# DUALITY PRINCIPLES AND REDUCTION THEOREMS

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Abstract. We introduce a fairly general class of Banach function spaces $\Lambda_X$ given by $\|f\|_{\Lambda_X} := \|f^*_\mu\|_X$, where $f$ is defined on a totally $\sigma$–finite non-atomic measure space $(\mathcal{R},\mu)$, $f^*_\mu$ is the non-increasing rearrangement of $f$ with respect to $\mu$ and $X$ is certain rearrangement-invariant space over the interval $(0,\mu(\mathcal{R}))$. This class contains for example classical Lorentz spaces. We prove a general duality principle for these spaces and present several applications. In particular, we prove theorems which enable us to reduce weighted inequalities involving integral operators restricted to monotone functions to certain more manageable weighted inequalities. Reduction theorems are then applied to obtain a characterization of embeddings between $\Lambda_X$ spaces.

## 1. Introduction

Let $(\mathcal{R},\mu)$ be a totally $\sigma$–finite non-atomic measure space. Let $\mathfrak{M}(\mathcal{R},\mu)$ be the set of all $\mu$–measurable a.e. finite real functions on $\mathcal{R}$. By $\mathfrak{M}^+(\mathcal{R},\mu)$ we denote the subset of $\mathfrak{M}(\mathcal{R},\mu)$ consisting of non-negative functions. When $\mathcal{R}$ is an interval $(a,b)$, $-\infty \le a < b \le \infty$, and $\mu$ is the one-dimensional Lebesgue measure $m$, then we denote by $\mathfrak{M}^+(a,b;\downarrow)$ the subset of $\mathfrak{M}^+(a,b)$ consisting of non-increasing functions on $(a,b)$.

For $f \in \mathfrak{M}(\mathcal{R},\mu)$ and $a = \mu(\mathcal{R})$, the *non-increasing rearrangement* of $f$ is the function $f^*_\mu$ defined by

$$f^*_\mu(t) = \inf\{\lambda; \mu(\{x, |f(x)| > \lambda\}) \le t\}, \quad t \in (0,a).$$

When $-\infty \le a < b \le \infty$ and $(\mathcal{R},\mu) = ((a,b),m)$, we write $f^*$ rather than $f^*_m$. When $w$ is a *weight* (that is, a Lebesgue-measurable non-negative function) on $(a,b)$ and $(\mathcal{R},\mu) = ((a,b), w\,dm)$, we write $f^*_w$.

As usual, by $A \lesssim B$ and $A \gtrsim B$ we mean that $A \le CB$ and $B \le CA$, respectively, where $C$ is a positive constant independent of appropriate quantities involved in $A$ and $B$. We write $A \approx B$ when both of the estimates $A \lesssim B$ and $B \lesssim A$ are satisfied. We shall use throughout the convention $0 \cdot \infty = 0$, $\frac{0}{0} = 0$ and $\frac{\infty}{\infty} = 0$.

In the theory of operators acting on function spaces it is often necessary to consider inequalities restricted to non-increasing non-negative functions on an interval. A typical example of such situation is the investigation of behaviour of integral operators on a *classical Lorentz space.* A classical Lorentz space $\Lambda^p(w)(\mathcal{R},\mu)$, where $p \in [1,\infty)$ and $w$ is a weight on $(0,\infty)$, is the set of all functions $f \in \mathfrak{M}(\mathcal{R},\mu)$ such that

$$\|f\|_{\Lambda^p(w)} := \left(\int_0^\infty \left(f^*_\mu(t)\right)^p w(t)\,dt\right)^{1/p} < \infty.$$

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In the pioneering paper [1], Ariño and Muckenhoupt characterized when the *Hardy–Littlewood maximal operator* $M$ is bounded on a classical Lorentz space, that is, they characterized the weights $w$ for which the inequality

$$\int_0^\infty [(Mf)^*(t)]^p\, w(t)\, dt \lesssim \int_0^\infty [f^*(t)]^p\, w(t)\, dt$$

holds for every locally integrable $f$ on $\mathbb{R}^n$. Recall that the operator $M$ is defined at $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$(Mf)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)|\, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is extended over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and $|E|$ denotes the $n$-dimensional Lebesgue measure of $E \subset \mathbb{R}^n$. The authors first observed that $M$ is bounded on $\Lambda^p(w)$ if and only if the weighted Hardy-type integral inequality

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f(s)\, ds\right)^p w(t)\, dt\right)^{1/p} \lesssim \left(\int_0^\infty f(t)^p w(t)\, dt\right)^{1/p} \tag{1.1}$$

holds for all $f \in \mathfrak{M}^+(0,\infty;\downarrow)$, and then they characterized the class of weights for which this is true. It turns out that this class is considerably wider than the class of weights for which (1.1) holds for all $f \in \mathfrak{M}^+(0,\infty)$. In another fundamental paper, Sawyer ([18]) characterized the quantity

$$\sup_{f \in \mathfrak{M}^+(0,\infty;\downarrow)} \frac{\int_0^\infty f(t) g(t)\, dt}{(\int_0^\infty f(t)^p v(t)\, dt)^{1/p}}$$

for a given $g \in \mathfrak{M}^+(0,\infty)$. Again, the resulting characterization is quite different from the analogous one in which all $f \in \mathfrak{M}^+(0,\infty)$ are considered. Sawyer's result has several important applications; for example, it produces a description of the associate space of a classical Lorentz space, it gives a characterization of weights $v, w$ for which the inclusion $\Lambda^p(v) \subset \Lambda^q(w)$ is true, and it enables one to characterize pairs of weights for which an operator $T$ is bounded from one classical Lorentz space to another as long as certain a-priori estimate of $(Tf)^*$ in terms of $f^*$ is known. In particular, Sawyer extended the results of Ariño and Muckenhoupt to more operators than just $M$. In 1990's, many authors have considered inequalities involving monotone functions in connection with various problems. In 1993, Stepanov ([19]) found a simple proof of Sawyer's duality result and extended the range of admissible parameters. A remarkable fact was recently revealed in [5]: The boundedness between classical Lorentz spaces of the *fractional maximal operator* $M_\gamma$, defined at $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$(M_\gamma f)(x) = \sup_{Q \ni x} |Q|^{\frac{\gamma}{n}-1} \int_Q |f(y)|\, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is again extended over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, is equivalent to the boundedness of the operator

$$(Tf)(t) := \sup_{s \in (t,\infty)} v(s) \int_0^s f(y)\, dy, \quad t \in (0,\infty), \tag{1.2}$$

restricted to $f \in \mathfrak{M}^+(0,\infty;\downarrow)$, between weighted Lebesgue spaces. Similar operators appeared recently in many other research projects in interpolation theory ([7], [9], [6], [16]) or in the theory of optimal Sobolev embeddings ([8], [12], [14], [15]).

This is our main motivation to develop a reasonably general concept of duality principles and reduction theorems which would be applicable to integral operators including those of the type (1.2). That is the subject of this paper. Let us outline our approach.

We first characterize the quantity

$$\sup_{f\in\mathfrak{M}^+(0,a;\downarrow)} \frac{\int_0^a f(t)g(t)\,dt}{\|f\|_X},$$

where $a \in (0,\infty]$, $g \in \mathfrak{M}^+(0,a)$, and $X$ is a rearrangement-invariant space over the interval $(0,a)$ endowed with a weighted measure. The proof is quite elementary and self-contained; the key step is a lemma in the spirit of [17] which gives a simple expression for a non-increasing rearrangement of a non-increasing function with respect to a weighted measure. One of the main consequences of this characterization is a duality principle for a fairly wide class of function spaces $\Lambda_X(\mathcal{R},\mu)$, whose norm is defined at $f \in \mathfrak{M}(\mathcal{R},\mu)$ by

$$\|f\|_{\Lambda_X} := \|f^*_\mu\|_X.$$

We present several applications. Perhaps the most important of them is what we call a *reduction theorem* for linear operators. The reduction theorem enables us to replace an inequality *restricted to non-increasing functions* by certain more manageable inequalities (the idea is borrowed from [18]). Another application is an embedding theorem for the $\Lambda_X$ spaces. As an example, we apply our results to kernel operators of Hardy and Volterra type.

