ARSENIN – KUNUGUI THEOREM AND WEAK FORMS OF BOREL BIMEASURABILITY

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Abstract. Let f be a Borel measurable mapping of a Luzin (i.e. absolute Borel) space L onto a metric space M such that $f(F)$ is a Borel subset of M if F is closed in L. We show that then M is also Luzin, that $f^{-1}(y)$ is a K_σ set for all, except for countably many, $y \in M$, and that the Borel classes of the sets $f(F)$, F closed in L, are bounded by a countable cardinal. It gives a counterpart to the classical theorem of Arsenin and Kunugui that enables to state it in the form of an equivalence. As a particular case we get a theorem of Taimanov saying that the image of a Luzin space by a closed continuous mapping is a Luzin space. The method is based on a straightforward construction that gives a Hurewicz theorem and on the use of the Jankov – von Neumann selection theorem.

A classical theorem of Novikov says that a Borel measurable mapping $f : L \to Y$ of a Borel subset L of a Polish space X to a Polish space Y is *Borel bimeasurable* (i.e. $f(B)$ is Borel for every Borel subset B of L) if $f^{-1}(y)$ is at most countable for each $y \in Y$. Let us remark that moreover the graph of f is the countable union of graphs of one-to-one Borel measurable mappings.

R.Purves showed in [8] that a Borel measurable $f : L \to Y$, with L and Y as above, is Borel bimeasurable if and only if the set $\{y \in Y; f^{-1}(y) \text{ is uncountable}\}$ is at most countable. We give in Remark 3 below a slightly stronger formulation of this theorem.

Classical Arsenin – Kunugui theorem says that the projection to Y of a Borel subset B of the product $X \times Y$ of two Polish spaces is a Borel subset of Y if the *sections of* B defined by $B^y = \{x \in X; (x, y) \in B\}$ are K_σ for all $y \in Y$. Considering the projection of the graph, it follows that, if $f : L \to Y$ is Borel measurable with L and Y as above, then the set $f(L)$ is Borel in Y, whenever $f^{-1}(y)$ is K_σ for each $y \in Y$. Let us remark that also this theorem has a strengthening analogical to the result mentioned above. J.Saint-Raymond showed in [9] that the graph of f may be decomposed to countably many graphs of Borel measurable mappings with compact fibers. Then the result of Arsenin and Kunugui follows from a well-known result of Novikov about projections of Borel sets with compact sections.

We are investigating in what sense the assumption on fibers in the Arsenin – Kunugui theorem is the best possible, and we get a result that has similar form as the mentioned

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theorem of Purves. First of all we notice that the condition of the Arsenin – Kunugui theorem concerning preimages of points may be weakened to saying that

$$\{y \in Y; f^{-1}(y) \text{ is not } K_\sigma\} \text{ is at most countable.} \tag{A}$$

Moreover, the assumption (A) is sufficient for the following property (a "weak form of Borel bimeasurability") of f:

The image $f(B)$ of every Borel subset B of L, for which $B \cap f^{-1}(y)$ is an F_σ subset of $f^{-1}(y)$, is a Borel subset of Y. In particular, a still weaker, and more natural, form of Borel bimeasurability holds. The set $f(F)$ is Borel in Y if F is closed in L.

We are going to prove that (A) is also necessary for the mentioned "weak forms of Borel bimeasurability" of f.

The main ideas go back to the proofs of Purves' theorem (see Mauldin's proof in [6], or rather a simplification in the book [10, Theorem 4.12.5] of S.M.Srivastava). The measure-theoretic argument of the preliminary version of this paper was replaced by a topological one. G.Debs suggested this change which makes the presentation simpler besides the fact that it is perhaps more natural.

We use a particular construction of a closed homeomorphic copy of the space of irrationals in an analytic space A that is not K_σ proving thus in particular a Hurewicz theorem (see Remark 1 below). Further essential step in our proof is the use of the uniformization theorem of Jankov and von Neumann.

We would like to comment the history of the preparation of this paper. The first author obtained first the equivalence of (2) and (3) in our main Theorem 4 and obtained the equivalence of these properties with (1) for continuous mappings (in fact, for Borel measurable mappings with F_σ fibres). The last essential step of the proof of (1) implies (3) in Theorem 4 was done by the second author and it is, in a simplified form, incorporated in our Lemma 6 and its application. It should be noted that the question for the implication (1) implies (3) was posed to us by our colleague O.Kalenda when discussing the results of the first author at our seminar at Charles University.

1. Some needed notions and facts.

Let us first recall a few of needed notions and results of the classical descriptive set theory. Each of these results and other well-known facts used later on may be found e.g. in [1], [2], [4], [10], etc. At results that may be less known we choose one of more possible references.

All spaces will be separable and metric without stating explicitly otherwise. A completely metrizable separable space is called a *Polish space*. Since the terminology is not unified, we recall also some standard notions in our terminology.

We say that S is a *Suslin subset of* X if it is the result of the Suslin (or A-) operation using closed, or equivalently Borel, sets.

A space L is called a *Luzin space* if there is a continuous (or equivalently Borel measurable) bijection $f : F \to L$, where F is a closed subset of the space $\mathbb{N}^{\mathbb{N}}$. It is well-known that L is Luzin if and only if it is a Borel subset of some (or all) Polish spaces in that it is embedded.

A space A is called *analytic* if it is empty or if there is a continuous surjection of $\mathbb{N}^{\mathbb{N}}$ onto A, or equivalently, if A is Suslin in some (or all) Polish spaces in which it is embedded.

Let us point out only those of many well-known results that will be most important for us. Every Suslin subset of a space X has the Baire property in X. Every uncountable analytic space contains a homeomorphic copy of the Cantor set.

We also need

Jankov – von Neumann selection theorem (see e.g. [10, Theorem 5.5.2]). *Let $A \subset X \times Y$ be an analytic subset of the product of two Polish spaces, X and Y. Let $p_Y(A)$ be the image of A by the projection p_Y of $X \times Y$ to Y. Then there is a mapping $f : p_Y(A) \to X$ such that $(f(y), y) \in A$ for $y \in p_Y(A)$ and the preimage of every open subset of X belongs to the smallest σ-algebra generated by analytic sets in Y. In particular, f has the Baire property (see* [4, Chapter 2, Section 32, I] *for the definition).*

The proof of the following auxiliary statement is not difficult and can be found e.g. in [1, II, 17, Theorem].

