Fluids with pressure dependent viscosity: partial regularity of steady flows up to the boundary

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Abstract. We study partial regularity up to the flat boundary for stationary solutions to a class of systems of fluid mechanics modelling flows of incompressible fluids with shear and pressure dependent viscosity in 2D or 3D. We consider systems which are a kind of generalized Navier-Stokes system where stress tensor T has linear growth in symmetric velocity gradient and dependence of viscosity on the pressure is small with respect to the ellipticity constant

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1 Introduction

Let $\Omega \subset \mathbb{R}^d,\ (d = 2, 3),$ be a bounded domain with boundary $\partial\Omega$. We study a following problem : For given $f = (f_1, \cdots, f_d)\ :\ \Omega \longrightarrow \mathbb{R}^d$ and stress tensor $T(Dv, p)\ :\ \mathbb{R}^{d\times d} \times \mathbb{R} \longrightarrow \mathbb{R}^{d\times d}$ we look for $v = (v_1, \cdots, v_d)\ :\ \Omega \longrightarrow \mathbb{R}^d$ and $p : \Omega \longrightarrow \mathbb{R}$ solving

$$\begin{aligned}\sum_{k=1}^{d} v_k \frac{\partial v}{\partial x_k} - \operatorname{div} T(p, Dv) + \nabla p = f \ \text{ in } \Omega,\\ \operatorname{div} v = 0 \ \text{ in } \Omega,\\ v = 0 \ \text{ on } \partial\Omega,\end{aligned} \tag{1}$$

where Dv denotes the symmetric part of the velocity gradient ∇v, i.e.

$$Dv = \frac{1}{2}(\nabla v + \nabla^T v) \quad \text{with} \ \ D_{ij}v = \frac{1}{2}\left(\frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i}\right).$$

We assume throughout this section that

$$T(p, Dv) = \nu(p, |D|^2)\, Dv, \tag{2}$$

where a generalized viscosity ν is supposed to be continuously differentiable function of both variables. Moreover, there exist positive constants $\lambda_0,\ \lambda_1$ and ν_0 such that for arbitrary symmetric $d \times d$- matrices $\xi,\ D$ and any $p \in \mathbb{R}$ the following estimates hold

$$\begin{aligned}\lambda_0|\xi|^2 \le \frac{\partial T}{\partial D}(p, D)\xi : \xi \le \lambda_1|\xi|^2,\\ \left|\frac{\partial \nu}{\partial p}(p, |D|)D\right| \le \nu_0.\end{aligned} \tag{3}$$

The existence of solutions to the system (1) is proved in [8] under assumptions on growth conditions of T. In [8], authors prove that there exists a solution to system

$$
\begin{aligned}
(\text{div } v^\varepsilon)\frac{v^\varepsilon}{2} + \sum_{k=1}^{d} v_k \frac{\partial v^\varepsilon}{\partial x_k} - \text{div } T(p^\varepsilon, Dv^\varepsilon) + \nabla p^\varepsilon = f \ \text{ in } \Omega,\\
-\varepsilon \triangle p^\varepsilon + \varepsilon (p^\varepsilon - (p^\varepsilon)_\Omega) \text{div } v^\varepsilon = 0 \ \text{ in } \Omega,\\
\frac{\partial p^\varepsilon}{\partial n} = 0 \ \text{ and } \ v^\varepsilon = 0 \ \text{ on } \partial\Omega.
\end{aligned}
\tag{4}
$$

By allowing $\epsilon \to 0$, authors obtain solution to (1) from solutions to (4).
In a completely similar way, we get the following existence results to (1) under conditions (2), (3) and ν_0 small with respect to $\lambda_1, \ \lambda_2$.
The smoothness of u and p is a more delicate problem. As we deal with a system of nonlinear elliptic PDEs we can not expect full regularity in space dimensions $d \geq 3$. When proving partial regularity results for such model we come to so-called "blow up" system which has the form

$$
\begin{aligned}
-\text{div } (ADw) + B\nabla q = g \ \text{ in } \Omega,\\
\text{div } w = 0 \ \text{ in } \Omega
\end{aligned}
\tag{5}
$$

with a $d^2 \times d^2$ matrix $A = \left(A_{ij}^{\alpha\beta}\right)_{i,j,\alpha,\beta=1}^{d}$ and a $d \times d$ matrix $B = (B_{ij})_{i,j=1}^{d}$ given by

$$
A_{ik}^{jl} = \frac{\partial T_{ij}}{\partial \xi_{kl}}(a,b), \ i,j,k,l = 1,\cdots,d; \ \ B_{ij} = \delta_{ij} - \frac{\partial T_{ij}}{\partial \tau}(a,b), \ i,j = 1,\cdots,d
$$

where

$$
a = \lim_{R \to 0+} \frac{1}{\text{meas } (B_R(x_0))} \int_{B_R(x_0)} Du \, dx, \ \ b = \lim_{R \to 0+} \frac{1}{\text{meas } (B_R(x_0))} \int_{B_R(x_0)} p \, dx.
$$

Saying differently, behaviour of solutions to (5) predicts behaviour of solution to (1) in regular points x_0.
In this paper we extend the results of [19] to study the partial regularity of solutions to problem (1) up to the boundary. For simplicity, we consider only $m = 2$ and $\Omega = B_1^+(0) := B_1(0) \cap \{x \in \mathbb{R}^d; \ x_d > 0\}$. The results are obtained for ν_0 small enough with respect to $\lambda_0, \ \lambda_1$. For the case of solutions $u \in W_0^{1,2}(B_1^+(0))^d$, $p \in L^2(B_1^+(0))$ with supp u, supp $p \ \subset B_1^*(0) := B_1(0) \cap \{x \in \mathbb{R}^d; \ x_d \geq 0\}$ we prove first that $u \in W^{2,2}(B_1^+(0))^d$, $p \in W^{1,2}(B_1^+(0))$. Therefore, we can define an energy

$$
E^{u,p}(x,R) = \frac{1}{R^{\frac{d-2}{2}}} \|\nabla^2 u\|_{2,B_R^+(x)} + \frac{1}{R^{\frac{d-2}{2}}} \|\nabla p\|_{2,B_R^+(x)} + R^\alpha,
$$

where $x \in B_1^*(0), R \in (0, \text{dist } (x, \partial B_1^+(0) \backslash \Gamma))$ and $0 < \alpha < 1$.
Next we establish the estimates for decay of the energy in auxiliary linear system. Then we present the lemma for decay of energy of system (1) at any regular points $x \in B_1^*(0)$. As a consequence we obtain that if $\bar{x} \in \Gamma$ is such that

$$
\liminf_{R \longrightarrow 0^+} E^{u,p}(\bar{x}, R) = 0 \text{ and } \limsup_{R \longrightarrow 0^+} (|(u)_{B_R^+(\bar{x})}| + |(\nabla u)_{B_R^+(\bar{x})}| + |(p)_{B_R^+(\bar{x})}|) < \infty,
$$

then there exists $\delta > 0$ such that ∇u and p are Hölder continuous in $B_\delta^+(\bar{x})$.
Finally, we show that the $d - 2 + \epsilon$-Hausdorff measure of all singular points in Γ is zero for any $\epsilon > 0$.

2 Preliminaries

In this section, we introduce notations, definitions and also recall some well-known results used later.

Begin with some definitions and notations. Let Ω be a domain in $\mathbb{R}^d$ $(d \geq 2)$, $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $f \in W^{k,2}(\Omega)$, $v = (v_1, \cdots, v_d) \in W^{k,2}(\Omega)^d$ with $k \in \mathbb{N}$. We will use notations

$$\nabla f := (\frac{\partial f}{\partial x_j})_{j=1}^d,\ \nabla^2 f := (\frac{\partial^2 f}{\partial x_j \partial x_l})_{j,l=1}^d,\ Dv := \frac{1}{2}(\nabla v + \nabla^T v)$$
$$x = (x', x_d),\ x' = (x_1, \cdots, x_{d-1}),\ u = (u', u_d),\ u' = (u_1, \cdots, u_{d-1}),$$
$$\nabla f = (\nabla' f, \frac{\partial f}{\partial x_d}),\ \nabla' = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_{d-1}}),\ \frac{\partial v}{\partial x_j} := (\frac{\partial v_1}{\partial x_j} \cdots, \frac{\partial v_d}{\partial x_j}).$$
$$B_r(x_0) := \left\{x \in \mathbb{R}^d; |x - x_0| < r\right\},\ B_r^+(x_0) := B_r(x_0) \cap \{x \in \mathbb{R}^d; x_d > 0\},$$
$$B_r^*(x_0) := B_r(x_0) \cap \{x \in \mathbb{R}^d; x_d \geq 0\},\ x_0 \in \mathbb{R}^d,\ r > 0;$$
$$B_r := B_r(0),\ B'_r := \{y' \in \mathbb{R}^{d-1};\ |y'| < r\},$$
$$\Gamma_r := B_r(x_0) \cap \{x \in \mathbb{R}^d; x_d = 0\},\ r > 0,\ \Gamma := \Gamma_1,\ \Gamma_\infty := \{x \in \mathbb{R}^d; x_d = 0\}.$$

For an integrable function u we denote by $(u)_\Omega$ an integral mean value of u over Ω, i.e. $(u)_\Omega := \frac{1}{\text{meas }\Omega}\int_\Omega u(x)dx$. For $\Omega = B_{R_0}(x_0)$ the symbol $(u)_\Omega$ will be abbreviated to $(u)_{x_0,R_0}$.

If $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $y = (y_1, \cdots, y_m) \in \mathbb{R}^d$, and $M,\ N$ are $d \times d$ matrices we use the notations

$$x.y := \sum_{i=1}^d x_i y_i,\ x \otimes y := (x_i y_j)_{i,j=1}^d, Mx.y := \sum_{i,j=1}^d M_{ij} x_i y_j,\ M.N := \sum_{i,j=1}^d M_{ij} N_{ij}, \tag{6}$$

while for $d^2 \times d^2$ matrix A and $u, v \in W^{k,q}(\Omega)^d$ we write

$$A\nabla u : \nabla v := \sum_{\alpha,\beta,i,j=1}^d A_{ij}^{\alpha\beta} \frac{\partial u_i}{\partial x_\alpha}\frac{\partial v_j}{\partial x_\beta},\ \ ADu : Dv := \sum_{k,l,i,j=1}^d A_{ij}^{kl} D_{kl} u D_{ij} v \tag{7}$$

For points $x \in \mathbb{R}^d$ as well as for matrices $M = (M_{ij})_{i,j=1}^m$ we write

$$|x| := (\sum_{i=1}^d |x_i|^2)^{\frac{1}{2}}, |M| := (\sum_{i,j=1}^m |M_{ij}|^2)^{\frac{1}{2}}. \tag{8}$$

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $\mu > 0$, $k \in \mathbb{N}$, $0 < \alpha \leq 1$; $L^{2,\mu}(\Omega)$, $\mathcal{L}^{2,\mu}(\Omega)$ and $C^{0,\alpha}(\overline{\Omega})$ denote Morrey, Campanato and Hölder spaces with norms $\|u\|_{L^{2,\mu}(\Omega)}$, $\|u\|_{\mathcal{L}^{2,\mu}(\Omega)} = \|u\|_2 + [u]_{2,\mu;\Omega}$ and $\|u\|_{C^{k,\alpha}(\overline{\Omega})}$ respectively, where

$$\begin{aligned}
\|u\|_{L^{2,\mu}(\Omega)} &:= [\sup_{x\in\Omega, 0<\rho<\text{diam }\Omega} \rho^{-\mu}\int_{\Omega\cap B_\rho(x)} |u|^2\ dx]^{1/2},\\
[u]_{2,\mu;\Omega} &:= [\sup_{x\in\Omega, 0<\rho<\text{diam }\Omega} \rho^{-\mu}\int_{\Omega\cap B_\rho(x)} |u - (u)_{\Omega\cap B_\rho(x)}|^2\ dx]^{1/2},\\
\|u\|_{C^{k,\alpha}(\overline{\Omega})} &:= \max_{0\leq|\alpha|\leq k}[\sup_{x,y\in\Omega, x\neq y}\frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^\alpha}] + \|u\|_{C^k(\overline{\Omega})}.
\end{aligned} \tag{9}$$

where α is an d-tuple of nonnegative integers α_i, $|\alpha| = \sum_{i=1}^d \alpha_i$ and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$.

Now we recall the results on solvability of equations

$$\text{div } v = g, \ \nabla p = f.$$

Lemma 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, let Ω_0 be a nonemp- -ty subdomain of Ω and let $1 < q < \infty$, $q' = \frac{q}{q-1}$. Then it holds:*
a)There is a constant $C = C(q, \Omega) > 0$ such that for each $g \in L^q(\Omega)$ with $\int_\Omega g \, dx = 0$ there exists at least one $v \in W_0^{1,q}(\Omega)^d$ satisfying

$$\text{div } v = g \text{ in } \Omega, \ \|\nabla v\|_q \leq C\|g\|_q. \tag{10}$$

b)There is a constant $C = C(q, \Omega, \Omega_0) > 0$ such that for each $f \in W^{-1,q}(\Omega)^d$ satisfying condition $[f, v] = 0$ for all $v \in W^{1,q'}_{0,\text{div}}(\Omega)$ there exists a unique $p \in L^q(\Omega)$ satisfying

$$\nabla p = f \text{ in } \Omega, \ \int_{\Omega_0} p \, dx = 0 \text{ and } \|p\|_q \leq C\|f\|_{-1,q}. \tag{11}$$

See [24], Chapter 2, Lemma 2.1.1, 2.2.2.
Remark. It is easy to see that C does not depend on translations and rotations of the couple Ω, Ω_0. By scaling argument we deduce that for $\Omega = \Omega_0 = B_R(x_0)$ or $\Omega = \Omega_0 = B_R^+(x_0)(x_0 \in \Gamma_\infty)$ the constant C does not depend on R. In this case we denote C_{div} as the infimum of the constants C from the inequality (10).
In connection with estimates of full gradient and symmetric gradient we will use Korn's inequality

Lemma 2.2. *(Korn's inequality) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Then for every $u \in W_0^{1,2}(\Omega)$ it holds*

$$\|\nabla u\|_{2;\Omega} \leq \sqrt{2}\|Du\|_{2;\Omega}. \tag{12}$$

See [22], Chapter 6, Section 6.3, Theorem 3.1.

Due to the relation

$$\frac{\partial^2 u_i}{\partial u_j \partial u_k} = \frac{\partial D_{ik} u}{\partial x_j} + \frac{\partial D_{ij} u}{\partial x_k} - \frac{\partial D_{jk} u}{\partial x_i}$$

we have lemma

Lemma 2.3. *Let $u \in W^{2,q}(\Omega)$, then it holds*

$$\|\nabla^2 u\|_q \leq 3\|D(\nabla u)\|_q. \tag{13}$$

Denote $H^\alpha(\Sigma)$ Hausdorff α-dimensional measure of Σ. We recall the well-known results about estimates of Hausdorff measure of the singular set.

Lemma 2.4. *Let $f \in L^1_{loc}(\Omega)$, $0 \leq \alpha < d$. Set*

$$\Sigma^\alpha := \{x \in \Omega; \ \limsup_{\rho \to 0} \ \rho^{-\alpha} \int_{B_\rho(x) \cap \Omega} |f(y)| \, dy > 0\}.$$

Then

$$H^\alpha(\Sigma^\alpha) = 0. \tag{14}$$

In particular, if $u \in W^{1,q}_{loc}(\Omega)$, $q \leq d$ and

$$\Sigma := \{x \in \Omega; \ \limsup_{\rho \to 0} \ \rho^{q-d} \int_{B_\rho(x) \cap \Omega} |\nabla u(y)|^q \, dy > 0\},$$

then

$$H^{d-q}(\Sigma) = 0. \tag{15}$$

See [11], Chapter 4, Theorem 2.2.

Lemma 2.5. *Let $u \in W^{1,q}_{loc}(\Omega)$, $q \leq d$ and*

$$\Sigma' := \{x \in \Omega;\ \limsup_{\rho \to 0}\ |(u)_{x,\rho}| = \infty\},$$

then for any $\epsilon > 0$ we have

$$H^{d-q+\epsilon}(\Sigma') = 0. \tag{16}$$

See [11], Chapter 4, Theorem 2.1.

We conclude this part by providing a simple result from the linear algebra about a positive definite matrix.

Lemma 2.6. *Let M be $d \times d$ matrix : $\Omega \to \mathbb{R}^d \times \mathbb{R}^d$, $M \in L^\infty(\Omega)$. If M is a positive definite i.e. there exists a constant $\lambda > 0$ such that*

$$\sum_{i,j=1}^{d} M_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \ \textit{ in } \Omega \textit{ for all } \xi \in \mathbb{R}^d., \tag{17}$$

then M is regular and an inequality

$$|\mathrm{det\ M}| \geq \mathrm{C} \tag{18}$$

holds with a constant $C = C(\lambda, d) > 0$.