Finally, we give an alternative elementary proof of a different type of reduction theorem, applicable to operators that are not necessarily linear (such as those of the type (1.2)).

## 2. Preliminaries

**Definition 2.1.** Let $X \subset \mathfrak{M}(\mathcal{R},\mu)$ be a Banach space over $(\mathcal{R},\mu)$, endowed with a norm $\|\cdot\|$. Assume that the functional $\|\cdot\|$ is defined on the entire $\mathfrak{M}(\mathcal{R},\mu)$ and $X = \{f \in \mathfrak{M}(\mathcal{R},\mu);\ \|f\| < \infty\}$. We say that $X$ is a *Banach lattice* if $\|f\| \le \|g\|$ whenever $0 \le f \le g$ on $\mathcal{R}$. We say that $X$ is a *rearrangement-invariant Banach function space* (or shortly an *r.i. space*) *over* $(\mathcal{R},\mu)$ if the following four axioms hold:

($A_1$) $0 \le f_n \nearrow f$ on $\mathcal{R}$ implies $\|f_n\| \nearrow \|f\|$;
($A_2$) $\|\chi_E\| < \infty$ whenever $E \subset \mathcal{R}$ and $\mu(E) < \infty$;
($A_3$) for every $E \subset \mathcal{R}$ with $\mu(E) < \infty$ there exists a $C > 0$ such that

$$\int_E f(x)\,d\mu(x) \lesssim \|f\| \quad \text{for all } f \in \mathfrak{M}(\mathcal{R},\mu).$$

($A_4$) $\|f\| = \|g\|$ for every $f, g$ such that $f^*_\mu = g^*_\mu$.

**Remark 2.2.** We note that, by [2, Chapter 2, Theorem 2.7], $(\mathcal{R},\mu)$ is *reasonant*, that is, for each $f$ and $g$ in $\mathfrak{M}(\mathcal{R},\mu)$, the identity

$$\int_0^{\mu(\mathcal{R})} f^*_\mu(t) g^*_\mu(t)\,dt = \sup \int_{\mathcal{R}} f(x)h(x)\,d\mu(x), \tag{2.1}$$

where the supremum is taken over all functions $h \in \mathfrak{M}(\mathcal{R},\mu)$ with $h_\mu^* = g_\mu^*$.

The concept of duality (with respect to the pairing $\int_{\mathcal{R}} f(x)g(x)\,d\mu(x)$) is in the context of r.i. spaces realized through the notion of an associate space.

**Definition 2.3.** Let $X$ be an r.i. space over $(\mathcal{R},\mu)$. We define the *associate space* $X'$ of $X$ as the set of all functions $f \in \mathfrak{M}(\mathcal{R},\mu)$ such that $\|f\|_{X'} < \infty$, where

$$\|f\|_{X'} := \sup_{\|g\|_X \le 1} \int_{\mathcal{R}} f(x)g(x)\,d\mu(x).$$

**Remark 2.4.** If $X$ is an r.i. space over $(\mathcal{R},\mu)$, then so is $X'$, and

$$\|f\|_{X'} = \sup_{\|g\|_X \le 1} \int_0^a f_\mu^*(t)g_\mu^*(t)\,dt \quad \text{for all } f \in X'.$$

We shall use the important concept of representation spaces. The following result is well known as the Luxemburg Representation Theorem (cf. [2, Chapter 2, Theorem 4.10]).

**Theorem 2.5.** *Let $X$ be an r.i. space over $(\mathcal{R},\mu)$ and $a = \mu(\mathcal{R})$. Then there is an r.i. space $\overline{X}$ over $((0,a),m)$ such that*

$$\|f\|_X = \|f_\mu^*\|_{\overline{X}} \quad \text{for all } f \in \mathfrak{M}(\mathcal{R},\mu).$$

The space $\overline{X}$ is called the *representation space* of $X$.

We shall often use the *Hölder inequality*

$$\int_{\mathcal{R}} f(x)g(x)\,d\mu(x) \le \|f\|_X\|g\|_{X'} \quad \text{for all } f,g \in \mathfrak{M}(\mathcal{R},\mu),$$

and the *fundamental identity*

$$\|\chi_E\|_X\|\chi_E\|_{X'} = t \quad \text{for every} \quad E \subset \mathcal{R},\ \mu(E)=t,\ t \in (0,a). \tag{2.2}$$

We shall also use the well-known inequality of Hardy, Littlewood and Pólya:

$$\int_{\mathcal{R}} f(x)g(x)\,d\mu(x) \le \int_0^a f_\mu^*(t)g_\mu^*(t)\,dt \quad \text{for all } f,g \in \mathfrak{M}(\mathcal{R},\mu). \tag{2.3}$$

**Definition 2.6.** Given an r.i. space $X$ over $(\mathcal{R},\mu)$, the *lower and upper Boyd indices $i_X$, $I_X$* are given by

$$i_X = \lim_{t\to 0_+} \frac{\log(1/t)}{\log h_X(t)}, \qquad I_X = \lim_{t\to\infty} \frac{\log(1/t)}{\log h_X(t)},$$

where $h_X(t)$ is defined for $t \in (0,\infty)$ by

$$h_X(t) = \sup_{f\in\mathfrak{M}^+(0,\infty)} \frac{\|(E_tf)\|_{\overline{X}}}{\|f\|_{\overline{X}}},$$

and $E_t$ is the dilation operator given at $f \in \mathfrak{M}^+(0,a)$ by $(E_tf)(s) = f(st)$, $0 < s,t < \infty$.

For a detailed treatment of (rearrangement-invariant) Banach function spaces, cf. [2].

Let $a \in (0,\infty]$ and let $w$ be a weight on $(0,a)$. Then, for $t \in (0,a)$, we denote $W(t) = \int_0^t w(s)\,ds$. We define the operators $A_w$ and $\bar{A}_w$ by

$$(A_wf)(t) := \frac{1}{W(t)} \int_0^t f(s)w(s)\,ds, \quad f \in \mathfrak{M}((0,a),w),\ t \in (0,a),$$

and

$$(\bar{A}_w f)(t) := \int_t^a f(s)\frac{w(s)}{W(s)}\,ds, \quad f \in \mathfrak{M}((0,a),w),\ t \in (0,a).$$

If, in particular, $w \equiv 1$, we get the usual integral average operator $A$ and its dual $\bar{A}$, given as

$$(Af)(t) = \frac{1}{t}\int_0^t f(s)\,ds, \quad f \in L^1_{\mathrm{loc}}(0,a),\ t \in (0,a),$$

and

$$(\bar{A}f)(t) = \int_t^a f(s)\frac{ds}{s}, \quad f \in L^1_{\mathrm{loc}}(0,a),\ t \in (0,a).$$

The definition of $\bar{A}_w$ of course depends on $a$ but it will be always clear from the context which particular value of $a$ is considered.

**Remark 2.7.** The operator $A_w$ is called *weighted average operator.* When $f \in \mathfrak{M}^+(0,a;\downarrow)$ and $w$ is a weight on $(0,a)$, then also $A_w f$ is non-increasing, and moreover

$$f(t) \le (A_w f)(t), \quad t \in (0,a). \tag{2.4}$$

## 3. Key lemma

The following lemma is a key step to our main results. It is a general version of (2.6) in [17].

