Uniformly Gâteaux differentiable norms in spaces with unconditional basis

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ABSTRACT. *It is shown that a Banach space X admits an equivalent uniformly Gâteaux differentiable norm if it has an unconditional basis and X^* admits an equivalent norm which is uniformly rotund in every direction.*

Let $(X, \|.\|)$ be a Banach space. Let S_X and B_X denote the unit sphere and the unit ball respectively, i. e. $S_X = \{x \in X; \|x\| = 1\}$ and $B_X = \{x \in X; \|x\| \le 1\}$. Let $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ denote the sets of positive integers, rational numbers and real numbers respectively. Let X^* denote the dual to the Banach space X and let $\|.\|^*$ denote the norm on X^* that is dual to the norm $\|.\|$ on X. A biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset X \times X^*$ is called an *unconditional basis* for a Banach space X if for each $x \in X$ $x = \sum_{\gamma\in\Gamma} f_\gamma(x)x_\gamma$ and the sum converges unconditionally. The norm $\|.\|$ on a Banach space X is said to be *uniformly rotund in every direction* (*URED* for short), if $\lim \|x_n - y_n\| = 0$ whenever $x_n, y_n \in S_X$ are such that $\lim \|x_n + y_n\| = 2$ and $x_n - y_n = \lambda_n z$ for some $z \in X, \lambda_n \in \mathbb{R}$. The norm $\|.\|$ on X is called *uniformly Gâteaux differentiable* (UG) if

$$\lim_{t\to 0} \frac{1}{t}\Big(\sup_{x\in S_X} \|x + th\| + \|x - th\| - 2\Big) = 0$$

for every $h \in S_X$. A compact space K is called a *uniform Eberlein compact* (UEC) if K is homeomorphic to a weakly compact subset of a Hilbert space in its weak topology. The space $(\Sigma(\mathbb{R}^\Gamma), \tau)$ is a subspace of a product space $\mathbb{R}^\Gamma$ with the product topology, such that $x \in (\Sigma(\mathbb{R}^\Gamma), \tau)$ iff $x(\gamma) \ne 0$ for at most countably many $\gamma \in \Gamma$.

The main result in this paper is the following theorem.

Theorem 1. *Let X be a Banach space with an unconditional basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$. If X^* admits an equivalent (not necessarily dual) URED norm, then B_{X^*} in its weak* topology is a UEC.*

By putting this implication together with other already known results (we refer to [BRW], [BS], [DGZ, Chap. II], we obtain Theorem 2. Note that, except (i) $\Rightarrow$ (ii), all the remaining implications hold without the assumption of existence of an unconditional Schauder basis.

Theorem 2. *Let X be a Banach space with an unconditional basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$. Then the following are equivalent*
(i) X^* admits an equivalent (not necessarily dual) URED norm.
(ii) B_{X^*} in its weak topology is a UEC.*
(iii) X admits an equivalent UG norm.

In the proof of Theorem 1 we shall use the following statements.

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Fact 3. *Let $(X, \|.\|)$ be a Banach space and let $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ be a normalized unconditional basis for X. For $x \in X$ put $\|x\|_1 = \sup_{\alpha_\gamma=\pm1}\left\|\sum_{\gamma\in\Gamma}\alpha_\gamma f_\gamma(x)x_\gamma\right\|$. Then*
(i) $\|.\|_1$ is an equivalent norm on X,
(ii) $\left\|\sum\limits_{\gamma\in F} a_\gamma x_\gamma\right\|_1 \le \left\|\sum\limits_{\gamma\in F} b_\gamma x_\gamma\right\|_1$, and $\left\|\sum\limits_{\gamma\in F} a_\gamma f_\gamma\right\|_1^* \le \left\|\sum\limits_{\gamma\in F} b_\gamma f_\gamma\right\|_1^*$, whenever $F\subset\Gamma$ is finite and $a_\gamma, b_\gamma\in\mathbb{R}$ satisfy $|a_\gamma|\le|b_\gamma|$ for every $\gamma\in F$,
(iii) $\|P_A\|_1 = 1$ for $A\subset\Gamma$ finite, where $P_A(x) = \sum_{\gamma\in A} f_\gamma(x)x_\gamma$,
(iv) $\|x_\gamma\|_1 = \|f_\gamma\|_1^* = 1$.

Proof. The proof is based on the similar ideas as [S, p. 499–505], where analogous statements for a countable set Γ are proved.

□

The following lemma is due to Troyanski [Tr].