Proof. Let a $\xi \in \mathbb{R}^d$ satisfies equations $M\xi = 0$. Then

$$0 = M\xi.\xi \geq \lambda|\xi|^2 \geq 0,$$

thus $\xi = 0$. It implies that M is regular.
Next, we have

$$|M^{-1}| = \sqrt{d} \sup_{\eta \in \mathbb{R}^d, |\eta|=1} |M^{-1}\eta|.$$

Choose in 17: $\xi = M^{-1}\eta$, then

$$M\xi.\xi = \eta.M^{-1}\eta \geq \lambda|M^{-1}\eta|^2$$

at the same time

$$M\xi.\xi = \eta.M^{-1}\eta \leq |\eta||M^{-1}\eta|.$$

Thus

$$\lambda|M^{-1}\eta|^2 \leq |\eta||M^{-1}\eta| \ \text{ or } \ |M^{-1}\eta| \leq \frac{|\eta|}{\lambda}.$$

Therefore,

$$|M^{-1}| \leq \frac{\sqrt{d}}{\lambda}.$$

As $|M^{-1}| \leq \frac{\sqrt{d}}{\lambda}$, we deduce det $|M^{-1}| \leq C$ with a constant $C = C(\lambda, d) > 0$. Consequently, $|\det M| \geq \frac{1}{C}$. The lemma is proved. ☐

3 Higher differentiability of solutions

The higher differentiability of solutions to the system (1) in the interior is studied in [19] for T having subquadratic growth m. In [19] under the suitable conditions the authors show that $v \in W^{2,2}_{\text{loc}}(\Omega)^d, p \in W^{1,2}_{\text{loc}}(\Omega)$. By a similar way we get the following result

Theorem 3.1. *(Interior estimates) Let the assumption* (2), (3) *be satisfied,* $\nu_0 < \frac{\lambda_0}{(\lambda_0 + C_{\text{div},\Omega}\lambda_1)C_{\text{div},\Omega}}$ *and* $f \in L^2(\Omega)$. *Let* $(v,p) \in W^{1,2}_0(\Omega)^d \times L^2(\Omega)$ *is a weak solution to the system* (1). *Then* $v \in W^{2,2}_{\text{loc}}(\Omega), p \in W^{1,2}_{\text{loc}}(\Omega)$. *Moreover, for any subdomain* $\Omega' \subset\subset \Omega$ *the inequality*

$$\|\nabla^2 v\|_{2;\Omega'} + \|\nabla p\|_{2;\Omega'} \le C \tag{19}$$

holds with a positive constant C *which depends on* $\|v\|_2, \|p\|_2, \|f\|_2$ *and* $\text{dist}\,(\Omega', \Omega)$.

In this section, we prove higher differentiability of solution up to the boundary. Suppose that $\Omega = B_1^+(0)$. Then we have the following theorem

Theorem 3.2. *Let the assumption* (2), .(3) *be satisfied,* $\nu_0 < \min\left(\frac{\lambda_0}{(\lambda_0 + \mathrm{C_{div}}\lambda_1)\mathrm{C_{div}}}, \frac{\lambda_0}{\lambda_0 + 3(\mathrm{d}-1)(\lambda_1 - \lambda_0)}\right)$ *with* (C_{div} *defined in Remark after Lemma* 2.1), $f \in L^2(B_1^+(0))$. *Let* $(v,p) \in W^{1,2}_0(B_1^+(0))^d \times L^2(B_1^+(0))$ *be a weak solution to the system* (1) *and* $\text{supp}\, v,\ \text{supp}\, p \subset B_1^*(0)$. *Then* $v \in W^{2,2}(B_1^+(0)), p \in W^{1,2}(B_1^+(0))$. *Moreover, we have an estimate*

$$\|\nabla^2 v\|_{2;B_1^+(0)} + \|\nabla p\|_{2;B_1^+(0)} \le C \tag{20}$$

with a positive constant C *which depends on* $\|v\|_2, \|p\|_2$, *and* $\|f\|_2$.

Proof. By the same procedure as in the proof of higher differentiability of solution to (1) in the interior of domain Ω (see Theorem 5.1 in [8]) we get

$$D\frac{\partial v}{\partial x_s} \in L^2(B_1^+(0))^{d^2},\ \frac{\partial p}{\partial x_s} \in L^2(B_1^+(0)) \text{ for all } s = 1, \cdots, d-1,$$

and we have estimate

$$\|D\nabla' v\|_2 + \|\nabla' p\|_2 \le C \tag{21}$$

with a constant $C = C(\|v\|_2, \|p\|_2, \|f\|_2) > 0$.
From the assumption $\text{supp}\ v \subset B_1^*(0)$, $v = 0$ on Γ, it follows that $\frac{\partial v}{\partial x_s} \in W^{1,2}_0(B_1^+(0))^d$ for all $s = 1, \cdots, d-1$. We have from Korn's inequality that $\nabla \frac{\partial v}{\partial x_s} \in L^2(B_1^+(0))^{d^2}$ for all $s = 1, \cdots, d-1$.
Since $\text{div}\, v = 0$, $\nabla \frac{\partial v}{\partial x_s} \in L^2(B_1^+(0))^{d^2}$ for all $s = 1, \cdots, d-1$, we get

$$\frac{\partial^2 v_d}{\partial^2 x_d} \in L^2(B_1^+(0)). \tag{22}$$

Therefore, it is sufficient to prove that $\frac{\partial D_{id} v}{\partial x_d} \in L^2(B_1^+(0))$, $i = 1, \cdots, d-1$ and $\frac{\partial p}{\partial x_d} \in L^2(B_1^+(0))$.
Next, Theorem 3.1 guarantees the existence of second derivatives of v and first derivatives of p which are locally square integrable on $B_1^+(0)$. Thus a.e. on $B_1^+(0)$ it holds

$$\begin{aligned}\sum_{k=1}^{d} v_k \frac{\partial v_i}{\partial x_k} - \sum_{j=1}^{d-1} \frac{\partial T_{ij}(p, Dv)}{\partial D}\frac{\partial Dv}{\partial x_j} - \sum_{j=1}^{d-1} \frac{\partial T_{ij}(p, Dv)}{\partial p}\frac{\partial p}{\partial x_j} \\ - \frac{\partial T_{id}(p, Dv)}{\partial x_d} + \nabla p = f \ \ ,\ \ i = 1, \cdots, d\end{aligned} \tag{23}$$

or

$$-\frac{\partial T_{id}(p,Dv)}{\partial D}\frac{\partial Dv}{\partial x_d}-\frac{\partial T_{id}(p,Dv)}{\partial p}\frac{\partial p}{\partial x_d}+\sum_{k=1}^{d}v_k\frac{\partial v_i}{\partial x_k}-\sum_{j=1}^{d-1}\frac{\partial T_{ij}(p,Dv)}{\partial D}\frac{\partial Dv}{\partial x_j}$$

$$-\sum_{j=1}^{d-1}\frac{\partial T_{ij}(p,Dv)}{\partial p}\frac{\partial p}{\partial x_j}+\nabla p=f\ \ ,\ \ i=1,\cdots,d. \tag{24}$$

Hence

$$\sum_{k=1}^{d-1}2\frac{\partial T_{id}(p,Dv)}{\partial D_{kd}}\frac{\partial D_{kd}v}{\partial x_d}+(\frac{\partial T_{id}(p,Dv)}{\partial p}-\delta_{id})\frac{\partial p}{\partial x_d}=\sum_{k=1}^{d}v_k\frac{\partial v_i}{\partial x_k}+F_i\ \ ,\ \ i=1,\cdots,d \tag{25}$$

where F_i are given by

$$F_i:=-\sum_{k,l=1}^{d-1}2\frac{\partial T_{id}(p,Dv)}{\partial D_{kl}}\frac{\partial D_{kl}v}{\partial x_d}-\frac{\partial T_{id}(p,Dv)}{\partial D_{dd}}\frac{\partial D_{dd}v}{\partial x_d}-\sum_{j=1}^{d-1}\frac{\partial T_{ij}(p,Dv)}{\partial D}$$

$$\frac{\partial Dv}{\partial x_j}-\sum_{j=1}^{d-1}\frac{\partial T_{ij}(p,Dv)}{\partial p}\frac{\partial p}{\partial x_j}+(1-\delta_{id})\frac{\partial p}{\partial x_i}-f_i\ \ ,\ \ i=1,\cdots,d. \tag{26}$$

It is easy to see that (21) and (22) implies

$$F_i\in L^2(B_1^+(0));\ i=1,\cdots,d. \tag{27}$$

Now, we consider the system (25) at first as a linear system in the unknowns $\frac{\partial D_{kd}v}{\partial x_d},\ \ k=1,\cdots,d-1,\ \frac{\partial p}{\partial x_d}$.
Denote $(R_{ik})_{i,k=1}^{d}$ the matrix of system (25) i.e.

$$R_{ik}=2\frac{\partial T_{id}(p,Dv)}{\partial D_{kd}}\ \text{if}\ \mathrm{k<d,i<d};\ \mathrm{R_{id}}=\frac{\partial T_{id}(p,Dv)}{\partial p}\ \text{if}\ \mathrm{k=d,i<d};$$

$$R_{dd}=\frac{\partial T_{dd}(p,Dv)}{\partial p}-1.$$

Multiply d^{th} row of R by $(1-\frac{\partial T_{dd}(p,Dv)}{\partial p})^{-1}\frac{\partial T_{id}(p,Dv)}{\partial p}$ and then subtract it from i^{th} row $(i=1,\cdots,d-1)$. We obtain a new matrix whose determinant det $(R)=(\frac{\partial T_{dd}(p,Dv)}{\partial p}-1)2^{d-1}\det S$ where S is a $(d-1)\times(d-1)$ matrix given by

$$S_{ik}=\frac{\partial T_{id}(p,Dv)}{\partial D_{kd}}+(1-\frac{\partial T_{dd}(p,Dv)}{\partial p})^{-1}\frac{\partial T_{id}(p,Dv)}{\partial p}\frac{\partial T_{dd}(p,Dv)}{\partial D_{kd}}.$$

Moreover, we have

$$\sum_{k=1}^{d-1}S_{ik}\frac{\partial D_{kd}v}{\partial x_d}=\sum_{k=1}^{d}v_k\frac{\partial v_i}{\partial x_k}+F_i+[\sum_{k=1}^{d}v_k\frac{\partial v_d}{\partial x_k}+F_d]$$

$$[1-\frac{\partial T_{dd}(p,Dv)}{\partial p}]^{-1}\frac{\partial T_{id}(p,Dv)}{\partial p}=:H_i,\ \ i=1,\cdots,d-1. \tag{28}$$

Next, we show that S is positive definite matrix. In fact, we have from (3)

$$\sum_{i,k=1}^{d-1} S_{ik}\zeta_i\zeta_k = \sum_{i,k=1}^{d-1} \frac{\partial T_{id}(p,Dv)}{\partial D_{kd}}\zeta_i\zeta_k + \sum_{i,k=1}^{d-1} [(1-\frac{\partial T_{dd}(p,Dv)}{\partial p})^{-1}$$

$$\frac{\partial T_{id}(p,Dv)}{\partial p}\frac{\partial T_{dd}(p,Dv)}{\partial D_{kd}}]\zeta_i\zeta_k \geq \frac{\lambda_0}{4}|\zeta|^2 - \sum_{i,k=1}^{d-1}\frac{\nu_0}{1-\nu_0}|\frac{\partial T_{dd}(p,Dv)}{\partial D_{kd}}||\zeta_i||\zeta_k|$$

for all $\zeta \in \mathbb{R}^{d-1}$.
On the other hand, for every $k \in \{1,\cdots,d-1\}$ by choosing $\xi_{dd} = \xi_{kd} = \xi_{dk} = 1, \xi_{ij} = 0$ if $(i,j) \neq (d,d),(d,k)$ and (k,d) the inequality (3) shows that

$$3\lambda_1 \geq 4\frac{\partial T_{kd}(p,Dv)}{\partial D_{kd}} + 4\frac{\partial T_{dd}(p,Dv)}{\partial D_{kd}} + \frac{\partial T_{dd}(p,Dv)}{\partial D_{dd}} \geq 3\lambda_0.$$

We also have

$$\lambda_1 \geq \frac{\partial T_{dd}(p,Dv)}{\partial D_{dd}} \geq \lambda_0;\ 2\lambda_1 \geq 4\frac{\partial T_{kd}(p,Dv)}{\partial D_{kd}} \geq 2\lambda_0.$$

Hence

$$|\frac{\partial T_{dd}(p,Dv)}{\partial D_{kd}}| \leq \frac{3}{4}(\lambda_1-\lambda_0).$$

Therefore, we obtain an inequality

$$\sum_{i,k=1}^{d-1} S_{ik}\zeta_i\zeta_k \geq \left[\frac{\lambda_0}{4} - \frac{3}{4}(d-1)\frac{\nu_0}{1-\nu_0}(\lambda_1-\lambda_0)\right]|\zeta|^2 = \lambda|\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^{d-1}. \tag{29}$$

Thanks to $\nu_0 < \frac{\lambda_0}{\lambda_0+3(d-1)(\lambda_1-\lambda_0)}$ we get $\lambda > 0$. From this we conclude that matrix S is positive definite. Thus there exists a positive constant C such that det $(N) \geq C$ (thanks to Lemma 2.6). Therefore

$$\det(R) \geq 2^{d-1}C \tag{30}$$

and system (25) can be solved for the unknowns $\frac{\partial D_{kd}v}{\partial x_d}$, $k = 1,\cdots,d-1$, $\frac{\partial p}{\partial x_d}$, for almost all $x \in B_1^+(0)$.
Sobolev embedding theorem shows that $L^6(B_1^+(0))^d \subset W_0^{1,2}(B_1^+(0))^d$. Using Young's inequality we get

$$\int_{B_1^+(0)} (v.\nabla v)^{\frac{3}{2}} \leq \frac{1}{4}\int_{B_1^+(0)} |v|^6\,dx + \frac{3}{4}\int_{B_1^+(0)}|\nabla v|^2\,dx \leq C.$$

It implies $v.\nabla v \in L^{\frac{3}{2}}(B_1^+(0))^d$. From this, (30) and the calculation of $\frac{\partial D_{kd}v}{\partial x_d}$, $k = 1,\cdots,d-1$, $\frac{\partial p}{\partial x_d}$ from (2.9) where $F_i \in L^2(B_1^+(0))$, $i = 1,\cdots,d$;
$v.\nabla v \in L^{\frac{3}{2}}(B_1^+(0))$, we deduce that

$$\frac{\partial D_{kd}v}{\partial x_d} \in L^{\frac{3}{2}}(B_1^+(0)),\ \ k = 1,\cdots,d-1,\ \frac{\partial p}{\partial x_d} \in L^{\frac{3}{2}}(B_1^+(0)).$$

It implies $v \in W_0^{2,\frac{3}{2}}(B_1^+(0))^d$.
(i) $d = 2$ Sobolev embedding theorem implies that $\nabla^2 v \in L^{\frac{3}{2}}(B_1^+(0))$, then $\nabla v \in L^6(B_1^+(0))$. Hence $v \in L^\infty(B_1^+(0))$ and $v.\nabla v \in L^2(B_1^+(0))^d$.
(ii) $d = 3$ Sobolev embedding theorem implies that $\nabla v \in L^3(B_1^+(0))$, $v \in L^6(B_1^+(0))$. Using Young's inequality we get

$$\int_{B_1^+(0)} (v.\nabla v)^2 \leq \frac{1}{3}\int_{B_1^+(0)} |v|^6\,dx + \frac{2}{3}\int_{B_1^+(0)}|\nabla v|^3\,dx \leq C.$$

It implies $v.\nabla v \in L^2(B_1^+(0))^d$ and $H_i \in L^2(B_1^+(0))$ for all $i = 1, \cdots, d-1$.
By a similar way, we obtain

$$\frac{\partial D_{kd} v}{\partial x_d} \in L^2(B_1^+(0)), \ \ k = 1, \cdots, d-1 \ , \ \frac{\partial p}{\partial x_d} \in L^2(B_1^+(0)). \tag{31}$$

Next, we estimate for $\frac{\partial D_{kd} v}{\partial x_d}, \ \ k = 1, \cdots, d-1, \ \frac{\partial p}{\partial x_d}$.
By setting $\zeta_k = \frac{\partial D_{id} v}{\partial x_d}, \ k = 1, \cdots, d-1;$, we get from (28)

$$\sum_{k=1}^{d-1} S_{ik}\zeta_k = H_i, \ i = 1, \cdots, d-1. \tag{32}$$

Thus

$$\sum_{i,k=1}^{d-1} S_{ik}\zeta_k\zeta_i = \sum_{i=1}^{d-1} H_i\zeta_i. \tag{33}$$

Consequently, $\lambda|\zeta|^2 \leq |H||\zeta|$ or

$$\lambda \sum_{k=1}^{d-1} |\frac{\partial D_{kd} v}{\partial x_d}| \leq |H| \ \text{ a.e. in } B_1^+(0).$$

Hence

$$\sum_{k=1}^{d-1} \|\frac{\partial D_{kd} v}{\partial x_d}\|_{2,B_1^+(0)} \leq C(\|v\|_2, \|p\|_2, \|f\|_2).$$

Now the d^{th} equation of system (25) implies

$$\|\frac{\partial p}{\partial x_d}\|_{2,B_1^+(0)} \leq C(\|v\|_2, \|p\|_2, \|f\|_2),$$

and we obtain the inequality (19). Theorem is proved. $\square$

4 Partial regularity of solutions

In [19] the interior partial regularity of solutions to (1) is studied for stress tensor with subquadratic growth $m \in (\frac{3d}{d+2}, 2)$. By the same way we get an analogous result for $m = 2$.