Lemma 1. *Let X be an analytic space, Y be an arbitrary space, and A be an analytic subset of $X \times Y$. Then the set $\overline{A}^{\text{hor}} = \{(x, y) \in X \times Y ; x \in \overline{A^y}\}$ is analytic.*

We are going to use many times the following well-known result. This is the topological argument that replaced a measure-theoretic one as we mentioned above.

Lemma 2. *Let A be a Baire space and X be a separable metric space. Let f_n, $n \in \mathbb{N}$, be mappings of A to X that have the Baire property.*

Then there is a dense G_δ subset P of A such that the restriction of each f_n to P is continuous on P.

In particular, if A_n, $n \in \mathbb{N}$, have the Baire property in A, then there is a dense G_δ subset P of A such that the sets $A_n \cap P$ are clopen in P.

Proof. It follows immediately from [4, Chapter 2, Section 32, II] and from the fact that A is a Baire space.

The particular case follows considering the characteristic functions of A_n's in place of f_n's. $\square$

The key results of Arsenin and Kunugui that we are investigating in this note are summarized in the following theorem.

Arsenin – Kunugui theorem (cf. [1, Chapter III, 32, Remark (b)]).

Let X and Y be Polish spaces and $B \subset X \times Y$ be a Borel set. Then the sets

(a) $\{y \in Y ; B^y$ is not $K_\sigma\}$ and

(b) $\{y \in Y ; B^y$ is not K_σ or B^y is empty$\}$

are analytic.

In particular, the projection of B to Y is Borel in Y if B^y is K_σ for every $y \in Y$.

Considering the Borel graph of f and the projection of it to Y, we may restate the very last claim as follows. Let $f : L \to Y$ be a Borel measurable mapping of a Luzin space L such that $f^{-1}(y)$ is K_σ for every $y \in Y$. Then $f(L)$ is a Borel subset of Y.

We need a strengthening of the point (a) of the Arsenin – Kunugui theorem. It concerns F_σ separation. We say that a set $S \subset Z$ can be F_σ *separated from a set* $T \subset Z$ in a space Z if there is an F_σ subset K of Z such that $S \subset K$ and $K \cap T = \emptyset$. Analogously, we say that S can be K_σ *separated from* T if there is a K_σ subset K of Z such that $S \subset K$ and $K \cap T = \emptyset$. The following result of Louveau and Saint-Raymond can be found in [5, Theorem 1(a) in Section 6.1].

Lemma 3. *Let A and B be two analytic subspaces of the product $X \times Y$ of two (Polish) spaces such that $A \cap B = \emptyset$. Then the set*

$$\{y \in Y; A^y \text{ cannot be separated from } B^y \text{ by an } F_\sigma \text{ set}\}$$

is analytic.

Proof. The proof of Louveau and Saint-Raymond is based on a deep theorem transferring the problem to the study of closed games. We would like to notice that an alternative proof can be obtained by following closely the methods from [1, Chapter IV], or [2, Theorem 35.38], or [10, Proposition 5.12.7] that are related to the Saint-Raymond's proof of Theorem in [9]. We indicate this possibility for the reader who may be more acquainted with the theory explained in any of the mentioned books.

Without any loss of generality we may suppose that X and Y are compact metric spaces.

Let $\varphi : \mathcal{N} \to A$ be a continuous surjection of $\mathcal{N}$ onto A. We put

$$F = \{(\nu, y) \in \mathcal{N} \times Y; y \text{ is the projection of } \varphi(\nu) \text{ to } Y\}.$$

It is simple to observe that $F \subset \mathcal{N} \times Y$ is closed and the mapping $f : F \to A$ defined by $f(\nu, y) = \varphi(\nu)$ is a continuous surjection which maps the sections $F^y \times \{y\}$ onto the sections $A^y \times \{y\}$.

We consider the Effros Borel structure $\mathcal{E}(F)$ on the hyperspace $\mathcal{F}(F)$ of non-empty closed subsets of F and a corresponding Polish topology on $\mathcal{F}(F)$.

We define the family $\mathcal{C}$ of elements S of $\mathcal{F}(F)$ for which $\overline{f(S)} \cap B = \emptyset$. The family is obviously hereditary, i.e. every closed non-empty subset of $C \in \mathcal{C}$ is in $\mathcal{C}$. It is also coanalytic as the preimage of the set $\mathcal{H} = \{H \in \mathcal{F}(X \times Y);\ H \cap B = \emptyset\}$ by the mapping $\Phi : \mathcal{F}(F) \to \mathcal{F}(X \times Y)$ defined by $\Phi(G) = \overline{f(G)}$. This follows using e.g. [1, Chapter II, Theorem 10] and [1, Chapter II, 13, Remark (c)]. Hence we may apply [2, Theorem 35.38] (or, to see it in more detail, cf. [10, Proposition 5.12.7] together with [2, 34.8]) to get that the set

$$\mathcal{S} = \{S \in \mathcal{F}(F); S \in \mathcal{C}_\sigma\}$$

is coanalytic.

It is simple to verify that the set $\mathcal{F}_0(F)$ of those $G \in \mathcal{F}(F)$ for which $G \subset F^y \times \{y\}$ for some $y \in Y$ is in $\mathcal{E}(F)$. Hence the set $\mathcal{D} = \mathcal{F}_0(F) \setminus \mathcal{S}$ is analytic. Notice that the set of our interest, namely the set

$$\tilde{Y} = \{y \in Y; A^y \text{ cannot be separated from } B^y \text{ by a } K_\sigma \text{ set}\},$$

is the image of $\mathcal{D}$ under the mapping that maps each $D \in \mathcal{D}$ to its projection in Y. Again this mapping is Borel measurable and thus $\tilde{Y}$ is analytic (cf. for example the end of the proof of [2, 35.38]). This concludes the proof. $\square$

2. Main results.

Our main result says that the condition (A) from the introduction is equivalent to some forms of Borel bimeasurability. In fact we prove

Theorem 4. *Let $f : L \to Y$ be Borel measurable, L be a Luzin space, and Y be a metric space. Then the following statements are equivalent.*

(1) For every closed set $F \subset L$ the set $f(F)$ is Borel in $f(L)$.

(2) For every Borel set $B \subset L$ such that the set $f^{-1}(y) \cap B$ is an F_σ subset of $f^{-1}(y)$ for each $y \in Y$, the image $f(B)$ is a Luzin space.

(3) The set $S = \{y \in Y; f^{-1}(y)$ is not $K_\sigma\}$ is at most countable.

It is trivial that (2) implies (1) in Theorem 4.