Lemma 4. *Let X be a Banach space and let $\|.\|$ be an equivalent URED norm on X. Then for any $\varepsilon > 0$ there exists a decomposition $\{S_i^{(\varepsilon)}\}_{i=1}^\infty$ of the unit sphere S_X such that for distinct $\{x_j\}_{j=1}^i \subset S_i^{(\varepsilon)}$ we have $\max\limits_{a_j=\pm1}\|\sum\limits_{j=1}^i a_j x_j\| > \varepsilon^{-1}$.*

We shall use the following topological characterization of uniform Eberlein compacts which can be found in [BRW].

Lemma 5. *A compact space K is UEC iff there exists a family $\{V_\delta, \delta\in\Delta\}$ of open F_σ subsets of K such that*
(i) The family $\{V_\delta, \delta\in\Delta\}$ separates points of K, i. e. for $x\ne y\in K$ there exists $\delta\in\Delta$ such that $x\in V_\delta$ and $y\notin V_\delta$, or $x\notin V_\delta$ and $y\in V_\delta$.
(ii) There exists a decomposition of Δ into a sequence $\{\Delta_n\}_{n=1}^\infty$ and natural numbers $\{k(n)\}_{n=1}^\infty$ such that $\{V_\delta, \delta\in\Delta_n\}$ is $k(n)$-finite, i. e. any $x\in K$ belongs to at most $k(n)$ sets of the family $\{V_\delta, \delta\in\Delta_n\}$.

The core of our note is the following lemma.

Lemma 6. *Let $(X, \|.\|_1)$ be a Banach space, $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ be a normalized unconditional basis, $\|.\|_1$ be a norm as in Fact 3, $\|.\|_2$ be an equivalent URED norm on X^*. Then there is a bounded linear, weak* to pointwise continuous, and one-to-one operator $T : (X^*, w^*) \to (\Sigma(\mathbb{R}^\Gamma), \tau)$ such that for any $\varepsilon > 0$ there exists a decomposition $\{\Gamma_i^{(\varepsilon)}\}_{i=1}^\infty$ of Γ such that*

$$card\,\{\gamma\in\Gamma_i^{(\varepsilon)}, |T(x^*)(\gamma)| > \varepsilon\} < i.$$

for all $x^*\in S_{(X^*,\|.\|_1^*)}$ and $i\in\mathbb{N}$.

Proof. Let $k > 1$ be such that $k\|x^*\|_2 \ge \|x^*\|_1^* \ge k^{-1}\|x^*\|_2$. Let $\tilde f_\gamma = \frac{f_\gamma}{\|f_\gamma\|_2}$ for all $\gamma\in\Gamma$. Put $\Gamma_i^{(\varepsilon)} = \{\gamma\in\Gamma, \tilde f_\gamma\in S_i^{(\varepsilon/k^2)}\}$, where according to Lemma 4, $\{S_i^{(\varepsilon/k^2)}\}_{i=1}^\infty$ is the decomposition of $S_{(X^*,\|.\|_2)}$ such that for all distinct $\{x_j^*\}_{j=1}^i \subset S_i^{(\varepsilon/k^2)}$ we have $\max_{a_j=\pm1}\|\sum_{j=1}^i a_j x_j^*\|_2 > k^2/\varepsilon$. Let $T\colon (X^*, w^*) \to (\mathbb{R}^\Gamma, \tau)$ be defined for $x^*\in X^*$ by $T(x^*) = \{x^*(x_\gamma)\}_{\gamma\in\Gamma}$. Clearly, the operator T is weak* to pointwise continuous and one-to-one. Put $U^{(\varepsilon)}_{x^*,i} = \{\gamma \in \Gamma^{(\varepsilon)}_i ; |T(x^*)(\gamma)| > \varepsilon\}$ and let us suppose that there is $\varepsilon > 0$, $x^* \in B_{X^*}$ and $i \in \mathbb{N}$ such that $|U^{(\varepsilon)}_{x^*,i}| \geq i$. Let $A \subset U^{(\varepsilon)}_{x^*,i}$ be such that $\operatorname{card} A = i$. Then