Theorem 4.1. *(Interior case) Let the assumptions* (2), (3) *are satisfied,* $\nu_0 < \frac{\lambda_0}{(\lambda_0 + C_{\mathrm{div}}\lambda_1)C_{\mathrm{div}}}$ *and* $f \in \mathrm{BMO}(\Omega)$. *Let* $(w, q) \in W^{1,2}(\Omega)^d \times L^2(\Omega)$ *be a weak solution to the system* (1). *Then there exists a closed subset* $\Sigma \subset \Omega$ *so that* $|\Sigma| = 0$ *and* $\nabla v, p$ *are locally Hölder continuous in* $\Omega \setminus \Sigma$.

The aim of this section is to study Hölder continuity of solutions to system (1) near the boundary. Suppose that $\Omega = B_1^+(0)$. We recall that $\Gamma := B_1(0) \cap \{x \in \mathbb{R}^d; \ x_d = 0\}$, $B_1^+(0) := B_1(0) \cap \{x \in \mathbb{R}^d; \ x_d > 0\}$ and $B_1^*(0) := B_1(0) \cap \{x \in \mathbb{R}^d; \ x_d \geq 0\}$. We start with several auxiliary results about Caccioppoli's and Campanato's type inequalities for solutions to a class of generalized Stokes systems with constant coefficients.

We consider a generalized Stokes system

$$\begin{aligned} -\mathrm{div}\,(ADw) + (E - B)\nabla q = 0 \ \text{ in } B_1^+(0), \\ \mathrm{div}\, w = 0 \ \text{ in } B_1^+(0), \end{aligned} \tag{34}$$

where A is a $d^2 \times d^2$ matrix ; B is a $d \times d$ matrix ; $A,\ B$ are a constant symmetric matrices; and E is the identity $d \times d$ matrix.

We assume throughout this part that there exist positive constants λ_0, λ_1, $\nu_0 < 1$ such that

$$\begin{aligned}&\lambda_0|\xi|^2 \leq A\xi : \xi \ \text{ for all symmetric matries } \ \xi \in \mathbb{R}^{d\times d}\text{and}\\ &|A_{ij}^{kl}| \leq \lambda_1,\ i,j,k,l = 1,\cdots,d;\ \ |B| \leq \nu_0.\end{aligned} \tag{35}$$

Moreover, suppose that

$$\nu_0 < \min\,(\frac{\lambda_0}{\lambda_0 + \sqrt{2}\lambda_1}, \frac{\lambda_0}{\lambda_0 + 3(\mathrm{d}-1)(\lambda_1 - \lambda_0)}). \tag{36}$$

We first establish Caccioppoli's inequality for solutions to the system (34):

Lemma 4.1. *(Caccioppoli's inequality) Let $(w,q) \in W^{1,2}(B_1^+(0))^d \times L^2(B_1^+(0))$ be a weak solution of system* (34) *and $w = 0$ on Γ. Then for all $\tau \in (0,1),\ R \leq 1$, we have $(w,q) \in W^{2,2}(B_{\tau R}^+(0))^d \times W^{1,2}(B_{\tau R}^+(0))$ and the inequalities*

$$\begin{aligned}\|\nabla^2 w\|_{2,B_{\tau R}^+(0)} &\leq \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)},\\ \|\nabla q\|_{2,B_{\tau R}^+(0)} &\leq \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)}\end{aligned} \tag{37}$$

hold with constant $C = C(\lambda_0, \lambda_1, \nu_0, d) > 0$.

Proof. The assumption (36) implies that there exists $(E-B)^{-1}$. Therefore, we can denote

$$H := (E-B)^{-1} - E = \sum_{n=1}^{\infty} B^n.$$

It is obvious that

$$|(E-B)^{-1}| \leq \frac{1}{1-\nu_0},\ |H| \leq \frac{\nu_0}{1-\nu_0}. \tag{38}$$

Then the system (34) can be written as

$$\text{div } w = 0, \quad -\text{div } [(E+H)ADw] + \nabla q = 0 \ \text{ in } B_1^+(0), \tag{39}$$

where $[(E+H)A]_{ij}^{kl} = \sum_{n=1}^{d}(E+H)_{in}A_{nj}^{kl}$.
Let $\eta \in C_0^\infty(\mathbb{R}^d)$ is a cut-off function :

$$\begin{gathered}\text{supp } \eta \subset B_{\frac{R+\tau R}{2}}(0),\ 0 \leq \eta \leq 1 \text{ on } B^+_{\frac{R+\tau R}{2}}(0),\ \eta \equiv 1 \text{ on } B_{\tau R}^+(0);\\ |\nabla \eta| \leq \frac{C}{R(1-\tau)}, |\nabla^2 \eta| \leq \frac{C}{R^2(1-\tau)^2}.\end{gathered}$$

Then we get

$$\text{div } (\eta w) = g, \quad -\text{div } [(E+H)AD(\eta w)] + \nabla[\eta(q - (q)_{B_R^+(0)})] = f \ \text{ in } B_1^+(0), \tag{40}$$

where $g(x) := \nabla\eta.w$ and $f(x) := (f_1(x), \cdots, f_d(x))$ is given by

$$\begin{aligned}f_i = \frac{1}{2}[(E+H)A]_{ij}^{kl}[\frac{\partial\eta}{\partial x_l}\frac{\partial w_k}{\partial x_j} + \frac{\partial\eta}{\partial x_k}\frac{\partial w_l}{\partial x_j} + \frac{\partial^2\eta}{\partial x_j \partial x_l}w_k + \frac{\partial^2\eta}{\partial x_j\partial x_k}w_l + 2\frac{\partial\eta}{\partial x_j}D_{kl}w]\\ + \frac{\partial\eta}{\partial x_i}(q - (q)_{B_R^+(0)}),\ \ i = 1,\cdots,d.\end{aligned}$$

The properties of η show that

$$\begin{aligned}
\|\nabla g\|_{2,B_R^+(0)} &\leq \|\nabla\eta\|_{2,B_R^+(0)}\|\nabla w\|_{2,B_R^+(0)} + \|\nabla^2\eta\|_{2,B_R^+(0)}\|w\|_{2,B_R^+(0)} \\
&\leq \frac{C}{R(1-\tau)}\|\nabla w\|_{2,B_R^+(0)} + \frac{C}{R^2(1-\tau)^2}\|w\|_{2,B_R^+(0)} \\
\|f\|_{2,B_R^+(0)} &\leq C[\|\nabla\eta\|_{2,B_R^+(0)}\|\nabla w\|_{2,B_R^+(0)} + \|\nabla^2\eta\|_{2,B_R^+(0)}\|w\|_{2,B_R^+(0)}] \\
&+ \|\nabla\eta\|_{2,B_R^+(0)}\|q-(q)_{B_R^+(0)}\|_{2,B_R^+(0)}.
\end{aligned}$$

Applying poincaré's inequality and Lemma 2.1 we get

$$\|\nabla w\|_{2,B_R^+(0)} \leq CR\|\nabla w\|_{2,B_R^+(0)}, \|q-(q)_{B_R^+(0)}\|_{2,B_R^+(0)} \leq C\|\nabla w\|_{2,B_R^+(0)}.$$

It implies the inequalities

$$\|\nabla g\|_{2,B_R^+(0)} \leq \frac{C}{R(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)}, \|f\|_{2,B_R^+(0)} \leq \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)} \tag{41}$$

hold with a constant $C = C(\lambda_1, \nu_0, d) > 0$.

We consider $s = 1, \cdots, d-1$. Set the difference quotient in the x_s direction $\triangle_{\delta,s}(\eta w) = \frac{1}{\delta}[\eta w(x+\delta e_s) - \eta w(x)]$ with $0 < |\delta| < \frac{R(1-\tau)}{2}$.
Since (w, q) solve the system (39) we have

$$\text{div}\ \ u_\delta = \triangle_{\delta,s}g, \qquad -\text{div}\ \ [(E+H)ADu_\delta] + \nabla p_\delta = \triangle_{\delta,s}f \quad \text{in}\ \ B_R^+(0), \tag{42}$$

where $u_\delta := \triangle_{\delta,s}(\eta w),\ p_\delta := \triangle_{\delta,s}[\eta(q-(q)_{B_R^+(0)})]$.
Thanks to Lemma 2.1 the quotient $\triangle_{\delta,s}g(x)$ can be written in the form $G_\delta \in W_0^{1,2}(B_R^+(0))^d$ such that $\|\nabla G_\delta\|_{2,B_R^+(0)} \leq C\|\triangle_{\delta,s}g\|_{2,B_R^+(0)}$ with C not depending on δ and s.
On the other hand, a calculation yields

$$\triangle_{\delta,s}f = \frac{\partial}{\partial x_s}[\frac{1}{\delta}\int_0^\delta f(x+te_s)dt]. \tag{43}$$

This shows that $\triangle_{\delta,s}f$ has the form div $F_{\delta,s}$ with some $F_{\delta,s} \in L^2(B_R^+(0))^{d^2}$ defined by (43).
Using Hölder's inequalities we have

$$\|\nabla G_\delta\|_{2,B_R^+(0)} \leq C\|\triangle_{\delta,s}g\|_{2,B_R^+(0)}$$

with C not depending on δ and s.
Using a weak formulation of the system and setting test a function $\varphi = (w_\delta - G_\delta) \in W_{0,\text{div}}^{1,2}(B_R^+(0))$ we get

$$\begin{aligned}
&\int_{B_R^+(0)} AD(u_\delta - G_\delta) : D(u_\delta - G_\delta) + AD(G_\delta) : D(u_\delta - G_\delta) + (HA)D(u_\delta - G_\delta) \\
&\qquad : \nabla(u_\delta - G_\delta) + (HA)DG_\delta : \nabla(u_\delta - G_\delta)\, dx = -\int_{B_R^+(0)} F_\delta.[\nabla(u_\delta - G_\delta)]\, dx.
\end{aligned}$$

Denote $I := \|D(u_\delta - G_\delta)\|_{2,B_R^+(0)}$. Then the assumption (35) and Young's inequality give

$$\begin{aligned}
&I^2 \leq \frac{1}{\lambda_0}\int_{B_R^+(0)} AD(u_\delta - G_\delta) : D(u_\delta - G_\delta) \leq \\
&\frac{C}{\lambda_0}\left(\lambda_1\|\triangle_{\delta,s}g\|_{2,B_R^+(0)}I + \|f\|_{2,B_R^+(0)}\right) + \frac{\lambda_1}{\lambda_0}|H|\|\nabla(u_\delta - G_\delta)\|_{2,B_R^+(0)}I.
\end{aligned} \tag{44}$$

Korn's inequality implies $\|\nabla(u_\delta - G_\delta)\|_{2,B_R^+(0)} \le \sqrt{2}I$. Inserting this inequality into (44) we obtain

$$(1-\sqrt{2}\frac{\lambda_1}{\lambda_0}|H|)I^2 \le \frac{C}{\lambda_0}\left(\lambda_1\|\triangle_{\delta,s}g\|_{2,B_R^+(0)} + \|f\|_{2,B_R^+(0)}\right)I.$$

The assumption (36) and (38) imply that $1-\sqrt{2}\frac{\lambda_1}{\lambda_0}|H|>0$. Henceforth,

$$\|D(u_\delta - G_\delta)\|_{2,B_R^+(0)} \le C\left(\|\triangle_{\delta,s}g\|_{2,B_R^+(0)} + \|f\|_{2,B_R^+(0)}\right) \tag{45}$$

and

$$\begin{aligned}\|D\triangle_{\delta,s}(\eta w)\|_{2,B_R^+(0)} = \|Du_\delta)\|_{2,B_R^+(0)} &\le \|D(u_\delta - G_\delta)\|_{2,B_R^+(0)} + \|DG_\delta\|_{2,B_R^+(0)}\\ &\le C\left(\|\triangle_{\delta,s}g\|_{2,B_R^+(0)} + \|f\|_{2,B_R^+(0)}\right)\end{aligned} \tag{46}$$

with a constant $C = C(\lambda_0,\lambda_1,d)>0$.
Due to $\int_{B_R^+(0)} p_\delta\ dx = 0$, we can apply Lemma 2.1 for the system (42) and using (46) we have estimate

$$\|\triangle_{\delta,s}(\eta(q-(q)_{B_R^+(0)}))\|_{2,B_R^+(0)} \le C\left(\|\triangle_{\delta,s}g\|_{2,B_R^+(0)} + \|f\|_{2,B_R^+(0)}\right). \tag{47}$$

Let $\delta\to 0$ in inequalities (46), (47) , we deduce $D\frac{\partial(\eta w)}{\partial x_s} \in L^2(B_R^+(0))^{d^2}, \frac{\partial(\eta(q-(q)_{B_R^+(0)}))}{\partial x_s} \in L^2(B_R^+(0))^d,\ s=1,\cdots,d-1$. Thanks to (41) and Nirenberg's lemma -see [12], II.3, Proposition , we also have estimates

$$\begin{aligned}\|D\frac{\partial(\eta w)}{\partial x_s}\|_{2,B_R^+(0)} &\le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)},\\ \|\frac{\partial(\eta(q-(q)_{B_R^+(0)}))}{\partial x_s}\|_{2,B_R^+(0)} &\le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)},\ s=1,\cdots,d-1.\end{aligned} \tag{48}$$

Thus

$$\|D\eta\frac{\partial w}{\partial x_s}\|_{2,B_R^+(0)} \le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)},\ s=1,\cdots,d-1. \tag{49}$$

Using Korn's inequality we also have $D(\eta\frac{\partial w}{\partial x_s}) \in L^2(B_R^+(0))^{d^2}$. Since div $w=0$, $\nabla(\eta\frac{\partial w}{\partial x_s}) \in L^2(B^+(0,R))^{d^2}$ for all $s=1,\cdots,d-1$, it implies that $\frac{\partial\eta\frac{\partial w_d}{\partial x_d}}{\partial x_d} \in L^2(B_{\tau R}^+(0)))$.
Therefore, it is sufficient to prove that $\frac{\partial D_{id}w}{\partial x_d} \in L^2(B_{\tau R}^+(0))$ for all $i=1,\cdots,d-1$ and $\frac{\partial q}{\partial x_d} \in L^2(B_{\tau R}^+(0))$.
Combining these inequalities and assertions, we can conclude that

$$(\nabla'\nabla w,\ \nabla' q) \in L^2(B_{\tau R}^+(0))^{d(d-1)} \times L^2(B_{\tau R}^+(0))^{d-1},\ \frac{\partial\frac{\partial w_d}{\partial x_d}}{\partial x_d} \in L^2(B_{\tau R}^+(0))$$

and

$$\begin{aligned}\|\nabla'\nabla w\|_{2,B_{\tau R}^+(0)} + \|\frac{\partial^2 w_d}{\partial^2 x_d}\|_{2,B_{\tau R}^+(0)} &\le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)},\\ \|\nabla' q\|_{2,B_{\tau R}^+(0)} &\le \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B_R^+(0)}\end{aligned} \tag{50}$$

hold with a constant $C = C(\lambda_0, \lambda_1, \nu_0, d) > 0$.
As in the proof of Theorem 3.2, we can show that $\frac{\partial D_{id} w}{\partial x_d} \in L^2(B^+_{\tau R}(0))$ for all $i = 1, \cdots, d-1$; $\frac{\partial q}{\partial x_d} \in L^2(B^+_{\tau R}(1))$ and inequalities

$$\begin{aligned} \|\frac{\partial D_{id} w}{\partial x_d}\|_{2,B^+_{\tau R}(0)} &\leq \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B^+_R(0)}, \quad i = 1, \cdots, d-1 \\ \|\frac{\partial q}{\partial x_d}\|_{2,B^+_{\tau R}(0)} &\leq \frac{C}{R^2(1-\tau)^2}\|\nabla w\|_{2,B^+_R(0)} \end{aligned} \tag{51}$$

hold with a constant $C = C(\lambda_0, \lambda_1, \nu_0, d) > 0$.
Finally, all above estimates implies $(\nabla^2 w,\ \nabla q) \in L^2(B^+_{\tau R}(0))^{d^2} \times L^2(B^+_{\tau R}(0))^d$ and we obtain the inequalities (37). □

Remark. Let $(w,\ q) \in W^{1,2}(B^+_1(0))^d \times L^2(B^+_1(0))$ be a weak solution of system (34) and $w = 0$ on Γ, then by the procedure of Lemma 4.1 and an induction argument, we can show that $w \in W^{k+2,2}(B^+_r(0))^d$, $q \in W^{k+1,2}(B^+_r(0)) (r < 1,\ k \in \mathbb{N})$.

