y_0. By the isomorphism of $\mathcal{L}^{2,n+2+2\alpha}(U(y_0), \delta)$ and $C^{0,\alpha}(\overline{U(y_0)}, \mathbb{R}^N, \delta)$ we finally obtain $u \in C^{0,\alpha}(\overline{U(y_0)}, \mathbb{R}^N, \delta)$. We have proved the theorem 3.

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