$$\begin{aligned} 1 \geq \|x^*\|^*_1 \geq \|P^*_A(x^*)\|^*_1 &= \Big\| \sum_{\gamma \in A} x^*(x_\gamma) f_\gamma \Big\|^*_1 \\ &\geq \min_{\gamma \in A} |x^*(x_\gamma)| \ \Big\| \sum_{\gamma \in A} f_\gamma \Big\|^*_1 > \varepsilon \Big\| \sum_{\gamma \in A} f_\gamma \Big\|^*_1 = \varepsilon \max_{a_\gamma = \pm 1} \Big\| \sum_{\gamma \in A} a_\gamma f_\gamma \Big\|^*_1 \\ &\geq \varepsilon k^{-1} \max_{a_\gamma = \pm 1} \Big\| \sum_{\gamma \in A} a_\gamma \tilde{f}_\gamma \Big\|^*_1 \geq \varepsilon k^{-2} \max_{a_\gamma = \pm 1} \Big\| \sum_{\gamma \in A} a_\gamma \tilde{f}_\gamma \Big\|_2 > 1, \end{aligned}$$

which is a contradiction. Hence, in particular, $T\colon (X^*, w^*) \to (\Sigma(\mathbb{R}^\Gamma), \tau)$.

$\square$

Proof of the Theorem 1.

We shall use the notation of Lemma 6. According to Lemma 5, in order to prove that (B_{X^*}, w^*) is UEC, we put $\Delta = \{(\gamma, r); \gamma \in \Gamma, r \in \mathbb{Q} \setminus \{0\}\}$. We put $V_{(\gamma,r)} = \big\{f \in B_{X^*}; T(f)(\gamma) > r\big\}$ for $r > 0$ and $V_{(\gamma,r)} = \big\{f \in B_{X^*}; T(f)(\gamma) < r\big\}$ for $r < 0$. Clearly, each $V_{(\gamma,r)}$ is weak*-open and F_σ-set. If $f, g \in B_{X^*}, f \neq g$ then there is $\gamma \in \Gamma$ such that $f(x_\gamma) \neq g(x_\gamma)$. Assume that $f(x_\gamma) < g(x_\gamma)$. We choose $0 \neq r \in \mathbb{Q}$ such that $f(x_\gamma) < r < g(x_\gamma)$. If $r > 0$, then $g \in V_{(\gamma,r)}$ and $f \notin V_{(\gamma,r)}$. If $r < 0$, then $f \in V_{(\gamma,r)}$ and $g \notin V_{(\gamma,r)}$. The case when $f(x_\gamma) > g(x_\gamma)$ can be treated similarly. Hence the family $\{V_{(\gamma,r)}; (\gamma, r) \in \Delta\}$ separates points of (B_{X^*}, w^*). Let $\{\Gamma^{(\varepsilon)}_i\}_{i=1}^\infty$ be the decomposition of Γ by Lemma 6. For $r \in \mathbb{Q} \setminus \{0\}$ and $i \in \mathbb{N}$ put

$$\Delta_{(r,i)} = \Gamma^{|r|}_i \times \{r\}.$$

Now fix one such (r, i) and consider any $f \in B_{X^*}$. If $(\gamma, r) \in \Delta_{(r,i)}$ and $f \in V_{(\gamma,r)}$, then $|T(f)(\gamma)| > |r|$. Therefore, by Lemma 6,

$$\mathit{card}\,\big\{V_{(\gamma,r)};\ (\gamma, r) \in \Delta_{(i,r)}, V_{(\gamma,r)} \ni f\big\} \leq \mathit{card}\,\big\{\gamma \in \Gamma^{|r|}_i;\ |T(f)(\gamma)| > |r|\big\} < i.$$

It means that the family $\{V_{(\gamma,r)}; (\gamma, r) \in \Delta_{(r,i)}\}$ is $(i-1)$–finite. Hence (B_{X^*}, w^*) is UEC by Lemma 5.

$\square$

Remarks.

1) The condition of the existence of an unconditional basis can not be dropped. Indeed, consider the space $X = C[0, \omega_1]$ of all continuous functions on the ordinal segment $[0, w_1]$. The space X^* is isometric to $l_1([0, \omega_1])$ and hence admits an equivalent URED norm (see e.g. [DGZ], Proposition II.7.7 and II.6.7). However, due to Talagrand's result (see e.g. [DGZ], Theorem VII.5.2 and Theorem 2.6.7), X^* admits no dual strictly convex norm, hence X admits no equivalent UG norm.

2) There is a Banach space X with an unconditional basis such that the dual space X^* admits an equivalent (non dual) strictly convex norm and X admits no equivalent Gâteaux smooth norm. See [AM].

3) There is a Banach space X with no equivalent UG norm such that the dual space X^* admits an equivalent dual URED norm. See [H].

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