**Lemma 3.1.** *Let $a \in (0,\infty]$ and let $w$ be a weight on $(0,a)$. Let $f \in \mathfrak{M}^+(0,a;\downarrow)$. Then*

$$f^*_w(t) = f(W^{-1}(t)), \quad 0 < t < a,$$

*where $W^{-1}$ is the generalized right-continuous inverse of $W$, given by $W^{-1}(t) = \inf\{s \in (0,a);\ W(s) > t\}$.*

*Proof.* Assume first that $f$ is strictly decreasing on $(0,a)$ and let $\lambda \in (0,\infty)$. Then

$$\{x; |f(x)| > \lambda\} = (0, f^{-1}(\lambda)),$$

whence, for $t \in (0,a)$,

$$\begin{aligned} f^*_w(t) &= \inf\{\lambda > 0; w(\{x; |f(x)| > \lambda\}) \le t\} \\ &= \inf\{\lambda > 0; W(f^{-1}(\lambda)) \le t\} = f(W^{-1}(t)). \end{aligned}$$

A routine argument extends the result to all $f \in \mathfrak{M}^+(0,a;\downarrow)$. □

As we shall see below, Lemma 3.1 has important consequences. Some of the most immediate ones are collected in the following corollary.

**Corollary 3.2.** *Let $a \in (0,\infty]$ and let $w$ be a weight on $(0,a)$. Let $X$ be an r.i. space over $((0,a),w)$.* (i) *For every $f \in \mathfrak{M}^+(0,a;\downarrow)$,*

$$\|f\|_X = \|f^*_w\|_{\overline{X}} = \|f(W^{-1})\|_{\overline{X}}. \tag{3.1}$$

(ii) *Let $i_X > 1$. Then*

$$\|A_w f\|_X \lesssim \|f\|_X \quad \text{for all } f \in \mathfrak{M}^+(0,a;\downarrow). \tag{3.2}$$

(iii) *Let $I_X < \infty$. Then*

$$\left\|\bar{A}_w f\right\|_X \lesssim \|f\|_X \quad \text{for all } f \in \mathfrak{M}^+(0,a;\downarrow). \tag{3.3}$$

*Proof.* (i) This follows from the Luxemburg Representation Theorem (Theorem 2.5) and Lemma 3.1.

(ii) By [2, Chapter 3, Theorem 5.15], the average operator $A$ is bounded on $\overline{X}$, more precisely,

$$\|Ag\|_{\overline{X}} \lesssim \|g\|_{\overline{X}} \quad \text{for all } g \in \mathfrak{M}^+(0, W(a)).$$

Thus, by the monotonicity of $A_w f$ and (3.1),

$$\begin{aligned}\|A_w f\|_X &= \left\|(A_w f)(W^{-1}(t))\right\|_{\overline{X}} = \left\|\frac{1}{t}\int_0^{W^{-1}(t)} f(s)w(s)\,ds\right\|_{\overline{X}} \\ &= \left\|\frac{1}{t}\int_0^t f(W^{-1}(s))\,ds\right\|_{\overline{X}} \lesssim \left\|f(W^{-1}(t))\right\|_{\overline{X}} = \|f\|_X.\end{aligned}$$

The proof of (iii) is analogous and thus left as an exercise. □

**Example 3.3.** Let $p,q \in (0,\infty]$. Let $a \in (0,\infty]$, let $w$ be a weight on $(0,a)$ and let $L^{p,q}((0,a),w)$ be the usual Lorentz space, given by

$$\|f\|_{L^{p,q}((0,a),w)} = \left\|t^{\frac{1}{p}-\frac{1}{q}} f_w^*(t)\right\|_{q,(0,W(a))}, \quad f \in \mathfrak{M}((0,a),w).$$

When $q \in (0,\infty)$, we get from Lemma 3.1 for every $f \in \mathfrak{M}^+(0,a;\downarrow)$,

$$\begin{aligned}\|f\|_{L^{p,q}((0,a),w)} &= \Big(\int_0^{W(a)} [f(W^{-1}(t))]^q t^{\frac{q}{p}-1}\,dt\Big)^{1/q} \\ &\approx \left(\int_0^a f(s)^q W(s)^{\frac{q}{p}-1} w(s)\,ds\right)^{1/q},\end{aligned}$$

which is just a complementary result to (2.6) in [17]. When $q=\infty$, we have analogously

$$\|f\|_{L^{p,\infty}((0,a),w)} = \sup_{0<t<a} W(t)^{1/p} f(t) \quad \text{for all } f \in \mathfrak{M}^+(0,a;\downarrow).$$

**Definition 3.4.** Let $(\mathcal{R},\mu)$ be a totally $\sigma$–finite non-atomic measure space with $a=\mu(\mathcal{R})$. Let $w$ be a weight on $(0,a)$ and let $X$ be an r.i. space over $((0,a),w)$. Define the space $\Lambda_X$ by

$$\Lambda_X = \Lambda_X(\mathcal{R},\mu) = \{f \text{ on } \mathcal{R};\ \|f\|_{\Lambda_X} := \|f_\mu^*\|_X < \infty\}.$$

Since $f_\mu^*$ is non-increasing on $(0,a)$, we have by Corollary 3.2 (i)

$$\|f\|_{\Lambda_X} = \|f_\mu^*\|_X = \|(f_\mu^*)_w^*\|_{\overline{X}} = \|f_\mu^*(W^{-1})\|_{\overline{X}}, \quad \text{for all } f \in \Lambda_X.$$

**Example 3.5.** For $X = L^{p,q}((0,a),w)$, $1 \le p,q \le \infty$, we have $\Lambda_X = \Lambda_w^{p,q}(\mu)$, the space introduced by Carro and Soria in [4].

We shall now give a characterization of an associate space of $\Lambda_X$ for those $X$ on which the operator $\bar{A}$ is bounded.

**Theorem 3.6.** *Let $(\mathcal{R},\mu)$ be a totally $\sigma$–finite non-atomic measure space with $a=\mu(\mathcal{R})$. Let $w$ be a weight on $(0,a)$ and let $X$ be an r.i. space over $((0,a),w)$. Assume that the operator $\bar{A}$ is bounded on $X$, that is,*

$$\left\|\int_t^a s^{-1} g(s)\,ds\right\|_X \le C\|g\|_X \quad \text{for all } g \in \mathfrak{M}^+(0,a). \tag{3.4}$$

*Then*

$$(\Lambda_X)' = \Lambda_{X'}.$$

*Proof.* First, by Definition 2.3, (2.1), (2.3), and the Hölder inequality, we get for every $f \in \Lambda_{X'}$

$$\|f\|_{(\Lambda_X)'} = \sup_{g \in \mathfrak{M}(\mathcal{R},\mu)} \frac{\int_0^a f_\mu^*(t) g_\mu^*(t)\,dt}{\|g_\mu^*\|_X} \le \|f_\mu^*\|_{X'} = \|f\|_{\Lambda_{X'}}.$$

Note that for this inequality the assumption (3.4) is not needed.

As for the converse, let $f \in (\Lambda_X)'$. Then, by (3.4), Fubini's theorem and (2.4),

$$\begin{aligned}
\|f\|_{(\Lambda_X)'} &= \sup_{g \in \mathfrak{M}(\mathcal{R},\mu)} \frac{\int_0^a f_\mu^*(t) g_\mu^*(t)\,dt}{\|g_\mu^*\|_X} \ge \sup_{h \in \mathfrak{M}^+(0,a)} \frac{\int_0^a \left(\int_t^a h(s)\,ds\right) f_\mu^*(t)\,dt}{\|\int_t^a h(s)\,ds\|_X} \\
&\gtrsim \sup_{h \in \mathfrak{M}^+(0,a)} \frac{\int_0^a h(s) \left(\int_0^s f_\mu^*(t)\,dt\right)\,ds}{\|t h(t)\|_X} \ge \left\| \frac{1}{t} \int_0^t f_\mu^*(s)\,ds \right\|_{X'} \\
&\gtrsim \|f_\mu^*\|_{X'} = \|f\|_{\Lambda_{X'}}.
\end{aligned}$$

□

Another characterization of the associate space of $\Lambda_X$ (under assumptions different than (3.4)) will be given in the next section.