Let $x \in \Gamma, 0 < R \leq \text{dist}\ \partial(x, B^+_1(0) \setminus \Gamma)$, we set

$$\begin{aligned} E^{w,q}_0(x, R) &:= \frac{1}{R^{\frac{d-2}{2}}}\|\nabla^2 w\|^2_{2,B^+_R(x)} + \frac{1}{R^{\frac{d-2}{2}}}\|\nabla q\|^2_{2,B^+_R(x)}, \\ E^{w,q}(x, R) &:= E^{w,q}_0(x, R) + R^\alpha \end{aligned} \tag{52}$$

where $0 < \alpha < 1$.

We have a following lemma for decay for the system (34)

Lemma 4.2. *Let $(w,\ q) \in W^{1,2}(B^+_1(0))^d \times L^2(B^+_1(0))$ be a weak solution of system (34) and $w = 0$ on Γ. Then for all $\tau, \alpha \in (0, 1)$, $R \leq 1$, there is a positive constant C^* such that*

$$E^{w,q}_0(0, \tau R) \leq C^* \tau E^{w,q}_0(0, R) \ \textit{and}\ E^{w,q}(0, \tau R) \leq C^* \tau^\alpha E^{w,q}(0, R) \tag{53}$$

where C^* depend only on $\lambda_0, \lambda_1, \nu_0, d$.

Proof. It is clear that (53) holds if $\tau \geq \frac{1}{2}$.

Let $0 < \tau < \frac{1}{2}$. Fix $k > \frac{d}{2}$. As all the tangent derivatives of w, q solve (39) and higher derivatives in normal direction are then calculated from (39) we can use repeatedly Lemma 4.1 for $j = 1, \cdots, k$ and $\nabla^j w, \nabla^j q$ on half balls with radii $\frac{1}{2}(1 + \frac{j-1}{k})$ and $\frac{1}{2}(1 + \frac{j}{k})$. Sobolev embedding implies $\nabla^2 w,\ \nabla q$ are essentially bounded on $B^+_{\frac{1}{2}}(0)$ and we get

$$\begin{aligned} \|\nabla^2 w\|_{L^\infty(B^+_{\frac{1}{2}}(0))} + \|\nabla q\|_{L^\infty(B^+_{\frac{1}{2}}(0))} &\leq C(\|\nabla^2 w\|_{k,2;B^+_{\frac{1}{2}}(0)} + \|\nabla q\|_{k,2;B^+_{\frac{1}{2}}(0)}) \\ &\leq C\|\nabla w\|_{2,B^+_1(0)}. \end{aligned} \tag{54}$$

with a constant $C = C(\lambda_0, \lambda_1, \nu_0, d) > 0$.
Thus inequalities

$$\begin{aligned} E^{w,q}_0(0, \tau) &= \frac{1}{\tau^{\frac{d-2}{2}}}\|\nabla^2 w\|_{2,B^+_\tau(0)} + \frac{1}{\tau^{\frac{d-2}{2}}}\|\nabla q\|_{2,B^+_\tau(0)} \\ &\leq C\tau(\|\nabla^2 w\|_{L^\infty(B^+_{\frac{1}{2}}(0))} + \|\nabla q\|_{L^\infty(B^+_{\frac{1}{2}}(0))}) \leq C^* \tau E^{w,q}_0(0, 1), \end{aligned}$$

hold with a constant $C^* = C(\lambda_0, \lambda_1, \nu_0, d) > 1$.
By simple scaling argument, from (34) we deduce

$$E^{w,q}_0(0, \tau R) \leq C^* \tau E^{w,q}_0(0, R).$$

Estimate for $E^{w,q}(0, \tau R)$ is obvious. □

Return to the nonlinear system (1).

Lemma 4.3. *(Decay for nonlinear system) Let the assumptions* (2), (3) *are satisfied,* $\nu_0 < \frac{\lambda_0}{(\lambda_0+C_0\lambda_1)C_0})$ *with* $C_0 = \max\,(C_{\rm div}, \sqrt{3(d-1)})$ *and* $f \in L^{2,\mu}(B_1^+(0))$ *for* $\mu > d-2\alpha$*. For all* $M \in (0,\infty)$ *and for all* $\tau \in (0,1)$*, there exists an* $\varepsilon > 0$ *and a constant* $C^* > 0$ *having the property : Let* $(v,p) \in W^{1,2}(B_1^+(0))^d) \times L^2(B_1^+(0))$ *with* $\operatorname{supp} v,\ \operatorname{supp} p \subset B_1^*(0)$ *be a weak solution of system* (1)*,* $v = 0$ *on* Γ*, let for* $x \in \Gamma, R \leq \operatorname{dist}\partial(\mathrm{x}, \mathrm{B}_1^+(0) \setminus \Gamma)$ *the inequalities*

$$E^{v,p}(x,R) < \varepsilon,\ |(\nabla v)_{B_R^+(x)}| + |(p)_{B_R^+(x)}| \leq M \tag{55}$$

hold. Then

$$E^{v,p}(x,\tau R) \leq 2C^*\tau^\alpha E^{v,p}(x,R). \tag{56}$$

Proof. We use the indirect method to prove this lemma. Assume that there exist $\tau,\ M, \varepsilon_h \to 0,\ x_h,\ R_h \to 0$ and weak solutions v_h, p_h to (1) whose supports are contained in $B_1^*(0)$ for which

$$E^{v_h,p_h}(x_h,R_h) = \varepsilon_h, |(\nabla v_h)_{B_{R_h}^+(x_h)}| + |(p_h)_{B_{R_h}^+(x_h)}| \leq M \tag{57}$$

and, at the same time,

$$E^{v_h,p_h}(x_h,\tau R_h) > 2C^*\tau^\alpha E^{v_h,p_h}(x_h,R_h). \tag{58}$$

Thanks to (57) we can assume

$$\begin{aligned}(p_h)_{B_{R_h}^+(x_h)} &\longrightarrow a \quad \text{in} \quad \mathbb{R},\\ (D^*v_h)_{B_{R_h}^+(x_h)} &\longrightarrow e \quad \text{in} \quad \mathbb{R}^{d^2}\end{aligned}$$

where D^*v_h is a symmetric $d \times d$ matrix given by

$$\begin{aligned}&D_{ij}^*v_h = 0 \,\text{if}\, i,j = 1,\cdots,d-1,\ D_{dd}^*v_h = \frac{\partial(v_h)_d}{\partial x_d},\\ &D_{id}^*v_h = D_{di}^*v_h = \frac{1}{2}\frac{\partial(v_h)_i}{\partial x_d}\ i,j = 1,\cdots,d-1.\end{aligned} \tag{59}$$

It is clear that e is a symmetric matrix and $|a| + |e| \leq M$.
Denote

$$A := \frac{\partial T(a,e)}{\partial D},\ B := \frac{\partial T(a,e)}{\partial p}. \tag{60}$$

Recall that for any a, e the ellipticity constant λ_0 and bounds λ_1, ν_0 for A, B remain the same as well as C^*.

Now we use a blow-up technique for p_h and v_h.

Step 1. Scaling and uniform estimates.
We consider functions w_h, q_h defined on $B_1^+(0)$, given by

$$\begin{aligned}w_h(y) :&= \frac{v_h(x_h + R_hy) - (\nabla v_h)_{B_{R_h}^+(x_h)}\ .(0,\cdots,0,y_d)^TR_h}{R_h\varepsilon_h},\\ q_h(y) :&= \frac{p(x_h + R_hy) - (p_h)_{B_{R_h}^+(x_h)}}{\varepsilon_h}, f_h(y) = \frac{R_h}{\varepsilon_h}f(x_h + R_hy).\end{aligned} \tag{61}$$

It follows that

$$
\begin{aligned}
\nabla_y w_h(y) &= \frac{\nabla_x v_h(x_h + R_h y) - (\nabla v_h)_{B^+_{R_h}(x_h)}(0,\cdots,0,1)^T}{\varepsilon_h},\\
\nabla_y q_h(y) &= \frac{R_h \nabla_x p_h(x_h + R_h y)}{\varepsilon_h},\ \nabla^2_y w_h(y) = \frac{R_h \nabla^2_x v_h(x_h + R_h y)}{\varepsilon_h},\\
D_y w_h(y) &= \frac{D_x v_h(x_h + R_h y) - (D^* v_h)_{B^+_{R_h}(x_h)}}{\varepsilon_h},
\end{aligned}
\tag{62}
$$

where $D^* v_h$ is in (59).
It easy to verify that

$$
w_h = \nabla' w_h = 0 \ \text{ on } \ \Gamma,\ (q_h)_{B^+_1(0)} = (\frac{\partial (w_h)_i}{\partial x_d})_{B^+_1(0)} = 0 \ \ i = 1,\cdots,d. \tag{63}
$$

The assumption (57) allows us to use Poincaré's inequality to have

$$
\begin{aligned}
\|\nabla^2 w_h(y)\|^2_{B^+_1(0)} &= \frac{1}{\varepsilon_h^2 R_h^{d-2}}\|\nabla^2 v_h\|^2_{B^+_{R_h}(x_h)} \le \frac{E^v_h(x_h,\tau R_h)^2}{\varepsilon_h^2} \le 1,\\
\|\nabla q_h(y)\|^2_{B^+_1(0)} &= \frac{1}{\varepsilon_h^2 R_h^{d-2}}\|\nabla p_h\|^2_{B^+_{R_h}(x_h)} \le \frac{E^v_h(x_h,\tau R_h)^2}{\varepsilon_h^2} \le 1,\\
\|\nabla w_h(y)\|^2_{B^+_1(0)} &\le C_P \|\nabla^2 w_h(y)\|^2_{B^+_1(0)} \le C_P,\\
\|q_h(y)\|^2_{B^+_1(0)} &\le C_P \|\nabla q_h(y)\|^2_{B^+_1(0)} \le C_P,\\
\|w_h(y)\|^2_{B^+_1(0)} &\le C_P \|\nabla w_h(y)\|^2_{B^+_1(0)} \le C_P^2,
\end{aligned}
\tag{64}
$$

where C_P is the constant from Poincaré's inequality on the half ball $B^+_1(0)$.
Hence $w_h,\ q_h$ are bounded in $W^{2,2}(B^+_1(0))^d,\ W^{1,2}(B^+_1(0))$. Then there exists a subsequence, still denoted by $w_h,\ q_h$ such that

$$
\begin{aligned}
(w_h, q_h) &\longrightarrow (w,q), \quad \text{weakly in} \quad W^{2,2}(B^+_1)^d \times W^{1,2}(B^+_1(0)),\\
(w_h, q_h) &\longrightarrow (w,q) \quad \text{strongly in} \quad W^{1,2}(B^+_1)^d \times L^2(B^+_1(0))^d,\\
w_h &\longrightarrow w,\ \ \nabla w_h \longrightarrow \nabla w,\ \ q_h \longrightarrow q \ \text{ a.e. on } \ (B^+_1(0)).
\end{aligned}
\tag{65}
$$

Next, putting instead of $p_h,\ v_h$ the rescaled qualities $q_h,\ w_h$ and $y = \frac{x-x_h}{R_h}$ into system (2.1), we obtain

$$
\begin{aligned}
&\int_{B^+_1(0)} \frac{1}{\varepsilon_h} T(q_h\varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}, Dw_h\varepsilon_h + (D^*v_h)_{B^+_{R_h}(x_h)}) : D_y\psi\, dy\\
&\quad + \int_{B^+_1(0)} \frac{1}{\varepsilon_h}[w_h R_h \varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B^+_{R_h}(x_h)}.(0,\cdots,0,y_d)^T R_h]\\
&\quad \otimes [w_h R_h \varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B^+_{R_h}(x_h)}.(0,\cdots,0,y_d)^T R_h].D_y\psi\, dy\\
&- \int_{B^+_1(0)} \frac{1}{\varepsilon_h}[q_h\varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}]\mathrm{div}\,_y\psi\, dy - \frac{R_h}{\varepsilon_h}\int_{B^+_1(0)} f_h\psi\, dy = 0 \quad \text{for all} \ \ \psi \in W^{1,2}_0(B^+_1(0))^d),\\
&\qquad \mathrm{div}\ w_h = \frac{1}{\varepsilon_h}(\frac{\partial (v_h)_d}{\partial x_d})_{B^+_{R_h}(x_h)} \ \text{ on } \ B^+_1(0).
\end{aligned}
\tag{66}
$$

We denote the four terms on the left hand side of (66) by $D_1,\ D_2,\ D_3,\ D_4$. Now we pass to the limit in (66) for $h \to 0$.

Step 2. Passage to the limit.
From the estimates (64), we deduce

$$\begin{aligned}
w_h R_h \varepsilon_h &\longrightarrow 0 \ \text{ strongly in } \ L^2(B_1^+(0))^d, \\
\nabla w_h \varepsilon_h &\longrightarrow 0 \ \text{ strongly in } \ L^2(B_1^+(0))^{d^2}, \\
q_h \varepsilon_h &\longrightarrow 0 \ \text{ strongly in } \ L^2(B_1^+(0)), \\
w_h R_h &\longrightarrow 0, \ \nabla w_h \varepsilon_h \longrightarrow 0, \ q_h \varepsilon_h \longrightarrow 0 \ \text{ a.e. on } \ B_1^+(0).
\end{aligned} \tag{67}$$

Moreover, we can assume that there exist functions $\tilde{w} \in L^2(B_1^+(0))^d$, $w^* \in L^2(B_1^+(0))^{d^2}$ such that $|w_h R_h \varepsilon_h| \leq \tilde{w}, |\nabla w_h \varepsilon_h| \leq w^*$ a.e. on $B_1^+(0)$. (Note that we pass to a not relabelled subsequence)
It implies that

$$\begin{aligned}
D w_h \varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)} &\longrightarrow a \ \text{ a.e. on } \ B_1^+(0), \\
q_h \varepsilon_h + (p_h)_{B^+_{R_h}(x_h)} &\longrightarrow e \ \text{ a.e. on } \ B_1^+(0).
\end{aligned} \tag{68}$$

We have

$$\begin{aligned}
D_1 &= \int_{B_1^+(0)} \frac{1}{\varepsilon_h} T(q_h \varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}, D w_h \varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)}).D_y \psi \, dy \\
&= \frac{1}{\varepsilon_h} \int_{B_1^+(0)} \left(\int_0^1 \frac{\partial}{\partial s} T(s q_h \varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}, s D w_h \varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)}) \, ds \right) .D\psi \, dy \\
&= \int_{B_1^+(0)} \left(\int_0^1 \frac{\partial T(s q_h \varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}, s D w_h \varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)})}{\partial D} \, ds \right) D w_h : D\psi(y) \, dy \\
&\quad + \int_{B_1^+(0)} \left(\int_0^1 \frac{\partial T(s q_h \varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}, s D w_h \varepsilon_h + (D^* v_h)_{B^+_{R_h}(x_h)})}{\partial p} \, ds \right) q_h . D\psi(y) \, dy.
\end{aligned} \tag{69}$$

Because of the continuity of T and the convergences (65), (68), we can pass to the limit in (69) and deduce that

$$D_1 \longrightarrow \int_{B_1^+(0)} A D w_h : D\psi(y) \, dy + \int_{B_1^+(0)} B q_h . D\psi(y) \, dy \ \text{ as } \ h \longrightarrow 0,$$

where A, B are given by (60).
We continue to pass to the limit in D_2 :

$$\begin{aligned}
D_2 = \int_{B_1^+(0)} & [w_h R_h + (\frac{\partial v_h}{\partial x_s})_{B^+_{R_h}(x_h)}.(0, \cdots, 0, y_d)^T \frac{R_h}{\varepsilon_h}] \otimes [w_h R_h \varepsilon_h \\
& + (\frac{\partial v_h}{\partial x_s})_{B^+_{R_h}(x_h)}.(0, \cdots, 0, y_d)^T R_h].D_y \psi \, dy.
\end{aligned} \tag{70}$$