Proof of (3) implies (2) in Theorem 4. We use the Arsenin – Kunugui theorem as restated for Borel measurable mappings above. Since $f(L)$ is analytic, we may consider its Polish completion P instead of Y. If B is as in (2), then the restriction of f to $B \setminus f^{-1}(S)$ fulfils the assumptions and $f(B) = f(B \setminus f^{-1}(S)) \cup (f(B) \cap S)$ is Borel in P and so it is a Luzin space. $\square$

So it remains to prove that (1) implies (3). We formulate the crucial part of the proof of Theorem 4 in the form of a lemma. We consider the set $\mathbb{S} = \{\emptyset\} \cup \bigcup_{k=1}^{\infty} \mathbb{N}^k$ endowed with the discrete topology. Further on $|s|$ denotes the *length* of s which is k if $s \in \mathbb{N}^k$ or zero if $s = \emptyset$. We recall that $s \subset t$ denotes that $s \in \mathbb{S}$ is an initial segment of t. We write sometimes $s|k$ instead of $(s_1, \dots, s_k)$.

Lemma 5. *Let $G, H \subset K \times Y_0$ be analytic subspaces of the product of a compact space (K, ρ) and a homeomorphic copy of the Cantor set (Y_0, σ), and let $G \cap H = \emptyset$. Let the set G^y cannot be K_σ separated from H^y for every $y \in Y_0$. Then there is a homeomorphism Φ of $\mathbb{N}^{\mathbb{N}} \times C$ onto a closed subset E of G, where $C \subset Y_0$ is a homeomorphic copy of the Cantor set. The map Φ is of the form $\Phi(\nu, y) = (\varphi(\nu, y), y)$. Moreover, $\overline{E} \setminus E \subset H$ is the union of pairwise disjoint graphs of countably many continuous functions of C into K. Also, $\overline{E^y} \setminus E^y$ is dense in $\overline{E^y}$ for each $y \in C$.*

Proof. Let $\xi : F \to G$ be a continuous bijection of a closed subset F of $\mathbb{N}^{\mathbb{N}}$ onto G. Let us consider the sets $F_s = \{\nu \in F; s \subset \nu\}$ for $s \in \mathbb{S}$. The sets $G_s = \xi(F_s)$, with $|s| = n$ fixed, form a partition of G to Borel subsets. Let us denote $G_\emptyset = G$. We point out that, if $\nu \in F \subset \mathbb{N}^{\mathbb{N}}$, then $\lim_{n\to\infty} \operatorname{diam} G_{\nu|n} = 0$, $G_{\nu|n+1} \subset G_{\nu|n}$, and $\bigcap_{n=1}^{\infty} G_{\nu|n} = \{\xi(\nu)\}$ is a singleton. We stress the following fact that we shall need later.

Fact ($\star$). Let $\nu \in \mathbb{N}^{\mathbb{N}}$. Let $G^*_{\nu|n}$ be an arbitrary subset of $G_{\nu|n}$ and let $G^*_{\nu|1} \cap \dots \cap G^*_{\nu|n} \neq \emptyset$ for each $n \in \mathbb{N}$. Then

$$\bigcap_{n\in\mathbb{N}} \overline{G^*_{\nu|1} \cap \dots \cap G^*_{\nu|n}} = \{\xi(\nu)\}.$$

Notice that $G^Y_s = \{y \in Y; G^y_s \times \{y\}$ cannot be K_σ separated from $H\}$ is analytic by Lemma 3 for every $s \in \mathbb{S}$. Using Lemma 2, we find a dense G_δ subset C_0 of Y_0 such that all sets $G_s^Y \cap C_0$, $s \in \mathbb{S}$, are clopen in C_0. We put

$$G_0 = \Big(K \times C_0\Big) \cap \Big(G \setminus \bigcup \{G_s \setminus [K \times G_s^Y]; s \in \mathbb{S}\}\Big).$$

It follows that G_0 is a Borel subset of G, and so it is a Luzin space. For each $y \in C_0$ and each $s \in \mathbb{S}$, the set $(G_s^y \cap G_0^y) \times \{y\}$, if it is non-empty, cannot be K_σ separated from H (we omitted at most countably many subsets of G^y that are K_σ separated from H). The important fact for us now is that, in particular, $(\overline{G_s^y \cap G_0^y}) \cap H^y \neq \emptyset$. Also $G_0^y \neq \emptyset$ since G^y is not K_σ separated from H for every $y \in Y_0$.

We are now going to construct the homeomorphism Φ. We describe the key construction, that is repeated during the induction later, in the following claim. We suppose that G_0, G_s for $s \in \mathbb{S}$, and C_0 are as above, i.e. in particular that

(I) C_0 is a dense G_δ subset of Y_0,

(II) $(\overline{G_s^y \cap G_0^y}) \cap H^y \neq \emptyset$ (in fact $(G_s^y \cap G_0^y) \times \{y\}$ cannot be K_σ separated from H) if $s \in \mathbb{S}$, $y \in C_0$, and $G_s^y \cap G_0^y \neq \emptyset$, and

(III) $G_0^y \neq \emptyset$ for every $y \in Y_0$.

Claim. *Let $C^{(1)}$ be a dense G_δ subset of C_0, let $\mathbf{s} : C^{(1)} \to \mathbb{S}$ be continuous and such that the set*

$$D = \bigcup_{y \in C^{(1)}} (G_{\mathbf{s}(y)}^y \cap G_0^y) \times \{y\}$$

fulfils that $D^y \neq \emptyset$ for $y \in C^{(1)}$, and let $b^{(1)} : C^{(1)} \to (0, \infty)$ be a continuous function.

Then there are a dense G_δ subset $C^{(2)}$ of $C^{(1)}$, a continuous mapping $h : C^{(2)} \to K$, continuous functions $b^{(1)} > a^{(1)} > b^{(2)} > a^{(2)} > \cdots > 0$ on $C^{(2)}$, and finally continuous mappings $\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \ldots$ of $C^{(2)}$ to $\mathbb{S}$ fulfilling the following conditions.

For every $y \in C^{(2)}$

$$h(y) \in \overline{D^y \cap G_0^y} \cap H^y, \tag{1}$$

$$\lim_{k \to \infty} a^{(k)}(y) = \lim_{k \to \infty} b^{(k)}(y) = 0, \tag{2}$$

$$\mathbf{s}(y) \subsetneqq \mathbf{s}^{(k)}(y), \tag{3}$$

$$G_{\mathbf{s}^{(k)}(y)}^y \cap G_0^y \neq \emptyset, \tag{4}$$

and finally

$$\bigcup_{y \in C^{(2)}} \Big(\overline{G_{\mathbf{s}^{(k)}(y)}^y} \times \{y\}\Big) \subset \Big\{(x, y) \in K \times C^{(2)}; \rho(x, h(y)) \in (a^{(k)}(y), b^{(k)}(y))\Big\}. \tag{5}$$

Proof of Claim. Notice that $\overline{D^y} \cap H^y = \overline{G^y_{\mathbf{s}(y)} \cap G^y_0} \cap H^y \neq \emptyset$ for all $y \in C^{(1)}$ by (II) and by the properties of D.