## 4. Duality principles

We now present our main results. First, we shall prove the following general duality principle.

**Theorem 4.1.** *Let $a \in (0,\infty]$, let $w$ be a weight on $(0,a)$ and let $X$ be a Banach function space over $((0,a),w)$. Assume that*

$$1 < i_X \le I_X < \infty. \tag{4.1}$$

*Let $g \in \mathfrak{M}^+(0,a)$. Then*

$$\begin{aligned}
\sup_{f \in \mathfrak{M}^+(0,a;\downarrow)} \frac{\int_0^a f(t) g(t)\,dt}{\|f\|_X} &\approx \left\| \int_{W^{-1}(t)}^a \frac{g(s)}{W(s)}\,ds \right\|_{\overline{X}'} \\
&\approx \left\| \frac{1}{t} \int_0^{W^{-1}(t)} g(s)\,ds \right\|_{\overline{X}'} + \frac{\int_0^a g(s)\,ds}{\|\chi_{(0,W(a))}\|_{\overline{X}}},
\end{aligned} \tag{4.2}$$

*where the last term is taken as zero when $W(a) = \infty$.*

*Proof.* Fix a function $g \in \mathfrak{M}^+(0,a)$. First, using (2.4), Fubini's theorem, a change of variables, Hölder's inequality and (3.1), we observe that, for $f \in \mathfrak{M}^+(0,a;\downarrow)$ and $g \in \mathfrak{M}^+(0,a)$,

$$\begin{aligned}
\int_0^a f(t) g(t)\,dt &\le \int_0^a \frac{1}{W(t)} \left( \int_0^t f(s) w(s)\,ds \right) g(t)\,dt \\
&= \int_0^a f(t) w(t) \left( \int_t^a \frac{g(s)}{W(s)}\,ds \right) dt \\
&= \int_0^{W(a)} f(W^{-1}(t)) \left( \int_{W^{-1}(t)}^a \frac{g(s)}{W(s)}\,ds \right) dt \\
&\le \|f(W^{-1})\|_{\overline{X}} \left\| \int_{W^{-1}(t)}^a \frac{g(s)}{W(s)}\,ds \right\|_{\overline{X}'} = \|f\|_X \left\| \int_{W^{-1}(t)}^a \frac{g(s)}{W(s)}\,ds \right\|_{\overline{X}'},
\end{aligned}$$

and therefore

$$\sup_{f\in\mathfrak{M}^+(0,a;\downarrow)}\frac{\int_0^a f(t)g(t)\,dt}{\|f\|_X}\le\Big\|\int_{W^{-1}(t)}^a\frac{g(s)}{W(s)}\,ds\Big\|_{\overline{X}'}. \tag{4.3}$$

Now, for $t\in(0,a)$, we denote $G(t)=\int_0^t g(s)\,ds$. Assume first that $W(a)<\infty$. Then, integrating by parts, using a change of variables and (2.2), we get

$$\begin{aligned}\int_{W^{-1}(t)}^a\frac{g(s)}{W(s)}\,ds&=\int_{W^{-1}(t)}^a\frac{dG(s)}{W(s)}=\frac{G}{W}\bigg|_{W^{-1}(t)}^a+\int_{W^{-1}(t)}^a\frac{G(s)}{W^2(s)}w(s)\,ds\\&\le\frac{G(a)}{W(a)}+\int_t^{W(a)}\frac{G(W^{-1}(s))}{s^2}ds\\&=\frac{G(a)}{\|\chi_{(0,W(a))}\|_{\overline{X}}\|\chi_{(0,W(a))}\|_{\overline{X}'}}+\int_t^{W(a)}\frac{G(W^{-1}(s))}{s^2}ds.\end{aligned} \tag{4.4}$$

By (4.1) and [2, Chapter 3, Theorem 5.15], the operator $\bar{A}$ is bounded on $\overline{X}'$. Thus,

$$\begin{aligned}\Big\|\int_{W^{-1}(t)}^a\frac{g(s)}{W(s)}\,ds\Big\|_{\overline{X}'}&\le\frac{G(a)}{\|\chi_{(0,W(a))}\|_{\overline{X}}}+\Big\|\int_t^{W(a)}\frac{G(W^{-1}(s))}{s^2}\,ds\Big\|_{\overline{X}'}\\&\lesssim\frac{G(a)}{\|\chi_{(0,W(a))}\|_{\overline{X}}}+\Big\|\frac{G(W^{-1}(t))}{t}\Big\|_{\overline{X}'}.\end{aligned} \tag{4.5}$$

When $W(a)=\infty$, we fix $\xi\in(0,a)$, replace $g$ by $g\chi_{(0,\xi)}$ and (accordingly) take $G(t)=\int_0^t g(s)\chi_{(0,\xi)}(s)\,ds$. Then we obviously have $\frac{G(a)}{W(a)}=0$ whence, on letting $\xi\to a_-$, we obtain, in place of (4.4),

$$\int_{W^{-1}(t)}^a\frac{g(s)}{W(s)}\,ds\le\int_t^{W(a)}\frac{G(W^{-1}(s))}{s^2}ds.$$

Thus, instead of (4.5) we get

$$\Big\|\int_{W^{-1}(t)}^a\frac{g(s)}{W(s)}\,ds\Big\|_{\overline{X}'}\lesssim\Big\|\frac{G(W^{-1}(t))}{t}\Big\|_{\overline{X}'}. \tag{4.6}$$

Next, note that for any function $h\in\mathfrak{M}^+(0,a)$ we have, by Lemma 3.1, (4.1) and [2, Chapter 2, Theorem 5.15],

$$\begin{aligned}\Big\|\int_t^a h(s)\,ds\Big\|_X&=\Big\|\int_{W^{-1}(t)}^a h(s)\,ds\Big\|_{\overline{X}}\\&=\Big\|\int_t^{W(a)}\frac{h(W^{-1}(s))}{w(W^{-1}(s))}\,ds\Big\|_{\overline{X}}\lesssim\Big\|\frac{h(W^{-1}(t))}{w(W^{-1}(t))}t\Big\|_{\overline{X}}.\end{aligned}$$

Therefore,

$$
\begin{aligned}
(4.7)\qquad \sup_{f\in\mathfrak{M}^+(0,a;\downarrow)} \frac{\int_0^a f(t)g(t)\,dt}{\|f\|_X} &\geq \sup_{h\in\mathfrak{M}^+(0,a)} \frac{\int_0^a \left(\int_t^a h(s)\,ds\right) g(t)\,dt}{\|\int_t^a h(s)\,ds\|_X} \\
&\gtrsim \sup_{h\in\mathfrak{M}^+(0,a)} \frac{\int_0^a h(s)G(s)\,ds}{\|\frac{h(W^{-1}(t))t}{w(W^{-1}(t))}\|_{\overline{X}}} \\
&= \sup_{h\in\mathfrak{M}^+(0,a)} \frac{\int_0^{W(a)} \frac{h(W^{-1}(s))G(W^{-1}(s))}{w(W^{-1}(s))}\,ds}{\|\frac{h(W^{-1}(t))t}{w(W^{-1}(t))}\|_{\overline{X}}} \\
&= \left\|\frac{G(W^{-1}(t))}{t}\right\|_{\overline{X}'}.
\end{aligned}
$$

When $W(a)<\infty$, we moreover have

$$
(4.8)\qquad \sup_{f\in\mathfrak{M}^+(0,a;\downarrow)} \frac{\int_0^a f(s)g(s)\,ds}{\|f\|_X} \geq \frac{\int_0^a g(t)\,dt}{\|\chi_{(0,a)}\|_X} = \frac{\int_0^a g(t)\,dt}{\|\chi_{(0,W(a))}\|_{\overline{X}}}.
$$

Combining (4.3), (4.5), (4.6), (4.7) and (4.8) we get (4.2). □

**Remark 4.2.** When $X=L^p$, $1<p<\infty$, the preceding theorem yields part of the result of [18, Theorem 1].

The first important corollary of Theorem 4.1 is a characterization of the associate space of $\Lambda_X$.