Now, inequality (57) implies $\frac{R_h}{\varepsilon_h} = \frac{R_h^\alpha}{\varepsilon_h} R_h^{1-\alpha} \to 0$ as $h \to 0$. Combining this with (57), (65), (67), we can summarize $D_2 \to 0$ as h tend to zero.
Next

$$D_3 = \int_{B_1^+(0)} \frac{1}{\varepsilon_h} [q_h \varepsilon_h + (p_h)_{B^+_{R_h}(x_h)}] \mathrm{div}_{\,y} \psi \, dy = \int_{B_1^+(0)} q_h \mathrm{div}_{\,y} \psi \, dy \longrightarrow \int_{B_1^+(0)} q \mathrm{div}_{\,y} \psi \, dy \quad \text{as } \ h \longrightarrow 0.$$

Recall the assumption $f \in L^{2,\mu}(B_1^+(0))$ for $\mu > d-2\alpha$, hence for all $\psi(y) \in W_0^{1,2}(B_1^+(0))^d$ and $\mu = d-2\alpha+\varepsilon$ for ε small enough we obtain

$$
\begin{aligned}
|D_4| = |\frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} f_h(y)\psi\, dy| &\leq \left(\int_{B_1^+(0)} |f_h|^2\, dy \int_{B_1^+(0)} |\psi|^2\, dy\right)^{\frac{1}{2}} \\
&= \left(\frac{R_h^{2\alpha}}{\varepsilon_h^2} R_h^{\varepsilon} \frac{1}{R_h^{d-2\alpha+\varepsilon}} \int_{B_{R_h}^+(x_h)} |f|^2 dx \int_{B_1^+(0)} |\psi|^2\, dy\right)^{\frac{1}{2}} \longrightarrow 0 \text{ as } h \longrightarrow 0. \quad (71)
\end{aligned}
$$

Finally, it is easily seen that div $w_h \to$ div $w = 0$ as $h \to 0$. It follows that $w,\ q$ solve the system (34) with A, B given by (60). On the other hand, it is easy to check that $A,\ B$ satisfy the assumptions of Lemma 4.2. Therefore, we have

$$
E^{w,q}(x, \tau R) \leq C^* \tau^{\alpha} E^{w,q}(x, R). \quad (72)
$$

Step 3. Strong convergence on $B_\tau^+(0)$

Thanks to Theorem 3.2 and the system (1), derivatives $\frac{\partial v_h(x)}{\partial x_s},\ \frac{\partial p_h(x)}{\partial x_s}$ for $s = 1, \cdots, d$ satisfy a weak formulation

$$
\begin{aligned}
\int_{B_{R_h}^+(x_h)} [&\frac{\partial T(p_h(x), Dv_h(x))}{\partial D} D\frac{\partial v_h(x)}{\partial x_s} : D\varphi(x) + \frac{\partial T(p_h(x), Dv_h(x))}{\partial p_h}\frac{\partial p_h(x)}{\partial x_s} : D\varphi(x) \\
&+ \frac{\partial (v_h \otimes v_h)}{\partial x_s}.D_{jk}\varphi(x) - \frac{\partial p_h(x)}{\partial x_s}\text{div } \varphi(x)]\, dx = -\int_{B_{R_h}^+(x_h)} f(x)\frac{\partial \varphi(x)}{\partial x_s}\, dx
\end{aligned} \quad (73)
$$

for all $\varphi(x) \in W_0^{1,2}(B_{R_h}^+(x_h))^d$.

Substituting $y = \frac{x-x_h}{R_h}$ and using (61), (62); (73), we have

$$
\begin{aligned}
&\int_{B_1^+(0)} [\frac{\partial T(q_h\varepsilon_h + (p_h)_{B_{R_h}^+}, (D^*v_h)_{B_{R_h}^+} + Dw_h\varepsilon_h)}{\partial D} D\frac{\partial w_h(y)}{\partial y_s} : D\psi(y) \\
&+ \frac{R_h}{\varepsilon_h}\{[\frac{\partial w_h}{\partial y_s}\varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B_{R_h}^+}.(0, \cdots, 0, 1)^T] \otimes [w_h R_h \varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B_{R_h}^+}.(0, \cdots, 0, y_d)^T R_h] \\
&+ [w_h R_h \varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B_{R_h}^+} R_h (0, \cdots, 0, y_d)^T] \otimes [\frac{\partial w_h}{\partial y_s}\varepsilon_h + (\frac{\partial v)}{\partial x_s})_{B_{R_h}^+}.(0, \cdots, 0, 1)^T]\}D\psi(y) \\
&+ \frac{\partial T(q_h(y)\varepsilon_h + (p_h)_{B_{R_h}^+}, (D^*v_h)_{B_{R_h}^+} + Dw_h\varepsilon_h)}{\partial p}\frac{\partial q_h(y)}{\partial y_s} : D\psi(x) - \frac{\partial q_h}{\partial y_s}\text{div } \psi(y)]\, dy \\
&\qquad = -\frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} f_h(y)\frac{\partial \psi(y)}{\partial y_s}\, dy, \quad s = 1, \cdots, d, \quad (74)
\end{aligned}
$$

and div $w_h = \frac{R_h}{\varepsilon_h}(\frac{\partial (v_h)_d}{\partial x_d})_{B_{R_h}^+}$ on $B_1^+(0)$, where $\psi(y) = \varphi(x_h + R_h y) \in W_0^{1,2}(B_1^+(0))^d$, $B_{R_h}^+ := B_{R_h}^+(x_h)$.

The inequalities (58) and (57) imply

$$
E^{w_h,q_h}(0,\tau) > 2C^*\tau^{\alpha} E^{w_h,q_h}(0,1),\ E^{w_h,q_h}(0,1) = 1. \quad (75)
$$

Denote

$$
\begin{aligned}
A_h :&= \frac{\partial T(q_h\varepsilon_h + (p_h)_{B_{R_h}^+}, (Dw_h)_{B_{R_h}^+} + Dw_h\varepsilon_h)}{\partial D}, \\
B_h &= \frac{\partial T(q_h(y)\varepsilon_h + (p_h)_{B_{R_h}^+}, (Dw_h)_{B_{R_h}^+} + Dw_h\varepsilon_h}{\partial p}. \quad (76)
\end{aligned}
$$

It is clear that $A_h,\ B_h$ are uniformly bounded, $A_h \to A,\ B_h \to B$ a.e. on $B_1^+(0)$.
For simplicity, system (74) can rewritten in the following way

$$\int_{B_1^+(0)} \left(A_h D\frac{\partial w_h}{\partial y_s} : D\psi + (B_h - E)\frac{\partial q_h}{\partial y_s} : D\psi \right) dy + I_h \;=\; -\frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} f_h(y)\frac{\partial \psi(y)}{\partial y_s}\; dy \tag{77}$$

for all $\psi(y) \in W_0^{1,2}(B_1^+(0))^d$, where I_h is the second term on the left hand side of (74).

Set $u_h = \frac{\partial w_h}{\partial y_s}, \pi_h = \frac{\partial q_h}{\partial y_s};\ u = \frac{\partial w}{\partial y_s}, \pi = \frac{\partial q}{\partial y_s},\ s = 1, \cdots, d$. Then $u_h,\ q_h$ satisfy a weak formulation

$$\int_{B_1^+(0)} (A_h Du_h : D\psi + (B_h - E)\pi_h : D\psi)\, dy \;=\; -I_h - \frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} f_h(y)\frac{\partial \psi_h(y)}{\partial y_s}\; dy \tag{78}$$

for all $\psi \in W_0^{1,2}(B_1^+(0))^d$.
Thanks to (64) and (65), we have

$$\|\nabla u_h(y)\|^2_{2,B_1^+(0)} \le 1 \quad \|\pi_h(y)\|^2_{2,B_1^+(0)} \le 1 \quad \|u_h(y)\|^2_{2,B_1^+(0)} \le C_P \tag{79}$$

and

$$\begin{aligned} (u_h,\pi_h) &\longrightarrow (u,\pi), \quad \text{weakly in} \quad W^{1,2}(B_1^+(0))^d \times L^2(B_1^+(0)),\\ u_h &\longrightarrow u \quad \text{strongly in} \quad L^2(B_1^+)^d \;\; u_h \longrightarrow u \;\text{ a.e. on }\; (B_1^+(0)). \end{aligned} \tag{80}$$

Now consider $s = 1, \cdots, d-1$.
As $u_h = 0,\ u = 0$ on Γ, we can choose a test function $\psi_h = \eta^2(u_h - u)$, where $\eta \in C_0^\infty(\mathbb{R}^d)$ is a cut-off function : supp $\eta \subset B_{\frac{1+\tau}{2}}(0)$, $0 \le \eta \le 1$ on $\mathrm{B}^+_{\frac{1+\tau}{2}}(0)$, $\eta \equiv 1$ on $\mathrm{B}^+_\tau(0)$; $|\nabla \eta| \le \frac{C}{1-\tau}$.
Functions u, π solve the system (34), hence (78) can be written as

$$\int_{B_1^+(0)} (A_h D(u_h - u) : D\psi_h + (\pi_h - \pi)(B_h - E).D\psi_h)\, dy = -I_h - G_h$$

where

$$G_h := \int_{B_1^+(0)} (A_h Du : D\psi_h + \pi(B_h - E).D\psi_h)\, dy + \frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} f_h \frac{\partial \psi_h}{\partial y_s}\; dy.$$

Then

$$\int_{B_1^+(0)} \eta^2 A_h D(u_h - u) : D(u_h - u)\; dy = \int_{B_1^+(0)} (\pi_h - \pi)(E - B_h).D\psi_h\; dy - I_h - G_h - H_h,$$

where

$$H_h = \int_{B_1^+(0)} 2\eta A_h D(u_h - u) : [(u_h - u) \otimes \nabla\eta]\; dy.$$

Using the assumption (3), we get

$$\begin{aligned} \int_{B_1^+(0)} |\eta D(u_h - u)|^2 dy &\le \frac{1}{\lambda_0}\int_{B_1^+(0)} \eta^2 A_h D(u_h - u) : D(u_h - u)\; dy \\ &= \frac{1}{\lambda_0}[\int_{B_1^+(0)} (\pi_h - \pi)(E - B_h).D\psi_h\; dy - I_h - G_h - H_h]. \end{aligned} \tag{81}$$

Since $\psi_h = \eta^2(u_h - u),\ s = 1, \cdots, d-1$, (80), we deduce

$$\begin{aligned}
&(\psi_h) \longrightarrow 0, \quad \text{weakly in} \quad W^{1,2}(B_1^+(0)), \\
&(\psi_h) \longrightarrow 0 \quad \text{strongly in} \quad L^2(B_1^+(0))^d \ \text{ and a.e. on } \ B_1^+(0).
\end{aligned}$$

Thanks to $\frac{R_h}{\varepsilon_h} \to 0,\ \varepsilon_h \to 0$, (57), (64), (67) we obtain $I_h \to 0$ as $h \to 0$.
As u, π are satisfy the identity (34), we have

$$G_h \;=\; \int_{B_1^+(0)} ((A_h - A)Du : D\psi_h + \pi(B_h - B).D\psi_h)\, dy \;+\; \frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} f_h \frac{\partial \psi_h}{\partial y_s} \quad dy.$$

The coefficients $A_h - A,\ B_h - B$ are equibounded and tend to 0 a.e. on $B_1^+(0),\ Du \in L^2(B_1^+)^{d^2},\ \pi \in L^2(B_1^+)$ and $|\frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} f_h(y)\frac{\partial\psi_h}{\partial y_s}\ dy| \to 0$ as $h \to 0$, which implies that $G_h \to 0$ as $h \to 0$.
It is easily seen that $H_h \to 0$ as $h \to 0$.
Denote

$$W_h = \|\eta D(u_h - u)\|_{2;B_1^+(0)},\ c_h = (\eta(\pi_h - \pi))_{B_1^+(0)},\ Q_h = \|\eta(\pi_h - \pi) - c_h\|_{2;B_1^+(0)}. \tag{82}$$

Next, we have

$$\begin{aligned}
&|\int_{B_1^+(0)} (\pi_h - \pi)(E - B_h).D\psi_h\ dy| = |\int_{B_1^+(0)} (\pi_h - \pi)(E - B_h).D[\eta^2(u_h - u)]dy| \\
&\leq |2\int_{B_1^+(0)} \eta(\pi_h - \pi)(E - B_h).[(u_h - u) \otimes \nabla\eta]\ dy| + |\int_{B_1^+(0)} c_h\eta(E - B_h).D(u_h - u)\ dy| \\
&+|\int_{B_1^+(0)} \eta[\eta(\pi_h - \pi) - c_h]\mathrm{div}\ (u_h - u)dy| + |\int_{B_1^+(0)} [\eta(\pi_h - \pi) - c_h]B_h.D(u_h - u)\eta\ dy| \qquad (83) \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Using Hölder's inequality, (64) and (65), we get

$$I_1 \leq \frac{C}{1-\tau}\|(\pi_h - \pi)\|_{2;B_1^+(0)}\|(u_h - u)\|_{2;B_1^+(0)} \leq C\|(u_h - u)\|_{2;B_1^+(0)} \longrightarrow 0 \ \text{ as } \ h \longrightarrow 0.$$

As $E - B_h$ is uniformly bounded in $L^\infty(B_1^+(0)^{d^2},\ D(u_h - u)$ is uniformly bounded in $L^2(B_1^+(0))^d$ and $c_h \to 0$ as $h \to 0$, it gives $I_2 \to 0$ as $h \to 0$.
Obviously, $I_3 = 0$ due to div $w =$ div $w_h = 0,\ w, w_h \in W^{2,2}(B_1^+(0))$.
For last term I_4, we have estimates

$$I_4 \leq \nu_0 W_h Q_h.$$

With all these above estimates, we can conclude that

$$W_h^2 \leq \frac{\nu_0}{\lambda_0} Q_h W_h + o(h),\ \text{as } h \longrightarrow 0. \tag{84}$$

Our next aim is to estimate Q_h.
Let $\varphi_h \in W_0^{1,2}(B_1^+(0))^d$ be a solution to problem

$$\mathrm{div}\ \varphi_h = \eta(\pi_h - \pi) - c_h \text{ in } B_1^+(0). \tag{85}$$

Recall that $\|\nabla\varphi_h\|_{2,B_1^+(0)} \leq C_{\mathrm{div}}\|\eta(\pi_h - \pi) - c_h\|_{2,B_1^+(0)}$. Combining this inequality and (79), we can assert that φ_h are bounded uniformly with respect to h in $W^{1,2}(B_1^+(0))^d$.