Since $\mathbf{s}$ is continuous, D is Borel, and, by Lemma 1, the set $\overline{D}^{\text{hor}}$ is analytic. Since H is analytic, also the set $\overline{D}^{\text{hor}} \cap H$ is analytic. Due to the Jankov – von Neumann selection theorem formulated above, there is a mapping $h : C^{(1)} \to K$ having the Baire property such that its graph is a subset of $\overline{D}^{\text{hor}} \cap H$ and such that it is defined on all of $C^{(1)}$.

Using Lemma 2, we find a dense G_δ set $C_h \subset C^{(1)}$ such that the restriction of h to C_h is continuous.

For every $k \in \mathbb{N}$ the set $\{(x, y) \in K \times C_h; \rho(x, h(y)) \in (\frac{1}{k}, b^{(1)}(y))\}$ is open in $K \times C_h$ as $b^{(1)}$ and h are continuous on C_h. Since D is analytic, the set $D \cap \{(x, y) \in K \times C_h; \rho(x, h(y)) \in (\frac{1}{k}, b^{(1)}(y))\}$ is also analytic, and its projection $C_h(k)$ to C_h is analytic, and so it has the Baire property. We choose, using Lemma 2, a dense G_δ subset C_{a^1} of C_h such that $C_h(k) \cap C_{a^1}$ is clopen in C_{a^1} for every $k \in \mathbb{N}$. The sets $C_h(k)$ cover C_h and we put $a^{(1)} = \frac{1}{k}$ on $C_h(k) \setminus \bigcup_{l<k} C_h(l)$. It is simple to notice that $a^{(1)}$ restricted to C_{a^1} is continuous.

Having $a^{(k)}$ on C_{a^k}, we put $b^{(k+1)}(y) = a^{(k)}/2$ for $y \in C_{a^k}$ and repeating the above procedure we get a continuous $a^{(k+1)}$ on some dense G_δ subset $C_{a^{k+1}}$ of C_{a^k}. Put $C_a = \bigcap_{k=1}^{\infty} C_{a^k}$. Now we shall consider the restriction of the above functions to C_a which is a dense G_δ subset of $C^{(1)}$.

We are going to define the mappings $\mathbf{s}^{(k)}$ on C_a first. Let $k \in \mathbb{N}$ be fixed. Also let $s_0 \in \mathbb{S}$ be fixed and we shall define $\mathbf{s}^{(k)}$ on the set $C_a(s_0) = C_a \cap \mathbf{s}^{-1}(s_0)$. Let $\mathbb{S}^{(k)}(s_0) = \{s \in \mathbb{S}; s_0 \subsetneqq s, \overline{G_s} \cap (K \times C_a) \subset U^{(k)}\}$, where $U^{(k)} = \{(x, y) \in K \times C_a; \rho(x, h(y)) \in (a^{(k)}(y), b^{(k)}(y))\}$. Since $U^{(k)}$ is open in $K \times C_a$ by the continuity of the functions $a^{(k)}$, $b^{(k)}$, and h, and ξ is continuous with $\xi(F_{s_0}) = G_{s_0}$, the family $\{G_s; s \in \mathbb{S}^{(k)}(s_0)\}$ covers $G_{s_0} \cap U^{(k)}$. As moreover the intersection $G^y_{s_0} \cap G^y_0 = D^y$ is non-empty for every $y \in C_a(s_0)$, the family of sets $C^s_a(s_0) = \{y \in C_a(s_0); G^y_s \cap G^y_0 \neq \emptyset\}$, $s \in \mathbb{S}^{(k)}(s_0)$, is a cover of $C_a(s_0)$. Each element $C^s_a(s_0)$ is analytic as the projection of an analytic set intersected with the G_δ set $C_a(s_0)$. Let s_i, $i \in \mathbb{N}$, be any one-to-one sequence of all elements of $\mathbb{S}$. Let us put $\mathbf{s}^{(k)} \equiv s_i$ on the set $C^{s_i}_a(s_0) \setminus \bigcup_{j<i} C^{s_j}_a(s_0)$.

Defining $\mathbf{s}^{(k)}$ in the same way on each $C_a(s_0)$, we get the asked map $\mathbf{s}^{(k)} : C_a \to \mathbb{S}$ such that $\overline{G^y_{\mathbf{s}^{(k)}(y)}} \subset (U^{(k)})^y$ and $G^y_{\mathbf{s}^{(k)}(y)} \cap G^y_0$ is non-empty for every $y \in C_a$. It follows that the closure of $G^y_{\mathbf{s}^{(k)}(y)} \cap G^y_0$ intersects H^y by our assumption (II) for every $y \in C_a$. Using Lemma 2, there is a dense G_δ subset $C^{(2)}$ of C_a such that all $\mathbf{s}^{(k)}$ are continuous on $C^{(2)}$ and the properties from our claim are satisfied. $\square$

Now we shall continue with the proof of Lemma 5 by induction over $n \in \mathbb{N}$.

We are going to use Claim to construct, for each $n \in \mathbb{N}$, a dense G_δ subset C_{n-1} of Y_0 and continuous mappings $\mathbf{s}_{k_1,\ldots,k_{n-1}} : C_{n-1} \to \mathbb{S}$, $h_{k_1,\ldots,k_{n-1}} : C_n \to K$, $b_{k_1,\ldots,k_n} : C_n \to (0, \infty)$, $a_{k_1,\ldots,k_n} : C_n \to (0, \infty)$ such that the following conditions (a) – (e) hold.

We use, here and later on, the notation

$$D(k_1, \ldots, k_{n-1}) = \bigcup_{y \in C_{n-1}} \Big((G^y_{\mathbf{s}_{k_1,\ldots,k_{n-1}}(y)} \cap G^y_0) \times \{y\} \Big)$$

and the notation

$$U(k_1,\ldots,k_n)=\\ \{(x,y)\in K\times C_n;\rho(x,h_{k_1,\ldots,k_{n-1}}(y))\in(a_{k_1,\ldots,k_n}(y),b_{k_1,\ldots,k_n}(y))\}\cap U(k_1,\ldots,k_{n-1}),$$

where $U(\emptyset)=K\times C_0$ and $\mathbf{s}_\emptyset\equiv\emptyset$.