**Theorem 4.3.** *Let $(\mathcal{R},\mu)$ be a totally $\sigma$–finite non-atomic measure space with $a=\mu(\mathcal{R})$. Let $w$ be a weight on $(0,a)$ and let $X$ be an r.i. space over $((0,a),w)$. Assume further that (4.1) holds. Let $g\in\mathfrak{M}^+(\mathcal{R},\mu)$. Then*

$$
\begin{aligned}
\sup_{f\in\mathfrak{M}(\mathcal{R},\mu)} \frac{\int_{\mathcal{R}} f(x)g(x)\,d\mu(x)}{\|f\|_{\Lambda_X}} &\approx \left\|\int_{W^{-1}(t)}^a \frac{g_\mu^*(s)}{W(s)}\,ds\right\|_{\overline{X}'} \\
&\approx \left\|\frac{1}{t}\int_0^{W^{-1}(t)} g_\mu^*(s)\,ds\right\|_{\overline{X}'} + \frac{\int_0^a g_\mu^*(s)\,ds}{\|\chi_{(0,W(a))}\|_{\overline{X}}},
\end{aligned}
$$

*where the last term is taken as zero when $W(a)=\infty$.*

*Proof.* By (2.1), (2.3) and [2, Chapter 2, Corollary 7.8]

$$
\begin{aligned}
\sup_{f\in\mathfrak{M}(\mathcal{R},\mu)} \frac{\int_{\mathcal{R}} f(x)g(x)\,d\mu(x)}{\|f\|_{\Lambda_X}} &= \sup_{f\in\mathfrak{M}(\mathcal{R},\mu)} \frac{\int_0^a f_\mu^*(t)g_\mu^*(t)\,dt}{\|f_\mu^*\|_X} \\
&= \sup_{h\in\mathfrak{M}^+(0,a;\downarrow)} \frac{\int_0^a h(t)g_\mu^*(t)\,dt}{\|h\|_X},
\end{aligned}
$$

and the result follows from Theorem 4.1. □

Theorem 4.1 can be used to obtain the following reduction theorem concerning linear positive operators.

**Theorem 4.4.** *Let $a\in(0,\infty]$. Let $v,w$ be weights on $(0,a)$. Let $X$ be an r.i. space over $((0,a),w)$ and let $Y$ be an r.i. space over $((0,a),v)$. Let $T$ be a positive linear operator acting on measurable functions defined on $(0,a)$. Let $T^*$ be the* *dual operator of $T$ with respect to the pairing $\int_0^a f(t)g(t)\,dt$. Then the following statements are equivalent.*

(i)
$$\|Tf\|_Y \lesssim \|f\|_X \quad \text{for all } f \in \mathfrak{M}^+(0,a;\downarrow);$$

(ii)
$$\Big\| \int_{W^{-1}(t)}^a \frac{(T^*(gv))(s)}{W(s)}\,ds \Big\|_{\overline{X}'} \lesssim \|g\|_{Y'} \quad \text{for all } g \in \mathfrak{M}^+(0,a);$$

(iii)
$$\Big\| \frac{1}{t}\int_0^{W^{-1}(t)} (T^*(gv))(s)\,ds \Big\|_{\overline{X}'} + \frac{\int_0^a (T^*(gv))(s)ds}{\|\chi_{(0,W(a))}\|_{\overline{X}}} \lesssim \|g\|_{Y'}.$$

*for all $g \in \mathfrak{M}^+(0,a)$.*

*Proof.* First, note that

$$\begin{aligned} \sup_{f\in\mathfrak{M}^+(0,a;\downarrow)} \frac{\|Tf\|_Y}{\|f\|_X} &= \sup_{f\in\mathfrak{M}^+(0,a;\downarrow)} \sup_{g\in\mathfrak{M}^+(0,a)} \frac{\int_0^a (Tf)(s)g(s)v(s)\,ds}{\|f\|_X\|g\|_{Y'}} \\ &= \sup_{g\in\mathfrak{M}^+(0,a)} \frac{1}{\|g\|_{Y'}} \sup_{f\in\mathfrak{M}^+(0,a;\downarrow)} \frac{\int_0^a f(s)\,(T^*(gv))(s)\,ds}{\|f\|_X}, \end{aligned}$$

and the assertion follows from Theorem 4.1. □

In a similar vein we can obtain the following theorem which is of independent interest.

**Theorem 4.5.** *Let $a, w, v, X, Y$ and $T$ be as in* Theorem 4.4. *Then the following statements are equivalent.*

(i)
$$\|(Tf)(V^{-1})\|_{\overline{Y}} \lesssim \|f\|_X \quad \text{for all } f \in \mathfrak{M}^+(0,a;\downarrow);$$

(ii)
$$\Big\| \int_{W^{-1}(t)}^a \frac{(T^*(h(V)v))(s)}{W(s)}\,ds \Big\|_{\overline{X}'} \lesssim \|h\|_{\overline{Y}'} \quad \text{for all } h \in \mathfrak{M}^+(0,a);$$

(iii)
$$\Big\| \frac{1}{t}\int_0^{W^{-1}(t)} (T^*(h(V)v))(s)\,ds \Big\|_{\overline{X}'} + \frac{\int_0^a (T^*(h(V)v))(s)ds}{\|\chi_{(0,W(a))}\|_{\overline{X}}} \lesssim \|h\|_{\overline{Y}'}$$

*for every $h \in \mathfrak{M}^+(0,V(a))$.*

*Proof.* As in the proof of the preceding theorem,

$$\begin{aligned} \sup_{f\in\mathfrak{M}^+(0,a;\downarrow)} \frac{\|(Tf)(V^{-1})\|_{\overline{Y}}}{\|f\|_X} &= \sup_{f\in\mathfrak{M}^+(0,a;\downarrow)} \sup_{h\in\mathfrak{M}^+(0,V(a))} \frac{\int_0^{V(a)} (Tf)(V^{-1}(s))h(s)\,ds}{\|f\|_X\|h\|_{\overline{Y}'}} \\ &= \sup_{h\in\mathfrak{M}^+(0,V(a))} \frac{1}{\|h\|_{\overline{Y}'}} \sup_{f\in\mathfrak{M}^+(0,a;\downarrow)} \frac{\int_0^a f(s)\,(T^*(h(V)v))(s)\,ds}{\|f\|_X}, \end{aligned}$$

and the assertion follows from Theorem 4.1 again. □

**Theorem 4.6.** *Let $a, w, v, X, Y$ and $T$ be as in* Theorem 4.4. *Then the estimate*

$$\|(Tf)(V^{-1})\|_{\overline{Y}} \lesssim \|f\|_X \tag{4.9}$$

*holds for all $f \in \mathfrak{M}^+(0,a;\downarrow)$ if and only if both of the conditions*

$$\Big\| \Big[ T\Big( \int_{W(\cdot)}^{W(a)} \frac{h(s)}{s}\,ds \Big) \Big] \big(V^{-1}(t)\big) \Big\|_{\overline{Y}} \lesssim \|h\|_{\overline{X}} \quad \text{for all } h \in \mathfrak{M}^+(0,W(a)),$$

*and*

$$\|(T\chi_{(0,\xi)})(V^{-1}(t))\|_{\overline{Y}} \lesssim \|\chi_{(0,W(\xi))}\|_{\overline{X}} \quad \text{for all } \xi \in (0,a),$$