Moreover, $\eta(\pi_h - \pi) - c_h \longrightarrow 0$ weakly in $L^2(B_1^+(0))$ and the solution operator of problem (85) is linear and continuous from $L^2(B_1^+(0))$ to $W^{1,2}(B_1^+(0))^d$ so that

$$\begin{aligned}&(\varphi_h) \longrightarrow 0, \quad \text{weakly in} \quad W^{1,2}(B_1^+(0))^d,\\ &(\varphi_h) \longrightarrow 0 \quad \text{strongly in} \quad L^2(B_1^+(0))^d.\end{aligned} \tag{86}$$

Choosing test function $\phi_h = \eta\varphi_h$ in the formulation (78), we get

$$\begin{aligned}\int_{B_1^+(0)} \pi_h \operatorname{div} \phi_h \, dy = \int_{B_1^+(0)} A_h Du_h : D\phi_h + \pi_h B_h . D\phi_h dy + I_h \\ + \frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h(y) \frac{\partial \phi_h(y)}{\partial y_s} \, dy,\end{aligned} \tag{87}$$

where I_h is the second term on the left hand side of (74) with test function φ_h replaced by ϕ_h.
From the definition of ϕ_h we have

$$\begin{aligned}\int_{B_1^+(0)} \pi_h \operatorname{div} \phi_h \, dy = \int_{B_1^+(0)} [\eta(\pi_h - \pi) - c_h] \operatorname{div} \varphi_h \, dx + \int_{B_1^+(0)} (\pi_h - \pi) \nabla\eta . \varphi_h \, dy \\ + \int_{B_1^+(0)} \pi \operatorname{div} \phi_h \, dy.\end{aligned}$$

Hence

$$\begin{aligned}Q_h^2 = \int_{B_1^+(0)} A_h Du_h : D\phi_h + \pi_h B_h . D\phi_h dy + I_h + \frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h(y) \frac{\partial \phi_h(y)}{\partial y_s} \, dy \\ - \int_{B_1^+(0)} (\pi_h - \pi) \nabla\eta . \varphi_h \, dy - \int_{B_1^+(0)} \pi \operatorname{div} \phi_h \, dy.\end{aligned} \tag{88}$$

By similar estimates as in the above proof, we can see that

$$I_h + \frac{R_h}{\varepsilon_h} \int_{B_1^+(0)} f_h(y) \frac{\partial \phi_h(y)}{\partial y_s} \, dy \longrightarrow 0 \ \text{ as } \ h \longrightarrow 0.$$

As $(\varphi_h) \to 0$ strongly and $\operatorname{div} \varphi_h \to 0$ weakly in $L^2(B_1^+(0))^d$, we have that $\operatorname{div} \phi_h \to 0$ weakly in $L^2(B_1^+(0))^d$; moreover π_h, π are the uniformly bounded in $L^2(B_1^+(0))$ and $(\pi_h - \pi) \to 0$ weakly in $L^2(B_1^+(0)$. Thus

$$\int_{B_1^+(0)} (\pi_h - \pi) \nabla\eta . \varphi_h \, dy + \int_{B_1^+(0)} \pi \operatorname{div} \phi_h \, dy \longrightarrow 0 \ \text{ as } \ h \longrightarrow 0.$$

Next we have

$$\begin{aligned}S_h = \int_{B_1^+(0)} A_h Du_h : D\phi_h + \pi_h B_h . D\phi_h \, dy = \int_{B_1^+(0)} A_h \eta D(u_h - u) : D\varphi_h \, dy \\ + \int_{B_1^+(0)} \eta A_h Du : D\varphi_h \, dy + \int_{B_1^+(0)} A_h Du_h : (\nabla\eta \otimes \varphi_h) \, dy \\ + \int_{B_1^+(0)} [\eta(\pi_h - \pi) - c_h] B_h . D\varphi_h \, dy + \int_{B_1^+(0)} (\pi\eta + c_h) B_h . D\varphi_h \, dy \\ + \int_{B_1^+(0)} \pi_h B_h . (\nabla\eta \otimes \varphi_h) dy.\end{aligned} \tag{89}$$

Employing Hölder's inequality, we get

$$
\begin{aligned}
|\int_{B_1^+(0)} A_h\eta D(u_h-u):D\varphi_h\ dy| &\le \lambda_1 C_{\rm div} W_h Q_h,\\
|\int_{B_1^+(0)} [\eta(\pi_h-\pi)-c_h]B_h.D\varphi_h\ dy| &\le \nu_0 C_{\rm div} Q_h^2. \qquad (90)
\end{aligned}
$$

Due to $\|Du_h\|_{B_1^+(0)} \le 1$, (3), (86) and the lower semicontinuity of the L^2 norm with respect to weak convergence, we have $\|Du\|_{B_1^+(0)} \le 1$. It implies

$$\int_{B_1^+(0)} \eta A_h Du : D\varphi_h\ dy + \int_{B_1^+(0)} A_h Du_h : (\nabla\eta\otimes\varphi_h)\ dy \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Combining the fact $\pi, \pi_h \in L^2_{B_1^+(0)}$, $c_h \to 0$ with (86) yields

$$\int_{B_1^+(0)} (\pi\eta + c_h)]B_h.D\varphi_h\ dy + \int_{B_1^+(0)} \pi_h B_h.(\nabla\eta\otimes\varphi_h)dy - \int_{B_1^+(0)} c_h B_h.D\varphi_h\ dy \longrightarrow 0 \quad \text{as } h \longrightarrow 0.$$

Using all the above estimates, we may conclude that

$$Q_h^2 \le \lambda_1 C_{\rm div} Q_h W_h + \nu_0 C_{\rm div} Q_h^2 + o(h), \text{ as } h \longrightarrow 0.$$

Applying Young's inequality, we have

$$\lambda_1 C_{\rm div} Q_h W_h \le \frac{\lambda_1 C_{\rm div}}{2(\lambda_1 C_{\rm div}+\lambda_0)} Q_h^2 + \frac{\lambda_1 C_{\rm div}(\lambda_1 C_{\rm div}+\lambda_0)}{2} W_h^2,$$

and thus

$$(1-\nu_0 C_{\rm div} - \frac{\lambda_1 C_{\rm div}}{2(\lambda_1 C_{\rm div}+\lambda_0)})Q_h^2 \le \frac{\lambda_1 C_{\rm div}(\lambda_1 C_{\rm div}+\lambda_0)}{2} W_h^2 + o(h), \text{ as } h \longrightarrow 0.$$

Due to the assumption $\nu_0 < \frac{\lambda_0}{(\lambda_0+C_0\lambda_1)C_0}$ and $C_0 \ge C_{\rm div}$, we deduce

$$Q_h^2 \le (\lambda_1 C_{\rm div}+\lambda_0)^2 W_h^2 + o(h), \text{ as } h \longrightarrow 0.$$

Recall that (84) implies

$$W_h^2 \le \frac{\nu_0}{\lambda_0} Q_h W_h + o(h) \le \frac{\nu_0(\lambda_1 C_{\rm div}+\lambda_0)}{\lambda_0} W_h^2 + o(h).$$

It is clear that $\frac{\nu_0(\lambda_1 C_{\rm div}+\lambda_0)}{\lambda_0} \le \frac{\nu_0(\lambda_1 C_0+\lambda_0)}{\lambda_0} \le \frac{1}{C_0} < 1$. Hence

$$W_h,\ Q_h \longrightarrow 0 \text{ as } h \longrightarrow 0 \text{ i.e.}$$

$$\|\eta D(u_h-u)\|_{2;B_1^+(0)} + \|\eta(\pi_h-\pi)-c_h\|_{2;B_1^+(0)} \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Thus $D(u_h-u) \to 0$ strongly in $L^2(B_\tau^+(0))^d$. Moreover, as $(u_h-u) \to 0$ strongly in $L^2(B_\tau^+(0))^d$, it implies

$$\|D[\eta(u_h-u)]\|_{2;B_1^+(0)}, \quad \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Applying Korn's inequality, we have

$$\nabla[\eta(u_h-u)] \longrightarrow 0 \quad \text{strongly in} \quad L^2(B_1^+(0))^d \text{ as } h \longrightarrow 0.$$

We also have $\|\eta(\pi_h - \pi)\|_{2;B_1^+(0)} \to 0$ as $h \to 0$. Henceforth, for all $s = 1, \cdots, d-1$ it holds

$$\begin{aligned}
\eta\nabla(\frac{\partial(w_h - w)}{\partial y_s}) &\longrightarrow 0 \quad \text{strongly in} \quad L^2(B_1^+(0))^{d^2} \\
\eta D(\frac{\partial(w_h - w)}{\partial y_s}) &\longrightarrow 0 \quad \text{strongly in} \quad L^2(B_1^+(0))^{d^2} \\
\eta\frac{\partial(p_h - p)}{\partial y_s} &\longrightarrow 0 \quad \text{strongly in} \quad L^2(B_1^+(0)).
\end{aligned} \tag{91}$$

Thanks to the condition div $w = 0$, we also have

$$\frac{\partial[\eta\frac{\partial(w_h - w)_d}{\partial y_d}]}{\partial y_d} \to 0 \;\text{ strongly in }\; L^2(B_1^+(0)) \;\text{ as }\; h \longrightarrow 0. \tag{92}$$

Next we shall prove $D_{kd}(\frac{\partial(w_h-w)}{\partial y_d}) \to 0$ strongly in $L^2(B_\tau^+(0))$ for all $k = 1, \cdots, d-1$ and $\frac{\partial(p_h-p)}{\partial y_d} \to 0$ strongly in $L^2(B_\tau^+(0))$ (this is sufficient to make $w_h \to\ w$ strongly in $W^{2,2}(B_\tau^+(0))^d,\ p_h \to\ p$ strongly in $W^{1,2}(B_\tau^+(0))$.
In fact, because v_h, p_h are solutions to the system (1), in an analogous way as in the proof of Theorem 3.2, we have

$$\sum_{j=1}^d \frac{\partial T_{ij}(p_h, Dv_h)}{\partial D}\frac{\partial Dv_h}{\partial x_j} + \sum_{j=1}^d \frac{\partial T_{ij}(p_h, Dv_h)}{\partial p}\frac{\partial p_h}{\partial x_j} - \nabla p_h = -f + v_h \otimes \frac{\partial(v_h)_i}{\partial x} \tag{93}$$

for all $i = 1, \cdots, d$.
Then we put instead of $p_h,\ v_h,\ Dv_h$ the rescaled quantities $q_h,\ w_h,\ Dw_h$ and we obtain

$$\sum_{j=1}^d (A_h)_{ij}\frac{\partial Dw_h}{\partial y_j} + \sum_{j=1}^d (B_h)_{ij}\frac{\partial q_h}{\partial y_j} - \nabla q_h = -\frac{R_h}{\varepsilon_h}(f_h)_i + (J_h)_i \tag{94}$$

for all $i = 1, \cdots, d$, where $A_h,\ B_h,\ f_h$ were already defined in the beginning of this proof and

$$\begin{aligned}
(J_h)_i = \frac{R_h}{\varepsilon_h}[w_h R_h \varepsilon_h + (\frac{\partial v_h}{\partial x_s})_{B^+_{R_h}} R_h(0, \cdots, 0, y_d)^T] \otimes [\frac{\partial(w_h)_i}{\partial y_s}\varepsilon_h \\
+ (\frac{\partial(v_h)_i}{\partial x_s})_{B^+_{R_h}}.(0, \cdots, 0, 1)^T]; \;\text{ for all }\; i = 1, \cdots, d.
\end{aligned}$$

Recall that $w,\ q$ are solution to system (34), thus

$$\begin{aligned}
\sum_{j=1}^d (A_h)_{ij}\frac{\partial D(w_h - w)}{\partial y_j} + \sum_{j=1}^d (B_h)_{ij}\frac{\partial(q_h - q)}{\partial y_j} - \nabla(q_h - q) \\
= -(\bar{G}_h)_i + (J_h)_i \quad \text{for all} \;\; i = 1, \cdots, d,
\end{aligned} \tag{95}$$

with $(\bar{G}_h)_i := \sum_{j=1}^d (A_h - A)_{ij}\frac{\partial Dw}{\partial y_j} + \sum_{j=1}^d (B_h - B)_{ij}\frac{\partial q}{\partial y_j} + \frac{R_h}{\varepsilon_h}(f_h)_i$ for all $i = 1, \cdots, d$.
The system can be written as

$$\sum_{k=1}^{d-1} 2(A_h)_{id}^{kd}\frac{\partial D_{kd}(w_h - w)}{\partial y_d} + (B_{id} - \delta_{id})\frac{\partial(q_h - q)}{\partial y_d} = (\bar{F}_h)_i - (\bar{G}_h)_i + (J_h)_i \tag{96}$$

for all $i = 1, \cdots, d$, where $\bar{F}_h$ are given by

$$(\bar{F}_h)_i := -\sum_{k,l=1}^{d-1} 2(A_h)_{id}^{kl} \frac{\partial D_{kl}(w_h - w)}{\partial y_d} - (A_h)_{dd}^{dd} \frac{\partial D_{dd}(w_h - w)}{\partial y_d}$$
$$- \sum_{j=1}^{d-1} (A_h)_{ij} \frac{\partial D(w_h - w)}{\partial y_j} - \sum_{j=1}^{d-1} B_{ij} \frac{\partial (q_h - q)}{\partial x_j} + (1 - \delta_{id}) \frac{\partial (q_h - q)}{\partial y_i} \quad (97)$$

for all $i = 1, \cdots, d$. Denote

$$K_h := \bar{F}_h - \bar{G}_h + J_h$$

Let us start with limit procedure for $h \to 0$. A_h, B_h are uniformly bounded, $A_h \to A$, $B_h \to B$ a.e. on $B_1^+(0)$. Next thanks to (64), (65), (67), (91), (92) and $\| \frac{R_h}{\varepsilon_h} \| f_h(y) \|_{2;B_1^+(0)} \to 0$ we deduce

$$\|\bar{F}_h\|_{2;B_\tau^+(0)},\ \|\bar{G}_h\|_{2;B_\tau^+(0)},\ \|J_h\|_{2;B_\tau^+(0)} \longrightarrow 0 \text{ as } h \longrightarrow 0. \quad (98)$$

Henceforth, $\|K_h\|_{2;B_\tau^+(0)} \to 0$ as $h \to 0$.
By multiplying d^{th} row of system (96) by $[1 - (B_h)_{dd}]^{-1} (B_h)_{id}$ and then adding the product to i^{th} row $(i = 1, \cdots, d-1)$, we obtain

$$\sum_{k=1}^{d-1} \bar{S}_{ik} \frac{\partial D_{kd} v}{\partial x_d} = \frac{1}{2} \{K_i + K_d [1 - (B_h)_{dd}]^{-1} (B_h)_{id}\}, \qquad i = 1, \cdots, d-1 \quad (99)$$

where $\bar{S}_{ik} = (A_h)_{id}^{kd} + [1 - (B_h)_{dd}]^{-1} (B_h)_{id} (A_h)_{dd}^{kd}$.
By the same argument as in the proof of Theorem 3.2, we get that $\bar{S}$ is the positive definite matrix. Namely, we have

$$\sum_{i,k=1}^{d-1} \bar{S}_{ik} \zeta_i \zeta_k \geq \left[\frac{\lambda_0}{4} - \frac{3}{4}(d-1) \frac{\nu_0}{1 - \nu_0} (\lambda_1 - \lambda_0) \right] |\zeta|^2 = \lambda |\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^{d-1}, \quad (100)$$

where we denote $\lambda := [\frac{\lambda_0}{4} - \frac{3}{4}(d-1) \frac{\nu_0}{1-\nu_0} (\lambda_1 - \lambda_0)]$. Note that $\lambda > 0$ because of the assumptions of Lemma 4.3.
In a similar way as in the proof of Theorem 3.2 we can conclude

$$\lambda \sum_{k=1}^{d-1} \| \frac{\partial [D_{kd}(v_h - v)]}{\partial y_d} \|_{2;B_\tau^+(0)} \leq \sum_{i=1}^{d-1} \frac{1}{2} \|K_i + K_d [1 - (B_h)_{dd}]^{-1} (B_h)_{id}\|_{2;B_\tau^+(0)}.$$

It is easily seen that $\sum_{i=1}^{d-1} \frac{1}{2} \|K_i + K_d [1 - (B_h)_{dd}]^{-1} (B_h)_{id}\|_{2;B_\tau^+(0)} \to 0$ as $h \to 0$, $\lambda > 0$. Hence

$$\| \frac{\partial [D_{kd}(v_h - v)]}{\partial y_d} \|_{2;B_\tau^+(0)} \longrightarrow 0 \text{ as } h \longrightarrow 0, \ \ k = 1, \cdots, d-1.$$

From the d^{th}-equation of system (96), it follows

$$\| \frac{\partial (p_h - p)}{\partial y_d} \|_{2;B_\tau^+(0)} \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Thus we proved that $w_h \to w$ strongly in $W^{2,2}(B_\tau^+(0))^d$, $q_h \to q$ strongly in $W^{1,2}(B_\tau^+(0)))$ as $h \to 0$. Hence

$$E^{w_h, q_h}(0, \tau) \longrightarrow E^{w,q}(0, \tau) \text{ as } h \longrightarrow 0.$$

Combining (75), (72) and using the lower semicontinuty of the L^2 norm with respect to weak convergence, we get

$$2C^* \tau^\alpha < E^{w_h, q_h}(0, \tau) \longrightarrow E^{w,q}(0, \tau) \leq C^* \tau^\alpha E^{w,q}(0, 1) \leq C^* \tau^\alpha \liminf_{h \longrightarrow 0} E^{w_h, q_h}(0, 1) \leq C^* \tau^\alpha,$$

which is a contradiction. $\square$

Next, we prove an iterated result

Lemma 4.4. *Let the assumptions of Lemma 4.3 hold. If $v \in W^{1,2}(B_1^+(0))^d$, $p \in L^2(B_1^+(0)))$ be a weak solution of system (1) with* supp v, supp $p \subset B_1^*(0)$ *and $v = 0$ on Γ. Then for all $M > 0$, $\gamma \in (0, \alpha)$, there exists $\tau \in (0, 1)$, $\varepsilon_1 > 0$ such that for any $x \in \Gamma$, $R \leq dist\, \partial(x, \Omega\backslash\Gamma)$ satisfying*

$$E^{v,p}(x, R) < \varepsilon_1,\ |(\nabla v)_{B_R^+(x)}| + |(p)_{B_R^+(x)}| \leq \frac{M}{2}, \tag{101}$$

we have the inequalities

$$\begin{aligned} &E^{v,p}(x, \tau^k R) \leq \tau^{k\gamma} E^{v,p}(x, R), \\ &|(\nabla v)_{B^+_{\tau^{k-1}R}(x)}| + |(p)_{B^+_{\tau^{k-1}R}(x)}| \leq M. \end{aligned} \tag{102}$$

for all $k \in \mathbb{N}$.