For each $n\in\mathbb{N}$ and $y\in C_n$

(a) $C_n\subset C_{n-1}$;

(b) $h_{k_1,\ldots,k_{n-1}}(y)\in\overline{D^y(k_1,\ldots,k_{n-1})}\cap H^y$;

(c) $b_{k_1,\ldots,k_{n-1},1}\equiv\frac{1}{n}>a_{k_1,\ldots,k_{n-1},1}>b_{k_1,\ldots,k_{n-1},2}>\ldots>0$, with $\lim\limits_{k\to\infty}b_{k_1,\ldots,k_{n-1},k}=0$;

(d) $\mathbf{s}_{k_1,\ldots,k_{n-1}}(y)\subsetneqq\mathbf{s}_{k_1,\ldots,k_n}(y)$ for every $y\in C_n$;

(e) $\overline{D^y(k_1,\ldots,k_n)}\subset U^y(k_1,\ldots,k_n)$.

We proceed by induction over $n\in\mathbb{N}$.

In the first step we put $C^{(1)}=C_0$, $\mathbf{s}(y)=\mathbf{s}_\emptyset(y)=\emptyset$, $D=D(\emptyset)=G_0$, and $b^{(1)}(y)=b_1(y)=1$ for $y\in C_0$.

We use Claim and (III), and we put $C_1=C^{(2)}$, $h_\emptyset\equiv h$, $b_k\equiv b^{(k)}$ and $a_k\equiv a^{(k)}$, and finally $\mathbf{s}_k\equiv\mathbf{s}^{(k)}$ on C_1.

Obviously, the conditions (a) to (e) are fulfilled by Claim for $n=1$. This finishes the first step.

Let all C_n, $\mathbf{s}_{k_1,\ldots,k_n}$, $b_{k_1,\ldots,k_n}$, $a_{k_1,\ldots,k_n}$ be already constructed for a fixed $n\in\mathbb{N}$.

We use Claim independently for every choice of $(k_1,\ldots,k_n)$ putting $C^{(1)}=C_n$, $\mathbf{s}\equiv\mathbf{s}_{k_1,\ldots,k_n}$, and $b^{(1)}\equiv\frac{1}{n+1}$.

We obtain $C(k_1,\ldots,k_n)=C^{(2)}$, $h_{k_1,\ldots,k_n}\equiv h$, $b_{k_1,\ldots,k_n,k}\equiv b^{(k)}$, $a_{k_1,\ldots,k_n,k}\equiv a^{(k)}$, and $\mathbf{s}_{k_1,\ldots,k_n,k}\equiv\mathbf{s}^{(k)}$ on $C(k_1,\ldots,k_n)$.

Finally, we define $C_{n+1}=\bigcap\limits_{(k_1,\ldots,k_n)\in\mathbb{N}^n}C(k_1,\ldots,k_n)$ and the construction of the objects fulfilling (a) – (e) is finished.

Now we put $C_\infty=\bigcap_{n=1}^\infty C_n$. Since C_∞ is a dense G_δ subset of Y_0, it contains a homeomorphic copy C of the Cantor set.

Further on, we consider the functions $h_{k_1,\ldots,k_{n-1}}$, $a_{k_1,\ldots,k_n}$, $b_{k_1,\ldots,k_n}$, $\mathbf{s}_{k_1,\ldots,k_n}$ on C only. By $U(k_1,\ldots,k_n)$ and by $D(k_1,\ldots,k_n)$ we understand the intersections of the corresponding sets with the compact set $K\times C$.

We put

$$V(k_1,\ldots,k_n)=\\ \{(x,y)\in K\times C;\rho(x,h_{k_1,\ldots,k_{n-1}}(y))\in\langle a_{k_1,\ldots,k_n}(y),b_{k_1,\ldots,k_n}(y)\rangle\}\cap V(k_1,\ldots,k_{n-1}),$$

where $V(\emptyset)=K\times C$,

$$E^*=\bigcap_{n=1}^{\infty}\bigcup_{(k_1,\ldots,k_n)\in\mathbb{N}^n}\Big(V(k_1,\ldots,k_n)\cup\mathrm{graph}(h_\emptyset)\cup\mathrm{graph}(h_{k_1})\cup\cdots\cup\mathrm{graph}(h_{k_1,\ldots,k_n})\Big),$$

and

$$E=\bigcap_{n=1}^{\infty}\bigcup_{(k_1,\ldots,k_n)\in\mathbb{N}^n}V(k_1,\ldots,k_n).$$

Since the families

$$\{V(k_1,\dots,k_n);(k_1,\dots,k_n)\in\mathbb{N}^n\},\ n\in\mathbb{N},$$

are disjoint by (c) and by the inclusions $V(k_1,\dots,k_{n+1})\subset V(k_1,\dots,k_n)$, we may also write

$$E=\bigcup_{\nu\in\mathbb{N}^{\mathbb{N}}}\bigcap_{n=1}^{\infty}V(\nu|n).$$

The choice $b(k_1,\dots,k_{n-1},1)\equiv\frac{1}{n}$ and further properties of the functions $b_{k_1,\dots,k_n}$ and $a_{k_1,\dots,k_n}$ in (c) with the definition of $V(k_1,\dots,k_n)$ imply that

$$\operatorname{diam}V^y(k_1,\dots,k_n)\le\frac{2}{n}\text{ for every }y\in C.$$

For each choice of $y\in C$ and $(k_1,\dots,k_n)\in\mathbb{N}^n$, we have that

$$D^y(k_1,\dots,k_n)\neq\emptyset$$

as the right-hand side of (b) is non-empty.

Let us consider any $y\in C$ and $\nu\in\mathbb{N}^{\mathbb{N}}$. Using Fact ($\star$) above, notice that $(G^y_{s_{\nu|n}(y)}\cap G^y_0)\times\{y\}\subset G_{s_{\nu|n}(y)}$, we get that

$$\bigcap_{n=1}^{\infty}\overline{G^y_{s_{\nu|n}(y)}}\times\{y\}=\bigcap_{n=1}^{\infty}\overline{(D^y_{\mathbf{s}_{\nu|n}(y)}\cap G^y_0)}\times\{y\}$$

is the singleton $\{\xi(\mu)\}\in G$, where μ is the only element of $F\subset\mathbb{N}^{\mathbb{N}}$ such that $\mathbf{s}_{\nu|n}(y)\subset\mu$ for every $n\in\mathbb{N}$. It exists and is unique by (d) above. We shall use the notation $(\varphi(\nu,y),y)$ for such a $\xi(\mu)$. Due to (e) and to the estimate on diameters of $V^y(k_1,\dots,k_n)$ above, we also have

$$\{\varphi(\nu,y)\}=\bigcap_{n=1}^{\infty}U^y(\nu|n)=\bigcap_{n=1}^{\infty}V^y(\nu|n).$$

Thus $E\subset G$ and, due to the choice of $h_{k_1,\dots,k_n}$'s, $E^*\cap G=E$.