*are satisfied.*

*Proof.* By the Fubini theorem, we get for $g \in \mathfrak{M}^+(0,W(a))$ and $h \in \mathfrak{M}^+(0,V(a))$,

$$\begin{aligned}
\int_0^{W(a)} \frac{g(t)}{t} &\left( \int_0^{W^{-1}(t)} \left(T^*(h(V)v)\right)(s)\,ds \right) dt \\
&= \int_0^a \left(T^*(h(V)v)\right)(s) \left( \int_{W(s)}^{W(a)} \frac{g(t)}{t}\,dt \right) ds \\
&= \int_0^a h(V(s))v(s) \left[ T\left( \int_{W(\cdot)}^{W(a)} \frac{g(t)}{t}\,dt \right) \right](s)\,ds \\
&= \int_0^{V(a)} h(s) \left[ T\left( \int_{W(\cdot)}^{W(a)} \frac{g(t)}{t}\,dt \right) \right](V^{-1}(s))\,ds.
\end{aligned}$$

Moreover, by a change of variables, we get for each fixed $\xi \in (0,a)$,

$$\begin{aligned}
\int_0^{\xi} \left(T^*(h(V)v)\right)(s)\,ds &= \int_0^{\xi} h(V(s))v(s) \left(T\chi_{(0,\xi)}\right)(s)\,ds \\
&= \int_0^{V(\xi)} h(s) \left(T\chi_{(0,\xi)}\right)(V^{-1}(s))\,ds.
\end{aligned}$$

Thus, the assertion follows from Theorem 4.5. □

Our aim now is to apply the results obtained to a kernel operator of Volterra type.

**Corollary 4.7.** *Let $a, w, v, X$ and $Y$ be as in* Theorem 4.4. *Let $\varrho$ and $\sigma$ be two weights on $(0,a)$. Let $k(t,s)$ be a measurable positive function on $(0,a)\times(0,a)$. Let $T$ be the Volterra operator given as*

$$(Tf)(t) = \varrho(t)\int_0^t k(t,s)\sigma(s)f(s)\,ds, \quad f \in \mathfrak{M}^+(0,a),\ t \in (0,a). \tag{4.10}$$

*Then (4.9) holds for all $f \in \mathfrak{M}^+(0,a;\downarrow)$ if and only if all of the following three conditions are satisfied for all $h \in \mathfrak{M}^+(0,W(a))$ and $\xi \in (0,a)$:*

$$\left\| \varrho(V^{-1}(t)) \left( \int_0^{V^{-1}(t)} k(V^{-1}(t),y)\sigma(y)\,dy \right) \left( \int_{W(V^{-1}(t))}^{W(a)} \frac{h(s)}{s}\,ds \right) \right\|_{\overline{Y}} \lesssim \|h\|_{\overline{X}};$$

$$\left\| \varrho(V^{-1}(t)) \int_0^{W(V^{-1}(t))} \frac{h(s)}{s} \left( \int_0^{W^{-1}(s)} k(V^{-1}(t),y)\sigma(y)\,dy \right) ds \right\|_{\overline{Y}} \lesssim \|h\|_{\overline{X}};$$

$$\left\| \varrho(V^{-1}(t)) \int_0^{V^{-1}(t)} k(V^{-1}(t),s)\chi_{(0,\xi)}(s)\sigma(s)\,ds \right\|_{\overline{Y}} \lesssim \|\chi_{(0,W(\xi))}\|_{\overline{X}}.$$

*If, moreover, the kernel $k$ satisfies*

$$k(\xi,\eta) \approx k(\xi,\theta) + k(\theta,\eta) \quad \text{whenever} \quad 0<\eta<\theta<\xi<a, \tag{4.11}$$

*then the second of the above three conditions can be replaced by the two estimates*

$$\left\|\varrho(V^{-1}(t))\int_0^{W(V^{-1}(t))}\frac{h(s)}{s}k(V^{-1}(t),W^{-1}(s))\left(\int_0^{W^{-1}(s)}\sigma(y)\,dy\right)ds\right\|_{\overline{Y}}\lesssim\|h\|_{\overline{X}}$$

*and*

$$\left\|\varrho(V^{-1}(t))\int_0^{W(V^{-1}(t))}\frac{h(s)}{s}\left(\int_0^{W^{-1}(s)}k(W^{-1}(s),y)\sigma(y)\,dy\right)ds\right\|_{\overline{Y}}\lesssim\|h\|_{\overline{X}},$$

*satisfied for every* $h\in\mathfrak{M}^+(0,W(a))$.

*Proof.* We have

$$\begin{aligned}&\left[T\left(\int_{W(\cdot)}^{W(a)}\frac{h(y)}{y}\,dy\right)\right]\left(V^{-1}(t)\right)\\&\quad=\varrho(V^{-1}(t))\int_0^{V^{-1}(t)}k(V^{-1}(t),s)\sigma(s)\left(\int_{W(s)}^{W(a)}\frac{h(y)}{y}\,dy\right)ds.\end{aligned}$$

We now split the innermost integral,

$$\int_{W(s)}^{W(a)}=\int_{W(s)}^{W(V^{-1}(t))}+\int_{W(V^{-1}(t))}^{W(a)},$$

and use the Fubini theorem. The first part of the assertion now follows from Theorem 4.6. The second part is easily obtained on observing that, by (4.11),

$$k(V^{-1}(t),y)\approx k(V^{-1}(t),W^{-1}(s))+k(W^{-1}(s),y)$$

whenever $0<y<W^{-1}(s)<V^{-1}(t)$. $\square$

It would be desirable to characterize (4.9) by conditions which would not involve arbitrary functions. In a particular case when the pair $(X,Y)$ satisfies the so-called $L$–condition, this is possible using results of [10].

**Definition 4.8.** We say that a pair of Banach function spaces $(X,Y)$ over the same measure space $(\mathcal{R},\mu)$ satisfies the $L$*–condition* (we write $(X,Y)\in L$) if there exists a Banach sequence space $\ell$ with a standard basis $\{e_k\}_{k\in\mathbb{N}}$ and positive constants $C,c$ such that for every sequence of disjoint sets $\{E_k\}\subset\mathcal{R}$, $\bigcup_k E_k=\mathcal{R}$, we have

$$\left\|\sum_k e_k\|\chi_{E_k}f\|_X\right\|_\ell\le C\|f\|_X\quad\text{for all } f\in X$$

and

$$\left\|\sum_k e_k\|\chi_{E_k}f\|_Y\right\|_\ell\ge c\|f\|_Y\quad\text{for all } f\in Y.$$

The $L$–condition in the sense of our definition was (as far as we know) first introduced by Berezhnoi ([3, Definition 2]) as a generalization of the concept of $p$-convexity that had been used before (cf. [13, Part II, Chapter 1d]). Unfortunately, it is a very restrictive condition.

The following result is [10, Theorem 3.1].