Remark. Here we choose $\tau \in (0, 1)$,
$\varepsilon_1 > 0$ such that $2C^*\tau^{\alpha-\gamma} \leq 1$, $\varepsilon_1 = \min\ (\varepsilon,\ \frac{M(\tau^d\kappa_d)^{\frac{1}{2}}(1-\tau^\gamma)}{C_P})$, C_P is the constant in Poincaré's inequality for both half ball and ball, κ_d is measure of unit half ball in $\mathbb{R}^d$.

Proof. We shall prove this assertion by an induction.

In fact, for $k = 1$ the second part of the assertion is exactly (101) and Lemma 4.3 implies that

$$E^{v,p}(x, \tau R) \leq 2C^*[\tau^\alpha E^{v,p}(x, R)] \leq \tau^\gamma E^{v,p}(x, R).$$

Therefore, we obtain the assertion (102) for $k = 1$.
Suppose that (102) holds for $i = 1, \cdots, k$ and we will prove that it holds also for $i = k + 1$. Applying the Poincaré 's inequality, we get

$$\begin{aligned} |(\nabla v)_{B^+_{\tau^i R}(x)} - (\nabla v)_{B^+_{\tau^{i-1}R}(x)}| &\leq \frac{1}{(\tau^i R)^d \kappa_d} \int_{B^+_{\tau^i R}(x)} |\nabla v - (\nabla v)_{B^+_{\tau^{i-1}R}(x)}| \\ &\leq \frac{1}{((\tau^i R)^d \kappa_d)^{\frac{1}{2}}} \|\nabla v - (\nabla v)_{B^+_{\tau^{i-1}R}(x)}\|_{2;B^+_{\tau^{i-1}R(x)}} \\ &\leq \frac{C_P}{(\tau^d\kappa_d)^{\frac{1}{2}}} \frac{1}{(\tau^{i-1})R^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2;B^+_{\tau^{i-1}R}(x)},\ i = 1, \cdots, k. \end{aligned} \tag{103}$$

Hence

$$\begin{aligned} |(\nabla v)_{B^+_{\tau^k R}(x)}| &\leq \sum_{i=1}^k |(\nabla v)_{B^+_{\tau^i R}(x)} - (\nabla v)_{B^+_{\tau^{i-1}R}(x)}| + |(\nabla v)_{B_R^+}(x)| \\ &\leq \frac{C_P}{(\tau^d\kappa_d)^{\frac{1}{2}}} \sum_{i=1}^k \frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2;B^+_{\tau^{i-1}R}(x)} + |(\nabla v)_{B_R^+(x)}|. \end{aligned} \tag{104}$$

By an analogous way, we have

$$\begin{aligned} |(p)_{B^+_{\tau^k R}(x)}| &\leq \sum_{i=1}^k |(p)_{B^+_{\tau^i R}(x)} - (p)_{B^+_{\tau^{i-1}R}(x)}| + |(p)_{B_R^+(x)}| \\ &\leq \frac{C_P}{(\tau^d\kappa_d)^{\frac{1}{2}}} \sum_{i=1}^k \frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}} \|\nabla p\|_{2;B^+_{\tau^{i-1}R}(x)} + |(p)_{B_R^+(x)}|. \end{aligned} \tag{105}$$

Adding (104) to (105) and using the hypothesis of induction in (102) we obtain

$$
\begin{aligned}
|(\nabla v)_{B^+_{\tau^k R}(x)}| + |(p)_{B^+_{\tau^k R}(x)}| &\leq \frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \sum_{i=1}^{k} E^v(x, \tau^{i-1}R) + \frac{M}{2} \\
&\leq \frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_1 \sum_{i=1}^{k} (\tau^\gamma)^{i-1} + \frac{M}{2} \leq \frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_1 \frac{1}{1-\tau^\gamma} + \frac{M}{2}.
\end{aligned} \tag{106}
$$

Thanks to the choice of ε_1 it holds $\frac{C_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_1 \frac{1}{1-\tau^\gamma} \leq \frac{M}{2}$ so that

$$
|(\nabla v)_{B^+_{\tau^k R}(x)}| + |(p)_{B^+_{\tau^k R}(x)}| \leq M.
$$

Thus the assumption of Lemma 4.3 is satisfied and we deduce

$$
E^{v,p}(x, \tau^{k+1}R) \leq \tau^\gamma E^{v,p}(x, \tau^k R) \leq \tau^{(k+1)\gamma} E^{v,p}(x, R).
$$

□

The partial regularity of solutions to the system (1) with subquadratic growth in the interior of domain Ω was already stated in [19], together with the decay lemma for system (1). In our case, by an analogous way we also have the same conclusion of Lemma 4.3.

Let $x \in B^+_1(0), 0 < R \leq \operatorname{dist} \partial(x, B^+_1(0))$ we set

$$
E^{w,q}(x,R) = E^{w,q}_0(x,R) + R^\alpha := \frac{1}{R^{\frac{d-2}{2}}} \|\nabla^2 w\|^2_{2,B_R(x)} + \frac{1}{R^{\frac{d-2}{2}}} \|\nabla q\|^2_{2,B_R(x)} + R^\alpha, \tag{107}
$$

where $0 < \alpha < 1$.

Lemma 4.5. *(Decay for nonlinear system - interior case) Let the assumptions of Lemma 4.3 hold, $(v,p) \in W^{1,2}(B^+_1(0))^d) \times L^2(B^+_1(0))$ with $\operatorname{supp} v$, $\operatorname{supp} p \subset B^*_1(0)$ be a weak solution of system (1), $v = 0$ on Γ. Then for all $M \in (0,\infty)$ and for all $\tau \in (0,1)$, there exists an $\varepsilon > 0$ so that for any $x \in B^+_1(0), R \leq \operatorname{dist}(\mathrm{x}, \partial \mathrm{B}^+_1(0))$ satisfying*

$$
E^{v,p}(x,R) < \varepsilon, \ |(v)_{B_R(x)}| + |(\nabla v)_{B_R(x)}| + |(p)_{B_R(x)}| \leq M, \tag{108}
$$

there is a constant C^* such that

$$
E^{v,p}(x, \tau R) \leq 2C^*[\tau^\alpha E^{v,p}(x,R)]. \tag{109}
$$

Remark. We will write the same notation M, τ, ε, C^* in Lemma 4.3 and Lemma 4.5. Further, we also obtain the result in a ball with radius $\tau^k R$, where k is the arbitrary nonnegative integer number.

Lemma 4.6. *(Interior case) Let the assumptions of Lemma 4.3 hold, let $v \in W^{1,2}(B^+_1(0))^d$, $p \in L^2(B^+_1(0)))$ with $\operatorname{supp} v$, $\operatorname{supp} p \subset B^*_1(0)$ is a weak solution of system (1). Then for all $M > 0$, $\gamma \in (0,\alpha)$, there exists $\tau \in (0,1)$, $\varepsilon_2 > 0$ so that for any $x \in B^+_1(0), R \leq \min$ (dist $(x, \partial\Omega)$, $\frac{1}{2}$) satisfying*

$$
E^{v,p}(x,R) < \varepsilon_2, \ |(v)_{B_R(x)}| + |(\nabla v)_{B_R(x)}| + |(p)_{B_R(x)}| \leq \frac{M}{4}, \tag{110}
$$

we have for all $k \in \mathbb{N}$ the inequalities

$$
E^{v,p}(x, \tau^k R) \leq \tau^{k\gamma} E^{v,p}(x,R), |(p)_{B_{\tau^{k-1}R}(x)}| + |(\nabla v)_{B_{\tau^{k-1}R}(x)}| + |(p)_{B_{\tau^{k-1}R}(x)}| \leq M. \tag{111}
$$

Remark. Here we choose $\tau \in (0,1),\ \varepsilon_2 > 0$ such that $2C^*\tau^{\alpha-\gamma} \le 1$, $\varepsilon_2 = \min\,(\varepsilon,\ \frac{M(2\tau^d\kappa_d)^{\frac{1}{2}}(1-\tau^\gamma)}{4C'_P})$, where $C'_P = \max\,(\mathrm{C_P}, \mathrm{C_P^2})$.

Proof. We shall prove this lemma by an induction

In fact, for $k=1$ the second assertion follows from (110). The Lemma 4.5 implies that

$$E^{v,p}(x,\tau R) \le 2C^*[\tau^\alpha E^{v,p}(x,R)] \le \tau^\gamma E^{v,p}(x,R),$$

where we defined γ as in the previews case .
Suppose that (111) holds for $i = 1,\cdots,k$ and we will prove that it holds also for $i = k+1$. Applying Poincaré 's inequality, we get

$$\begin{aligned} |(\nabla v)_{x,\tau^i R} - (\nabla v)_{x,\tau^{i-1}R}| &\le \frac{1}{2(\tau^i R)^d \kappa_d}\int_{B_{\tau^i R}(x)} |\nabla v - (\nabla v)_{x,\tau^{i-1}R}| \\ &\le \frac{1}{(2(\tau^i R)^d\kappa_d)^{\frac{1}{2}}}\|\nabla v - (\nabla v)_{x,\tau^{i-1}R}\|_{2;B_{\tau^{i-1}R}(x)} \\ &\le \frac{C_P}{(2\tau^d\kappa_d)^{\frac{1}{2}}}\frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}}\|\nabla^2 v\|_{2;B_{\tau^{i-1}R}(x)},\ i = 1,\cdots,k. \end{aligned} \tag{112}$$

Using repeatedly (112) for i then

$$\begin{aligned} |(\nabla v)_{x,\tau^k R}| &\le \sum_{i=1}^{k} |(\nabla v)_{x,\tau^i R} - (\nabla v)_{x,\tau^{i-1}R}| + |(\nabla v)_{x,R}| \\ &\le \frac{C_P}{(2\tau^d\kappa_d)^{\frac{1}{2}}}\sum_{i=1}^{k}\frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}}\|\nabla^2 v\|_{2;B_{\tau^{i-1}R}(x)} + |(\nabla v)_{B_R(x)}|. \end{aligned} \tag{113}$$

By an analogous way, we have

$$\begin{aligned} |(p)_{x,\tau^k R}| &\le \sum_{i=1}^{k} |(p)_{B_{\tau^i R}(x)} - (p)_{B_{\tau^{i-1}R}(x)}| + |(p)_{B_R(x)}| \\ &\le \frac{C_P}{(2\tau^d\kappa_d)^{\frac{1}{2}}}\sum_{i=1}^{k}\frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}}\|\nabla p\|_{2;B_{\tau^{i-1}R}(x)} + |(p)_{B_R(x)}|. \end{aligned} \tag{114}$$

Next, let use the Poincaré 's inequality for estimates of $|(v)_{B_{\tau^k R}(x)}|$

$$\begin{aligned} |(v)_{x,\tau^i R} - (v)_{x,\tau^{i-1}R}| &\le \frac{1}{2(\tau^i R)^d\kappa_d}\int_{B_{\tau^i R}(x)} |v(z) - (v)_{x,\tau^{i-1}R} - (\nabla v)_{x,\tau^{i-1}R}(z-x)|\,dz \\ + \tau^i R|(\nabla v)_{x,\tau^{i-1}R}| &\le \frac{1}{(2(\tau^i R)^d\kappa_d)^{\frac{1}{2}}}\|v(z) - (v)_{x,\tau^{i-1}R} - (\nabla v)_{x,\tau^{i-1}R}(z-x)\|_{2;B_{\tau^{i-1}R}(x)} \\ + \tau^i R|(\nabla v)_{x,\tau^{i-1}R}| &\le \frac{C_P}{(2\tau^d\kappa_d)^{\frac{1}{2}}}\frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}}\|\nabla v - (\nabla v)_{x,\tau^{i-1}R}\|_{2;B_{\tau^{i-1}R}(x)} + \tau^i R|(\nabla v)_{x,\tau^{i-1}R}| \\ &\le \frac{C_P^2\tau^{i-1}R}{(2\tau^d\kappa_d)^{\frac{1}{2}}}\frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}}\|\nabla^2 v\|_{2;B_{\tau^{i-1}R}(x)} + \tau^i R|(\nabla v)_{x,\tau^{i-1}R}|. \end{aligned} \tag{115}$$

Thus

$$\begin{aligned} |(v)_{x,\tau^k R}| &\le \sum_{i=1}^{k} |(v)_{x,\tau^i R} - (v)_{x,\tau^{i-1}R}| + |(v)_{x,R}| \\ &\le \sum_{i=1}^{k}\left(\frac{C_P^2\tau^{i-1}R}{(2\tau^d\kappa_d)^{\frac{1}{2}}}\frac{1}{(\tau^{i-1}R)^{\frac{d-2}{2}}}\|\nabla^2 v\|_{2;B_{\tau^{i-1}R}(x)} + \tau^i R|(\nabla v)_{x,\tau^{i-1}R}|\right) + |(\nabla v)_{x,R}|. \end{aligned} \tag{116}$$

Summing (113), (114), (116) and using the hypothesis we obtain

$$|(v)_{x,\tau^k R}| + |(\nabla v)_{x,\tau^k R}| + |(p)_{x,\tau^k R}| \le \frac{C'_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \sum_{i=1}^{k} E^v(x, \tau^{i-1} R) + \sum_{i=1}^{k} (\tau^i R |(\nabla v)_{x,\tau^{i-1}R}|)$$
$$+ \frac{M}{4} \le \frac{C'_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_2 \sum_{i=1}^{k} (\tau^\gamma)^{i-1} + \frac{MR}{1-\tau} + \frac{M}{4} \le \frac{C'_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_2 \frac{1}{1-\tau^\gamma} + \frac{MR}{1-\tau} + \frac{M}{4} \tag{117}$$

where $C'_P = \max\,(C_P, C_P^2)$.
Thanks to the choice of ε_2, R it holds $\frac{C'_P}{(2\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_2 \frac{1}{1-\tau^\gamma} \le \frac{M}{4}$, $\frac{R}{1-\tau} \le \frac{1}{2}$ so that

$$|(v)_{x,\tau^k R}| + |(\nabla v)_{x,\tau^k R}| + |(p)_{x,\tau^k R}| \le M.$$

Applying Lemma 4.5, we obtain

$$E^{v,p}(x, \tau^{k+1} R) \le \tau^\gamma E^{v,p}(x, \tau^k R) \le \tau^{(k+1)\gamma} E^{v,p}(x, R).$$

□

Next, we present the main result of this part to show that in regular points solutions belong to Hölder's space up to the boundary.

Theorem 4.2. *Let the assumptions of Lemma* 4.3 *hold,* $v \in W^{1,2}(B_1^+(0))^d$, $p \in L^2(B_1^+(0)))$ *with* $\operatorname{supp} v$, $\operatorname{supp} p \subset B_1^*(0)$ *be a weak solution of system* (1) *and* $v = 0$ *on* Γ. *If* $\bar{x} \in \Gamma$ *is such that*

$$\liminf_{R \longrightarrow 0^+} E^{v,p}(\bar{x}, R) = 0, \ \limsup_{R \longrightarrow 0^+} |(v)_{B_R^+(\bar{x})}| + |(\nabla v)_{B_R^+(\bar{x})}| + |(p)_{B_R^+(\bar{x})}| < \infty, \tag{118}$$

then there exists $\delta > 0$ *such that* ∇v *and* p *are Hölder continuous on* $\overline{B_\delta^+(\bar{x})}$.