Now we shall show that E^* is closed in $K\times C$, and thus it is compact. It is enough to show that for every n the set

$$E^*_n=\bigcup_{(k_1,\dots,k_n)\in\mathbb{N}^n}\Big(V(k_1,\dots,k_n)\cup\operatorname{graph}(h_\emptyset)\cup\operatorname{graph}(h_{k_1})\cup\dots\cup\operatorname{graph}(h_{k_1,\dots,k_n})\Big)$$

is closed. Let $n\in\mathbb{N}$ be fixed and $(x_m,y_m)\in E^*_{k_1,\dots,k_n}$, $m\in\mathbb{N}$, be a sequence such that $\lim\limits_{m\to\infty}(x_m,y_m)=(x,y)\in K\times C$.

If there were infinitely many n-tuples $(k_1,\dots,k_n)\in\mathbb{N}$ such that $V(k_1,\dots,k_n)$ contains an element of $(x_m,y_m),m\in\mathbb{N}$, then there are an $i\in\{1,\dots,n-1\}$ and $k^0_1,\dots,k^0_i\in\mathbb{N}$ such that for infinitely many $k\in\mathbb{N}$ there is an element (x_m,y_m) in $V(k^0_1,\dots,k^0_i,k)$. Similarly, if there were infinitely many graphs of functions $h_\emptyset$ and $h_{k_1,\dots,k_j}$, $j = 1,\dots,n$, containing an element of the sequence (x_m, y_m) , then there are an $i \in \{1,\dots,n-1\}$ and $k_1^0,\dots,k_i^0 \in \mathbb{N}$ such that for infinitely many $k \in \mathbb{N}$ there is an element (x_m, y_m) in $\operatorname{graph}(h_{k_1^0,\dots,k_i^0,k})$.

We may notice that the sequences of functions $a_{k_1^0,\dots,k_i^0,k}$ and $b_{k_1^0,\dots,k_i^0,k}$ converge uniformly to zero on the compact set C when $k \in \mathbb{N}$ tends to the infinity. The mappings $h_{k_1^0,\dots,k_i^0,k}$ converge uniformly to $h_{k_1^0,\dots,k_i^0}$ on C. As the functions $h_{k_1^0,\dots,k_i^0}$, $a_{k_1^0,\dots,k_i^0,k}$, and $b_{k_1^0,\dots,k_i^0,k}$ are continuous on C, we get that $\lim\limits_{m\to\infty}(x_m, y_m) = (h_{k_1^0,\dots,k_i^0}(y), y) \in E_n^*$.

Otherwise, there is a fixed sequence $k_1^1,\dots,k_n^1$ and a subsequence of $(x_m, y_m), m \in \mathbb{N}$, that is contained in $V(k_1^1,\dots,k_n^1)$ or in one of the sets $\operatorname{graph}(h_\emptyset)$, $\operatorname{graph}(h_{k_1^1})$, $\dots$, $\operatorname{graph}(h_{k_1^1,\dots,k_n^1})$. It follows by the definition of the set $V(k_1^1,\dots,k_n^1)$ and the continuity of $a_{k_1^1,\dots,k_n^1}$ and $b_{k_1^1,\dots,k_n^1}$ that the set $V(k_1^1,\dots,k_n^1)$ is closed, and since the functions $h_\emptyset, h_{k_1^1,\dots,k_i^1}$ for $i = 1,\dots,n$ are continuous, there graphs are closed, too. Hence $(x, y) \in E_n^*$.

Hence, E^* is closed and it follows easily by our construction that $E^* = \overline{E}$ and $\overline{E^y} \setminus E^y$ is dense in $\overline{E^y}$.

The mappings $\varphi(\cdot, y)$, $y \in C$, are one-to-one since the families $\{V(\nu|n); \nu \in \mathbb{N}^\mathbb{N}\}$ are disjoint for every $n \in \mathbb{N}$.

The mapping $\varphi : \mathbb{N}^\mathbb{N} \times C \to K$ is continuous. Indeed, let $\nu_0 \in \mathbb{N}^\mathbb{N}$, $y_0 \in C$, and $\varepsilon > 0$ be arbitrary. We may find $n_0 \in \mathbb{N}$ such that $\frac{2}{n_0} < \frac{\varepsilon}{3}$. For $\nu \in \mathbb{N}^\mathbb{N}$ such that $\nu|n_0 = \nu_0|n_0$ we have that $\rho(\varphi(\nu, y), h_{\nu|n_0}(y)) \le \frac{2}{n_0} < \frac{\varepsilon}{3}$ for every $y \in C$ by (c) and the definitions of $U(k_1,\dots,k_n)$ and φ. Now, as $h_{\nu|n_0}$ is continuous, we choose a $\delta > 0$ such that $\rho(h_{\nu|n_0}(y), h_{\nu|n_0}(y_0)) < \frac{\varepsilon}{3}$ if $\sigma(y, y_0) < \delta$. Finally,

$$\rho(\varphi(\nu, y), \varphi(\nu_0, y_0)) <$$

$$< \rho(\varphi(\nu, y), h_{\nu|n_0}(y)) + \rho(h_{\nu|n_0}(y), h_{\nu|n_0}(y_0)) + \rho(\varphi(\nu_0, y_0), h_{\nu|n_0}(y_0)) < \varepsilon.$$

Thus φ is continuous.

We define $\Phi(\nu, y) = (\varphi(\nu, y), y)$. Obviously, $\Phi : \mathbb{N}^\mathbb{N} \times C \to G$ is a continuous injection such that $\Phi(\cdot, y)$ maps $\mathbb{N}^\mathbb{N}$ onto $E^y \times \{y\}$. We claim that Φ^{-1} is continuous on E. Indeed, let $(x_0, y_0) \in E$ with $\Phi^{-1}(x_0, y_0) = (\nu_0, y_0)$, $n_0 \in \mathbb{N}$, and $\varepsilon > 0$ be chosen. There is a $\delta > 0$, $\delta < \varepsilon$ such that $(x, y) \in U(\nu_0|n_0)$ if $\rho(x, x_0) < \delta$ and $\sigma(y, y_0) < \delta$ as $U(\nu_0|n_0)$ is open in $K \times C$. Then, for $(x, y) \in U(\nu_0|n_0) \cap E$, we have $\Phi^{-1}(x, y) = (\nu, y)$, where $\nu|n_0 = \nu_0|n_0$ and $\sigma(y, y_0) < \varepsilon$. This proves the continuity of Φ^{-1}.