**Theorem 4.9.** *Let $a \in (0,\infty]$. Let $X, Y$ be two Banach function spaces over an interval $(0,a)$ and assume that $(X,Y) \in L$. Let $\alpha, \beta$ be two non-decreasing functions on $(0,a)$ such that $0 \le \alpha(t) \le \beta(t) \le a$ for every $t \in (0,a)$. Let $\varrho, \sigma$ be two weights on $(0,a)$. Assume that $k(t,s)$ is a non-negative function defined a.e. on $\{(t,s);\, 0<t<a,\, \alpha(t) \le s \le \beta(t)\}$, non-decreasing in $t$, non-increasing in $s$, and such that*

$$k(t,s) \lesssim (k(t,\beta(y)) + k(y,s)) \quad \text{whenever } y \le t \text{ and } \alpha(t) \le s \le \beta(y). \tag{4.12}$$

(*Such $k$ we shall call an* admissible kernel.) *Let $T$ be a kernel operator given by*

$$(Tf)(t) = \varrho(t) \int_{\alpha(t)}^{\beta(t)} k(t,s)\sigma(s) f(s)\, ds, \quad f \in \mathfrak{M}^+(0,a).$$

*Then $T$ is bounded from $X$ into $Y$ if and only if both of the conditions*

$$\sup_{t \le s, \alpha(s) \le \beta(t)} \|\chi_{(t,s)}(y)\varrho(y)k(y,\beta(t))\|_Y \|\chi_{(\alpha(s),\beta(t))}(y)\sigma(y)\|_{X'} < \infty,$$

$$\sup_{t \le s, \alpha(s) \le \beta(t)} \|\chi_{(t,s)}(y)\varrho(y)\|_Y \|\chi_{(\alpha(s),\beta(t))}(y)k(t,y)\sigma(y)\|_{X'} < \infty$$

*are satisfied.*

Combining Theorem 4.9 with Corollary 4.7, we get the following result.

**Theorem 4.10.** *Let $a, w, v, X$ and $Y$ be as in* Theorem 4.4 *and let moreover $(X,Y) \in L$. Let $k$ be an admissible kernel and let $T$ be the Volterra operator given by* (4.10). *Then* (4.9) *holds if and only if all of the following five conditions are satisfied*:

$$\sup_{0<x<V(a)} \left\| \chi_{(0,x)}(t)\varrho(V^{-1}(t)) \int_0^{V^{-1}(t)} k(V^{-1}(t),s)\sigma(s)\, ds \right\|_{\overline{Y}} \times \left\| \chi_{(W(V^{-1}(x)),W(a))}(t) \frac{1}{t} \right\|_{\overline{X}'} < \infty,$$

$$\sup_{0<x<V(a)} \left\| \chi_{(x,V(a))}(t)\varrho(V^{-1}(t))k\left(V^{-1}(t),V^{-1}(x)\right) \right\|_{\overline{Y}} \times \left\| \chi_{(0,W(V^{-1}(x)))}(t) \frac{1}{t} \int_0^{W^{-1}(t)} \sigma(s)\, ds \right\|_{\overline{X}'} < \infty,$$

$$\sup_{0<x<V(a)} \left\| \chi_{(x,V(a))}(t)\varrho(V^{-1}(t)) \right\|_{\overline{Y}} \times \left\| \chi_{(0,W(V^{-1}(x)))}(t) k\left(V^{-1}(x),W^{-1}(t)\right) \frac{1}{t} \int_0^{W^{-1}(t)} \sigma(s)\, ds \right\|_{\overline{X}'} < \infty,$$

$$\sup_{0<x<V(a)} \left\| \chi_{(x,V(a))}(t)\varrho(V^{-1}(t)) \right\|_{\overline{Y}} \times \left\| \chi_{(0,W(V^{-1}(x)))}(t) \frac{1}{t} \int_0^{W^{-1}(t)} k(W^{-1}(t),s)\sigma(s)\, ds \right\|_{\overline{X}'} < \infty,$$

$$\left\| \varrho(V^{-1}(t)) \int_0^{V^{-1}(t)} k(V^{-1}(t),s)\chi_{(0,\xi)}(s)\sigma(s)\, ds \right\|_{\overline{Y}} \lesssim \|\chi_{(0,W(\xi))}\|_{\overline{X}}$$

*for every* $\xi \in (0,a)$.

In particular, for the two-weight Hardy operator

$$(Tf)(t) = \varrho(t) \int_0^t \sigma(s) f(s)\, ds, \quad f \in \mathfrak{M}^+(0,a), \tag{4.13}$$

where $\varrho, \sigma$ are weights on $(0,a)$, we have the following result.

**Corollary 4.11.** *Let* $a, w, v, X$ *and* $Y$ *be as in* Theorem 4.4 *and let moreover* $(X,Y) \in L$. *Let* $T$ *be the two-weight Hardy operator from* (4.13). *Then* (4.9) *holds if and only if all of the following three conditions are satisfied*:

$$\sup_{0<x<V(a)} \left\| \chi_{(0,x)}(t) \varrho(V^{-1}(t)) \int_0^{V^{-1}(t)} \sigma(s)\, ds \right\|_{\overline{Y}} \left\| \chi_{(W(V^{-1}(x)),W(a))}(t) \frac{1}{t} \right\|_{\overline{X}'} < \infty;$$

$$\sup_{0<x<V(a)} \left\| \chi_{(x,V(a))}(t) \varrho(V^{-1}(t)) \right\|_{\overline{Y}} \left\| \chi_{(0,W(V^{-1}(x)))}(t) \frac{1}{t} \int_0^{W^{-1}(t)} \sigma(s)\, ds \right\|_{\overline{X}'} < \infty;$$

$$\left\| \varrho(V^{-1}(t)) \int_0^{V^{-1}(t)} \sigma(s) \chi_{(0,\xi)}(s)\, ds \right\|_{\overline{Y}} \lesssim \| \chi_{(0,W(\xi))} \|_{\overline{X}}$$

*for every* $\xi \in (0,a)$.

## 5. Embedding theorems for $\Lambda_X$ spaces

Theorem 4.6 will now be applied to characterize embeddings of type $\Lambda_X \hookrightarrow \Lambda_Y$.

**Theorem 5.1.** *Let* $a \in (0,\infty]$, *let* $v, w$ *be two weights on* $(0,a)$, *let* $X$ *be an r.i. space over* $((0,a),w)$ *and let* $Y$ *be an r.i. space over* $((0,a),v)$. *Then the inequality*

$$\|f\|_Y \lesssim \|f\|_X$$

*holds for all* $f \in \mathfrak{M}^+(0,a;\downarrow)$ *if and only if both of the conditions*

$$\left\| \int_{W(V^{-1}(t))}^{W(a)} \frac{h(s)}{s}\, ds \right\|_{\overline{Y}} \lesssim \|h\|_{\overline{X}} \quad \text{for all } h \in \mathfrak{M}^+(0,W(a)), \tag{5.1}$$

*and*

$$\|\chi_{(0,V(\xi))}\|_{\overline{Y}} \lesssim \|\chi_{(0,W(\xi))}\|_{\overline{X}}, \quad \text{for all } \xi \in (0,a), \tag{5.2}$$

*are satisfied.*

*Proof.* We just apply Theorem 4.6 to the case when $T$ is the identity operator. $\square$

**Corollary 5.2.** *Let* $a, v, w, X$ *and* $Y$ *be as in* Theorem 5.1. *Then the embedding*

$$\Lambda_X \hookrightarrow \Lambda_Y \tag{5.3}$$

*holds if and only if both of the conditions* (5.1) *and* (5.2) *are satisfied.*

**Corollary 5.3.** *Let* $a, v, w, X$ *and* $Y$ *be as in* Theorem 5.1 *and assume moreover that* $(X,Y) \in L$. *Then* (5.3) *holds if and only if both of the conditions*

$$\sup_{0<x\le V(a)} \|\chi_{(0,x)}\|_{\overline{Y}} \left\| \chi_{(W(V^{-1}(x)),W(a))}(t) \frac{1}{t} \right\|_{\overline{X}'} < \infty$$

*and* (5.2) *are satisfied.*

## 6. A reduction theorem

Our final main goal is to prove a reduction theorem for the inequality

$$\left\| T\left(\int_0^t f(y)\,dy\right)\right\|_Y \lesssim \|f\|_X, \tag{6.1}$$

where $f \in \mathfrak{M}^+(0,a;\downarrow)$, $a \in (0,\infty]$, and the operator $T$ is not necessarily linear but instead it is supposed to be *monotone* (that is, if $Tf \le Tg$ on $(0,a)$ whenever $0 \le f \le g$ on $(0,a)$.)