Proof. Let $\bar{x} \in \Gamma$ satisfies (118). Then there are $R_0 \in (0, 1)$ and $M \in (0, \infty)$ such that

$$\sup_{R \le R_0} |(v)_{B_R^+(\bar{x})}| + |(\nabla v)_{B_R^+(\bar{x})}| + |(p)_{B_R^+(\bar{x})}| < \frac{M}{32}.$$

Take $\gamma \in (0, \alpha)$, $\tau \in (0, \min\,(\frac{1}{2}(2C^*)^{\frac{1}{\gamma-\alpha}}, \frac{1}{2}))$ and find $\varepsilon, \varepsilon_1, \varepsilon_2$ from Lemma 4.3, Lemma 4.4, Lemma 4.5, Lemma 4.6. Choose $\varepsilon_3 > 0$ so that it satisfies inequalities

$$[(\frac{2}{\tau})^{\frac{d-2}{2}} (\frac{2}{\tau R})^\gamma + 1] \varepsilon_3 \le \varepsilon_2, \frac{C'_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \frac{1}{1-\tau^\gamma} \varepsilon_3 \le \frac{M}{32} \ \text{ and } \ \frac{C'_P}{\sqrt{2\kappa_d}} (\frac{2}{\tau})^{\frac{d}{2}} \tau^{\frac{2-d}{2}} (\frac{1}{\tau R})^\gamma \varepsilon_3 \le \frac{M}{16}. \tag{119}$$

Then (118) guarantees the existence of $R \in (0, \min\,(\frac{1}{4}, R_0, \operatorname{dist}\,(\bar{x}, \partial B_1^+(0) \backslash \Gamma))$ such that

$$|E^{v,p}(\bar{x}, R)| < \varepsilon_3 \ \text{ and } \ |(v)_{B_R^+(\bar{x})}| + |(\nabla v)_{B_R^+(\bar{x})}| + |(p)_{B_R^+(\bar{x})}| < \frac{M}{32}. \tag{120}$$

It implies that there is $\delta \in (0, \frac{R}{4})$ such that for all $x \in B_\delta^*(\bar{x}) := B_\delta(\bar{x}) \cap B_1^*(0)$ inequalities

$$|E^{v,p}(x, R)| < \varepsilon_3 \ \text{ and } \ |(v)_{B_R^+(x)}| + |(\nabla v)_{B_R^+(x)}| + |(p)_{B_R^+(x)}| < \frac{M}{32} \tag{121}$$

where

$$B_R^+(x) := B_R(x) \cap B_1^+(0), E^{v,p}(x,R) := \frac{1}{R^{\frac{d-2}{2}}}\|\nabla^2 w\|_{2,B_R^+(x)} + \frac{1}{R^{\frac{d-2}{2}}}\|\nabla p\|_{2,B_R^+(x)} + R^\alpha. \tag{122}$$

Now we want to show that there exists a constant C_H such that

$$E_0^{v,p}(x,\rho) := \frac{1}{\rho^{\frac{d-2}{2}}}\|\nabla^2 v\|_{2,B_\rho^+(x)} + \frac{1}{\rho^{\frac{d-2}{2}}}\|\nabla p\|_{2,B_\rho^+(x)} \leq C_H\ \rho^\gamma \tag{123}$$

for all $x \in B_\delta^*(\bar{x})$ and $0 < \rho < \frac{R-\delta}{2}$.

Case 1 : $x \in B_\delta(\bar{x}) \cap \Gamma$.

We can find $k \in \mathbb{N}$ such that $\tau^{k+1}R < \rho \leq \tau^k R$. Then

$$E_0^{v,p}(x,\rho) \leq \tau^{\frac{2-d}{2}} E_0^{v,p}(x,\tau^k R). \tag{124}$$

Due to (121) and $\varepsilon_3 \leq \varepsilon_1$ so that we can apply Lemma 4.4 and have inequalities

$$E^{v,p}(x,\tau^k R) \leq \tau^{k\gamma} E^{v,p}(x,R) \leq \tau^{k\gamma}\varepsilon_3 \leq (\frac{1}{\tau R})^\gamma \varepsilon_3 \rho^\gamma. \tag{125}$$

Thus

$$E_0^{v,p}(x,\rho) \leq \tau^{\frac{2-d}{2}} E^v(x,\tau^k R_x) \leq \tau^{\frac{2-d}{2}} (\frac{1}{\tau R})^\gamma \varepsilon_3 \rho^\gamma. \tag{126}$$

Case 2 : $x \in B_\delta^+(\bar{x})$.

Denote x_0 be projection of x into Γ. It is clear that

$$x_0 \in B_\delta^+(\bar{x}),\ x_0 = (x',0),\ x_d = |x_0 - x|,\ 2x_d < 2\delta < R - \delta.$$

(i) if $x_d \leq \rho \leq \frac{R-\delta}{2}$, it follows $2\rho \leq R-\delta$ and $B_\rho^+(x) \subset B_{2\rho}^+(x_0)$. Henceforth,

$$E_0^{v,p}(x,\rho) \leq 2^{\frac{d-2}{2}} E_0^{v,p}(x_0,2\rho). \tag{127}$$

By the same way in case 1 we obtain

$$E_0^{v,p}(x_0,2\rho) \leq \tau^{\frac{2-d}{2}} (\frac{2}{\tau R})^\gamma \varepsilon_3 \rho^\gamma. \tag{128}$$

Hence

$$E_0^{v,p}(x,\rho) \leq\ [(\frac{2}{\tau})^{\frac{d-2}{2}} (\frac{2}{\tau R})^\gamma \varepsilon_3]\rho^\gamma. \tag{129}$$

(ii) $0 < \rho < x_d$. We start with proving estimates

$$|E^{v,p}(x,x_d)| < \varepsilon_2, \tag{130}$$

$$|(v)_{B_{x_d}(x)}| + |(\nabla v)_{B_{x_d}(x)}| + |(p)_{B_{x_d}(x)}| < \frac{M}{4}. \tag{131}$$

In fact, applying the inequality (129) for $\rho = x_d$ we get

$$E_0^{v,p}(x,x_d) \leq\ [(\frac{2}{\tau})^{\frac{d-2}{2}} (\frac{2}{\tau R})^\gamma \varepsilon_3] x_d^\gamma.$$

Thanks to choice ε_3 satisfies (119) we deduce the inequality (130)

Next we can find $l \in \mathbb{N}$ such that $\tau^{l+1}R < 2x_d \le \tau^l R$ and denote $r_0 = \tau^l R$. Then $B_{x_d}(x) \subset B^+_{r_0}(x_0)$ and applying Poincaré 's inequality, we get

$$\begin{aligned} |(\nabla v)_{x,x_d} - (\nabla v)_{B^+_{r_0}(x_0)}| &\le \frac{1}{2x_d^d\kappa_d}\int_{B_{x_d}(x)} |\nabla v - (\nabla v)_{B^+_{r_0}(x_0)}|\, dz \\ \le \frac{1}{(2x_d^d\kappa_d)^{\frac12}}\|\nabla v - (\nabla v)_{B^+_{r_0}(x_0)}\|_{2;B^+_{r_0}(x_0)} &\le \frac{C_P r_0^{\frac d2}}{(2x_d^d\kappa_d)^{\frac12}}\frac{1}{r_0^{\frac{d-2}{2}}}\|\nabla^2 v\|_{2;B^+_{r_0}(x_0)}. \end{aligned} \tag{132}$$

Hence

$$\begin{aligned} |((\nabla v)_{x,x_d}| &\le |(\nabla v)_{x,x_d} - (\nabla v)_{B^+_{r_0}(x_0)}| + |(\nabla v)_{B^+_{r_0}(x_0)}| \\ &\le \frac{C_P}{\sqrt{2\kappa_d}}(\frac{r_0}{x_d})^{\frac d2}\frac{1}{r_0^{\frac{d-2}{2}}}\|\nabla^2 v\|_{2;B^+_{r_0}(x_0)} + |(\nabla v)_{B^+_{r_0}(x_0)}|. \end{aligned} \tag{133}$$

By an analogous way, we have

$$\begin{aligned} |((p)_{x,x_d}| &\le |(p)_{x,x_d} - (p)_{B^+_{r_0}(x_0)}| + |(p)_{B^+_{r_0}(x_0)}| \\ &\le \frac{C_P}{\sqrt{2\kappa_d}}(\frac{r_0}{x_d})^{\frac d2}\frac{1}{r_0^{\frac{d-2}{2}}}\|\nabla p\|_{2;B^+_{r_0}(x_0)} + |(p)_{B^+_{r_0}(x_0)}|. \end{aligned} \tag{134}$$

Next we have following estimates for $|(v)_{x,x_d}|$

$$\begin{aligned} |(v)_{x,x_d} - (v)_{B^+_{r_0}(x_0)}| &\le \frac{1}{2x_d^d\kappa_d}\int_{B_{x_d}(x)} |v(z) - (v)_{B^+_{r_0}(x_0)} - (\nabla v)_{B^+_{r_0}(x_0)}(z-x)|\, dz \\ + x_d|(\nabla v)_{B^+_{r_0}(x_0)}| &\le \frac{1}{(2(x_d)^d\kappa_d)^{\frac12}}\|v(z) - (v)_{B^+_{r_0}(x_0)} - (\nabla v)_{B^+_{r_0}(x_0)}(z-x)\|_{2;B^+_{r_0}(x_0)} \\ + x_d|(\nabla v)_{B^+_{r_0}(x_0)}| &\le \frac{C_P r_0^{\frac d2}}{(2x_d^d\kappa_d)^{\frac12}}\frac{1}{r_0^{\frac{d-2}{2}}}\|\nabla v - (\nabla v)_{B^+_{r_0}(x_0)}\|_{2;B^+_{r_0}(x_0)} + x_d|(\nabla v)_{B^+_{r_0}(x_0)}| \\ &\le \frac{C_P^2 r_0^{\frac d2} r_0}{(2x_d^d\kappa_d)^{\frac12}}\frac{1}{r_0^{\frac{d-2}{2}}}\|\nabla^2 v\|_{B^+_{r_0}(x_0)} + x_d|(\nabla v)_{B^+_{r_0}(x_0)}|. \end{aligned} \tag{135}$$

Thus

$$\begin{aligned} |(v)_{x,x_d}| &\le |(v)_{x,x_d} - (v)_{B^+_{r_0}(x_0)}| + |(v)_{B^+_{r_0}(x_0)}| \\ &\le C_P^2(\frac{r_0^{d+2}}{2x_d^d\kappa_d})^{\frac12}\frac{1}{r_0^{\frac{d-2}{2}}}\|\nabla^2 v\|_{B^+_{r_0}(x_0)} + x_d|(\nabla v)_{B^+_{r_0}(x_0)}| + |(v)_{B^+_{r_0}(x_0)}| \\ &\le \frac{C_P^2}{\sqrt{2\kappa_d}}(\frac{r_0}{x_d})^{\frac d2}\frac{1}{r_0^{\frac{d-2}{2}}}\|\nabla^2 v\|_{B^+_{r_0}(x_0)} + x_d|(\nabla v)_{B^+_{r_0}(x_0)}| + |(v)_{B^+_{r_0}(x_0)}|. \end{aligned} \tag{136}$$

Summing (133), (134), (136) and using (121), we obtain

$$\begin{aligned} &|(v)_{x,x_d}| + |(\nabla v)_{x,x_d}| + |(p)_{x,x_d}| \\ \le \frac{C'_P}{\sqrt{2\kappa_d}}(\frac{r_0}{x_d})^{\frac d2}E_0^{v,p}(x_0,r_0) &+ x_d|(\nabla v)_{B^+_{r_0}(x_0)}| + \left(|(v)_{B^+_{r_0}(x_0)}| + |(\nabla v)_{B^+_{r_0}(x_0)}| + |(p)_{B^+_{r_0}(x_0)}|\right). \end{aligned} \tag{137}$$

As $x = x_0, \rho = r_0$ the inequality (126) gives

$$E_0^{v,p}(x_0, r_0) \le \tau^{\frac{2-d}{2}} (\frac{1}{\tau R})^\gamma \varepsilon_3.$$

Therefore, from (119) we deduce

$$\frac{C'_P}{\sqrt{2\kappa_d}} (\frac{r_0}{x_d})^{\frac{d}{2}} E_0^{v,p}(x_0, r_0) \le \frac{C'_P}{\sqrt{2\kappa_d}} (\frac{2}{\tau})^{\frac{d}{2}} E_0^{v,p}(x_0, r_0) \le \frac{M}{16}.$$

Thanks to (121), (119) it holds

$$|E^{v,p}(x_0, R)| < \varepsilon_3 \quad , \quad |(v)_{B_R^+(x_0)}| + |(\nabla v)_{B_R^+(x_0)}| + |(p)_{B_R^+(x_0)}| < \frac{M}{32}$$
$$\text{and} \quad \frac{C'_P}{(\tau^d \kappa_d)^{\frac{1}{2}}} \varepsilon_3 \frac{1}{1 - \tau^\gamma} \le \frac{M}{32}.$$

By the similar way in Lemma 4.6, we have for all $k \in \mathbb{N}$ the inequalities

$$|(p)_{B^+_{\tau^{k-1}R}(x_0)}| + |(\nabla v)_{B^+_{\tau^{k-1}R}(x_0)}| + |(p)_{B^+_{\tau^{k-1}R}(x_0)}| \le \frac{M}{8}.$$

Hence,

$$|(p)_{B^+_{r_0}(x_0)}| + |(\nabla v)_{B^+_{r_0}(x_0)}| + |(p)^+_{B_{r_0}(x_0)}| \le \frac{M}{8} \ \text{ and } \ x_d |(\nabla v)_{B^+_{r_0}(x_0)}| \le \frac{M}{16}.$$

From the above estimates we can conclude the assertion (131) holds.

Return to prove the inequality (123). There is $k \in \mathbb{N}$ such that $\tau^{k+1} x_d < \rho \le \tau^k x_d$. The assertion (131) holds so that we can apply Lemma 4.6 and obtain

$$E_0^{v,p}(x, \rho) \le \tau^{\frac{d-2}{2}} E_0^{v,p}(x, \tau^k x_d) \le \tau^{\frac{2-d}{2}} E^{v,p}(x, \tau^k x_d) \le \tau^{\frac{2-d}{2}} \tau^{k\gamma} E^{v,p}(x, x_d). \tag{138}$$

On the other hand, applying (129) for $\rho := x_d$ we have

$$E^{v,p}(x, x_d) \le \ [(\frac{2}{\tau})^{\frac{d-2}{2}} (\frac{2}{\tau R})^\gamma \varepsilon_3] x_d^\gamma + x_d^\gamma.$$

Thus

$$E_0^{v,p}(x, \rho) \le \tau^{\frac{2-d}{2}} [(\frac{2}{\tau})^{\frac{d-2}{2}} (\frac{2}{\tau R})^\gamma \varepsilon_3 + 1] \tau^{k\gamma} x_d^\gamma \le \tau^{\frac{2-d}{2}} [(\frac{2}{\tau})^{\frac{d-2}{2}} (\frac{2}{\tau R})^\gamma \varepsilon_3 + 1] \tau^{-\gamma} \rho^\gamma. \tag{139}$$

Summarize that the inequality (123) holds for all $x \in B_\delta^+(\bar{x})$, where C_H given

$$C_H \ = \ \max \ (\tau^{\frac{2-d}{2}} (\frac{1}{\tau R})^\gamma \varepsilon_3, (\frac{2}{\tau})^{\frac{d-2}{2}} (\frac{2}{\tau R})^\gamma \varepsilon_3, \tau^{\frac{2-d}{2}} [(\frac{2}{\tau})^{\frac{d-2}{2}} (\frac{2}{\tau R})^\gamma \varepsilon_3 \ + \ 1] \tau^{-\gamma}). \tag{140}$$

Therefore, $\nabla^2 v \in L^{2,d-2+\gamma}(B_\delta^+(x))^{d^3}$ and $\nabla p \in L^{2,d-2+\gamma}(B_\delta^+(x))^d$. It implies that $\nabla v \in \mathcal{L}^{2,d+\gamma}(B_\delta^+(x))^{d^2} \simeq C^{\frac{\gamma}{2}}(\overline{B_\delta^+(x)}^{d^2}$ and $p \in \mathcal{L}^{2,d+\gamma}(B_\delta^+(x)) \simeq C^{\frac{\gamma}{2}}(\overline{B_\delta^+(x)}$. Thus prove that ∇v, p are Hölder continuous on $\overline{B_\delta^+(x)}$. □

Denote

$$\begin{aligned}
&\Sigma_1^v = \{x \in \Gamma;\ \liminf_{R \to 0+} \frac{1}{R^{\frac{d-2}{2}}} \|\nabla^2 v\|_{2;B_R^+(x)} > 0\}, \\
&\Sigma_1^p = \{x \in \Gamma;\ \liminf_{R \to 0+} \frac{1}{R^{\frac{d-2}{2}}} \|\nabla p\|_{2;B_R^+(x)} > 0\}, \\
&\Sigma_2^v = \{x \in \Gamma;\ \limsup_{R \to 0+} |(v)_{2;B_R^+(x)}| + |(\nabla v)_{2;B_R^+(x)}| = \infty\}, \\
&\Sigma_2^p = \{x \in \Gamma;\ \limsup_{R \to 0+} |(p)_{2;B_R^+(x)}| = \infty\}.
\end{aligned} \tag{141}$$

Applying Lemma 2.4 and Lemma 2.5, for any $\epsilon > 0$ it holds

$$H^{d-2}(\Sigma_1^v) = H^{d-2}(\Sigma_1^p) = 0, H^{d-2+\epsilon}(\Sigma_2^v) = H^{d-2+\epsilon}(\Sigma_2^p) = 0. \tag{142}$$

Then we have

Theorem 4.3. *Let the assumptions in Theorem 4.2 are satisfied. Then there exists $\Gamma_0 \subset \Gamma$ relatively open in Γ so that $\mathcal{H}_{d-2+\epsilon}(\Gamma - \Gamma_0) = 0$ for any $\epsilon > 0$. Moreover, for any $x \in \Gamma_0$ there exists $\delta > 0$ so that ∇v and p are Hölder continuous on $\overline{B_\delta^+(x)}$.*

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