Hence $\Phi : \mathbb{N}^\mathbb{N} \times C \to E$ is a homeomorphism and Lemma 5 is proved. $\square$

At this moment we are able to prove the implication (2) implies (3). To get (1) implies (3) we need still a lemma.

Lemma 6. *Let Y be a separable metric space, $C \subset Y$ be a homeomorphic copy of $\{0,1\}^\mathbb{N}$ and $g_n : C \to Y$ be a mapping with the Baire property without fixed points for every $n \in \mathbb{N}$. Then there exists $D \subset C$ such that*

(a) *D is a homeomorphic copy of $\{0,1\}^\mathbb{N}$,*
(b) *g_n is continuous on D for every $n \in \mathbb{N}$,*
(c) *$g_n(D) \cap D = \emptyset$ for every $n \in \mathbb{N}$.*

Proof. We need the following Claim.

Claim. Let Y be a metric space and T be a homeomorphic copy of $\{0,1\}^{\mathbb{N}}$, $g : T \to Y$ be a continuous mapping without fixed points, $k \in \mathbb{N}$ and $V_1, \dots, V_k$ be open sets intersecting T. Then there exists a set $L \subset (V_1 \cup \dots \cup V_k) \cap T$ such that

(a) L is a homeomorphic copy of $\{0,1\}^{\mathbb{N}}$,
(b) L intersects each V_j, $j = 1, \dots, k$,
(c) g is continuous on L,
(d) $g(L) \cap L = \emptyset$.

Proof of Claim. We pick a point $x_j \in T \cap V_j$ for every $j \in \{1, \dots, k\}$ in such a way that $\{x_1, \dots, x_k\} \cap \{g(x_1), \dots, g(x_k)\} = \emptyset$. This can be done as follows. For $j = 1, \dots, k$ denote

$$N_j = \begin{cases} \emptyset & \text{if } g(T \cap V_j) \text{ is infinite,} \\ g(T \cap V_j) & \text{if } g(T \cap V_j) \text{ is finite;} \end{cases}$$

$$P = \bigcup_{j=1}^{k} N_j.$$

The set P is clearly finite. Choose $x_1 \in (V_1 \cap T) \setminus P$. Suppose that we have picked points $x_1, \dots, x_s$, $s < k$, such that

$$x_j \in (V_j \cap T) \setminus \big(P \cup g(\{x_1, \dots, x_{j-1}\}) \cup g^{-1}(\{x_1, \dots, x_{j-1}\})\big), \ j = 1, \dots, s.$$

Observe that the set

$$(V_{s+1} \cap T) \setminus \big(P \cup g(\{x_1, \dots, x_s\}) \cup g^{-1}(\{x_1, \dots, x_s\})\big)$$

is nonempty. Indeed, since otherwise we have that $g(V_{s+1} \cap T)$ is finite and, as $(V_{s+1} \cap T) \setminus (P \cup g(\{x_1, \dots, x_s\}))$ is infinite, there exists $j_0 \in \{1, \dots, s\}$ such that $x_{j_0} \in g(V_{s+1} \cap T) = N_{s+1} \subset P$, a contradiction. Our construction of $x_1, \dots, x_k$ clearly works.

This construction and continuity of g on each $V_j \cap T$ implies that, for every $j = 1, \dots, k$, we can find in sufficiently small neighbourhood of x_j a homeomorphic copy L_j of $\{0,1\}^{\mathbb{N}}$ such that $L_j \subset V_j \cap T$ and $\bigcup_{j=1}^{k} L_j \cap g(\bigcup_{j=1}^{k} L_j) = \emptyset$. Put $L = \bigcup_{j=1}^{k} L_j$. It is not difficult to see that L is a homeomorphic copy of $\{0,1\}^{\mathbb{N}}$ and has all required properties. This finishes the proof of Claim. □

To prove Lemma 6, we use first Lemma 2, and we find a homeomorphic copy $K_0 \subset C$ of $\{0,1\}^{\mathbb{N}}$ such that all g_n's are continuous on K_0. We construct inductively open sets $G_s \subset C$, $s \in \bigcup_{j=1}^{\infty} \{0,1\}^j \cup \{\emptyset\}$, and a sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, and $s \in \{0,1\}^n$ or $s = \emptyset$, we have

(i) $G_\emptyset = Y$,
(ii) $\overline{G_{s\hat{}0}} \cap \overline{G_{s\hat{}1}} = \emptyset$,
(iii) $\overline{G_{s\hat{}0}} \cup \overline{G_{s\hat{}1}} \subset G_s$,
(iv) $\operatorname{diam} G_s < 2^{-n}$ if $s \neq \emptyset$,

(v) $K_{n-1} \cap G_s \neq \emptyset$,
(vi) $K_n \subset \bigcup_{\overline{s} \in \{0,1\}^n} G_{\overline{s}}$,
(vii) g_n is continuous on K_n,
(viii) $g_n(K_n) \cap K_n = \emptyset$,
(ix) $K_n \subset K_{n-1}$.

In the n-th step of induction we choose open sets G_t, $t \in \{0,1\}^n$, such that (ii), (iii) with $s \in \{0,1\}^{n-1}$ (here $\{0,1\}^0$ denotes $\{\emptyset\}$) hold, and (iv), (v) with $t \in \{0,1\}^n$ hold. Using Claim for K_{n-1} in place of T and G_t, $t \in \{0,1\}^n$, in place of $V_1, \ldots, V_k$, we get K_n satisfying (vi) – (ix).