Examples of such operators are the identity operator $Tf := f$ or, more importantly, the operator

$$(Tg)(t) := \sup_{s\ge t} v(s)g(s),$$

where $v$ is a weight on $(0,a)$. Applied to this operator, the reduction theorem can be used to obtain information about the behaviour of the fractional maximal operator. For details see [11] and [5].

We shall give a direct proof which does not rely on duality results. On the other hand, a restriction on Boyd indices has to be assumed.

**Theorem 6.1.** *Let $a \in (0,\infty]$. Let $T$ be a positive quasilinear monotone operator. Assume that $X$ is an r.i. space over $((0,a),w)$ and let $Y$ be a Banach lattice over $(0,a)$. Suppose that*

$$1 < i_X \le I_X < \infty. \tag{6.2}$$

*Then the inequality* (6.1) *holds for all $f \in \mathfrak{M}^+(0,a;\downarrow)$ if and only if both of the inequalities*

$$\|T(tf(t))\|_Y \lesssim \|f\|_X \quad \text{for all } f \in \mathfrak{M}^+(0,a;\downarrow), \tag{6.3}$$

*and*

$$\left\| T\left(\int_0^t g(y)\,dy\right)\right\|_Y \lesssim \left\| g(s)\frac{W(s)}{sw(s)}\right\|_X \quad \text{for all } g \in \mathfrak{M}^+(0,a) \tag{6.4}$$

*are satisfied.*

*Proof.* We begin with proving the necessity of (6.3) and (6.4). First, for $f \in \mathfrak{M}^+(0,a;\downarrow)$ we obviously have

$$\int_0^t f(y)\,dy \ge tf(t), \quad t \in (0,a),$$

hence (6.3) immediately follows from (6.1) and the monotonicity of $T$. Let $g \in \mathfrak{M}^+(0,a)$ and set $h(t) = \int_t^a s^{-1}g(s)\,ds$. Then for $t \in (0,a)$,

$$\int_0^t g(s)\,ds = \int_0^t s^{-1}g(s)\left(\int_0^s dy\right)\,ds = \int_0^t \left(\int_y^t s^{-1}g(s)\,ds\right)\,dy \le \int_0^t h(s)\,ds.$$

Now, $h$ is obviously non-increasing, whence, by monotonicity of $T$, (6.1) and (3.3),

$$\begin{aligned}\left\| T\left(\int_0^t g(y)\,dy\right)\right\|_Y &\le \left\| T\left(\int_0^t h(y)\,dy\right)\right\|_Y \lesssim \|h\|_X \\ &= \left\| \int_t^a y^{-1}g(y)\,dy\right\|_X \lesssim \left\| g(t)\frac{W(t)}{tw(t)}\right\|_X,\end{aligned}$$

proving the necessity of (6.4).

To show the sufficiency part of the theorem, we can restrict ourselves to those $f$ which can be expressed as

$$f(t) = K + \int_t^a h(s)\,ds = f_1(t) + f_2(t), \quad h \in \mathfrak{M}^+(0,a),\ t \in (0,a),$$

where $K > 0$. First, for the constant function $f_1$ we have $\int_0^t f_1(y)\,dy = tf_1(t)$, hence, by (6.3), the inequality (6.1) is satisfied for $f = f_1$. Since the operator $T$ is quasilinear, it is enough to verify (6.1) also for $f_2$. We have

$$\begin{aligned}\int_0^t f_2(s)\,ds &= \int_0^t \left(\int_y^a h(s)\,ds\right)\,dy = \int_0^t \left(\int_y^t h(s)\,ds\right)\,dy + t\int_t^a h(s)\,ds\\ &= \int_0^t sh(s)\,ds + tf_2(t).\end{aligned}$$

By (6.3),

$$\|T(tf_2(t))\|_Y \lesssim \|f_2\|_X,$$

and therefore it will suffice to show

$$\left\| T\left(\int_0^s h(s)\,ds\right)\right\|_Y \lesssim \|f\|_X. \tag{6.5}$$

Next,

$$\begin{aligned}\int_0^t sh(s)\,ds &= \frac{1}{W(t)}\int_0^t sh(s)W(s)\,ds + \int_0^t sh(s)W(s)\left(\int_s^t \frac{w(y)}{W^2(y)}\,dy\right)\,ds\\ &\le \frac{t}{W(t)}\int_0^t h(s)W(s)\,ds + \int_0^t h(s)W(s)\left(\int_s^t \frac{y}{W^2(y)}\,w(y)\,dy\right)\,ds\\ &= \frac{t}{W(t)}\int_0^t \left(\int_y^t h(s)\,ds\right)w(y)\,dy + \int_0^t \frac{y}{W^2(y)}\left(\int_0^y h(s)W(s)\,ds\right)\,w(y)\,dy\\ &= \frac{t}{W(t)}\int_0^t f(y)w(y)\,dy + \int_0^t \frac{y}{W^2(y)}\left(\int_0^y\left(\int_z^y h(s)\,ds\right)w(z)\,dz\right)\,w(y)\,dy\\ &= \frac{t}{W(t)}\int_0^t f(y)w(y)\,dy + \int_0^t \frac{y}{W^2(y)}\left(\int_0^y f(z)w(z)\,dz\right)w(y)\,dy.\end{aligned}$$

By (6.3) (note that $A_w f$ is non-increasing, as mentioned in Remark 2.7), and (3.2),

$$\left\| T\left(\frac{t}{W(t)}\int_0^t f(s)w(s)\,ds\right)\right\|_Y \lesssim \|A_w f\|_X \lesssim \|f\|_X,$$

and, by (6.4) and (3.2),

$$\left\| T\left(\int_0^t \frac{y}{W^2(y)}\left(\int_0^y f(s)w(s)\,ds\right)\,w(y)\,dy\right)\right\|_Y \lesssim \|A_w f\|_X \lesssim \|f\|_X.$$

The last two estimates yield (6.5). The proof is complete. $\square$

Theorem 6.1 can be applied to obtain an important characterization of weights for which the operator from (1.2) is bounded from one weighted Lebesgue space into another. Details can be found in [11].

We now present a generalization of Theorem 6.1 to operators involving kernels. The proof uses the same ideas and is therefore omitted.

**Theorem 6.2.** *Let $a \in (0,\infty]$. Let $T$ be a positive quasilinear monotone operator. Let $k(t,s)$ be a non-negative measurable function on non-increasing in $s$ and non-decreasing in $t$, and satisfying*

$$k(t,s) \approx k(t,y) + k(y,s) \quad \textit{for every} \quad 0 < s \le y \le t < a.$$

*Assume that $X$ is an r.i. space on $(0,a)$ and let $Y$ be a Banach lattice over $(0,a)$. Suppose that* (6.2) *is satisfied. Then the inequality*

$$\left\| T\left( \int_0^t k(t,s) f(s)\, ds \right) \right\|_Y \lesssim \|f\|_X, \quad f \in \mathfrak{M}^+(0,a;\downarrow),$$

*holds if and only if all of the inequalities*

$$\left\| T\left( f(t) \int_0^t k(t,s)\, ds \right) \right\|_Y \lesssim \|f\|_X, \quad f \in \mathfrak{M}^+(0,a;\downarrow),$$

$$\left\| T\left( \int_0^t \frac{k(t,s) s w(s)}{W(s)} g(s)\, ds \right) \right\|_Y \lesssim \|g\|_X, \quad g \in \mathfrak{M}^+(0,a),$$

*and*

$$\left\| T\left( \int_0^t \frac{w(s)\left(\int_0^s k(s,y)\, dy\right)}{W(s)} g(s)\, ds \right) \right\|_Y \lesssim \|g\|_X, \quad g \in \mathfrak{M}^+(0,a),$$

*are satisfied.*

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