We put $K = \bigcap_{n=0}^{\infty} K_n$. It is not difficult to see that K has all required properties. □

Proof of Theorem 4, (1) implies (3). Let K be a metric compactification of L. Since $f(L)$ is analytic, its completion is Polish and we may notice that to prove that (1) implies (3) it is sufficient to suppose that Y is that Polish completion. The graph G of f is a Borel subset of $K \times Y$ as L is Borel in K and $f : L \to Y$ is Borel measurable. Let us suppose that (3) does not hold, i.e. that the set $S = \{y \in Y; G^y \text{ is not } K_\sigma\}$ is uncountable. By Arsenin-Kunugui theorem (a) stated above (or, also by Lemma 3 above with $A = G$ and $B = G^c$), S is analytic and so there is a homeomorphic copy Y_0 of $\{0,1\}^{\mathbb{N}}$ in S. Let $C \subset Y_0$, $E \subset (K \times C) \cap G$, and Φ be as in Lemma 5. So $\overline{E} \setminus E$ is covered by the graphs of some continuous mappings $\gamma_n : C \to K$, $n \in \mathbb{N}$. We put $g_n = f \circ \gamma_n : C \to Y$ and use Lemma 6 for them. Thus we find a homeomorphic copy $D \subset C$ of $\{0,1\}^{\mathbb{N}}$ such that each $f \circ \gamma_n : D \to Y$ is continuous and $f \circ \gamma_n(D) \cap D = \emptyset$. As Φ from Lemma 5 restricted to $\mathbb{N}^{\mathbb{N}} \times D$ is a homeomorphism onto the set $E_D = (K \times D) \cap E$ with the property that $\Phi(x,y) \in E_D^y$ for every $(x,y) \in \mathbb{N}^{\mathbb{N}} \times C$, there is a relatively closed subset H of E_D such that its projection N to D is not Borel in D. (It suffices to choose a closed subset of $\mathbb{N}^{\mathbb{N}} \times D$ with non-Borel projection N to D and take its image by Φ.) Finally, we consider the relatively closed subset $F = \overline{p_L(H)}^L$ in L, where p_L is the projection to L. Since H is a subset of the graph of f and of E_D, we see that $f(p_L(H)) = N \subset D$.

We claim that

$$(\star\star) \qquad f(F \setminus p_L(H)) = \bigcup_{n \in \mathbb{N}} f(\gamma_n(D) \cap F).$$

Indeed, if $y = f(x)$ with $x \in \gamma_n(D) \cap F$, then $x \notin p_L(H)$. Otherwise y would be an element of $f \circ \gamma_n(D) \cap D (= \emptyset)$. The other inclusion can be proved as follows. Let $y = f(x)$ with $x \in F \setminus p_L(H)$. There is a sequence $(x_n, y_n) \in H$ such that $x_n \to x$ by the definition of F and, as $\overline{E_D}$ is compact, we may suppose that (x_n, y_n) tends to $(x,y) \in \overline{E_D}$. As H is relatively closed in E_D and $x \notin p_L(H)$, we have that $(x,y) \in \overline{E_D} \setminus E_D \subset \overline{E} \setminus E$. Hence, there is an $n \in \mathbb{N}$ such that $\gamma_n(y) = x$ and so $y \in f(\gamma_n(D) \cap F)$ and $(\star\star)$ is proved.

It follows from $(\star\star)$ that $f(F \setminus p_L(H)) \subset \bigcup_{n \in \mathbb{N}} f \circ \gamma_n(D)$ and so $f(F \setminus p_L(H)) \cap D = \emptyset$. Therefore $f(F) \cap D = N$ and, as N is not Borel in the compact set D, the set $f(F)$ is not Borel in Y and the theorem is proved. □

Corollary 7. *Let $f : L \to Y$ be a Borel measurable mapping of a Luzin space L to a metric space Y. Then exactly one of the following possibilities holds.*

(a) There is an ordinal $\alpha < \omega_1$ such that $f(F)$ is of a Borel class less than α for each closed subset F of L.

(b) For every ordinal α, $2 \le \alpha < \omega_1$, there is a closed subset F of L such that $f(F)$ is a Borel subset of exact additive class $\mathbf{\Sigma}^0_\alpha$. If $3 \le \alpha < \omega_1$, then there is a closed set F such that $f(F)$ is of exact multiplicative class $\mathbf{\Pi}^0_\alpha$. Also there is a closed subset F of L such that $f(F)$ is analytic but not Borel.

Proof. (a) Since $f(L)$ is separable, we may assume that Y is separable. If $f(F)$ is Borel for every closed $F \subset L$, then the set $S = \{y \in Y;\ f^{-1}(y) \text{ is not } K_\sigma\}$ is countable by Theorem 4. Put

$$G = (\operatorname{graph} f) \cap (L \times (Y \setminus S)).$$

The set G is a Borel subset of $L \times Y$ such that G^y is K_σ for every $y \in Y$. According to Saint-Raymond's theorem [9] there exist Borel sets G_n, $n \in \mathbb{N}$, such that $G = \bigcup_{n=1}^\infty G_n$ and G_n^y is compact whenever $y \in Y$ and $n \in \mathbb{N}$. Let F be a closed subset of L. Then

$$f(F) = \left(\bigcup_{n=1}^\infty p_Y((F \times Y) \cap G_n) \right) \cup (f(F) \cap S),$$

where p_Y is the projection of $L \times Y$ onto Y. It is sufficient to show that if $B \subset L \times Y$ is a Borel set with B^y compact for every $y \in Y$, then there exists an $\alpha < \omega_1$ such that $p_Y((F \times Y) \cap B)$ is of Borel class less than α, whenever $F \subset L$ is closed.

Observe that, if $\{F_j\}_{j=1}^\infty$ is a decreasing sequence of closed subsets of L such that $\bigcap_{j=1}^\infty F_j = F$, then

$$p_Y \left(\bigcap_{j=1}^\infty (F_j \times Y) \cap B \right) = \bigcap_{j=1}^\infty p_Y((F_j \times Y) \cap B)$$

since B^y's are compact (cf. e.g. [1, II, 19, Theorem]).

Fix a countable open basis $\mathcal{B}$ of Y. Consider a countable family $\mathcal{F}$ of all closed sets of the form $Y \setminus (U_1 \cup \dots \cup U_m)$, where $m \in \mathbb{N}$ and $U_i \in \mathcal{B}$ for every $i = 1, \dots, m$. Since $\mathcal{F}$ is countable there exists an $\alpha < \omega_1$ such that for every $F \in \mathcal{F}$ we have $p_Y((F \times Y) \cap B)$ is of Borel class less than α. We see that each closed subset of L can be written as an intersection of a decreasing sequence of sets from $\mathcal{F}$. Thus we obtain that $p_Y((F \times Y) \cap B)$ is of Borel class less than $\alpha + 1$ whenever F is closed in L.

(b) Let K be a metric compactification of L, $E \subset K \times Y$, D, E_D be sets from the proof of implication (1) $\Rightarrow$ (3) of Theorem 4.

For every ordinal $2 \le \alpha < \omega_1$ there is a relatively closed subset $H \subset E_D$ such that its projection N to D is of exact class $\mathbf{\Sigma}^0_\alpha$ (Π^0_α, respectively). Put $F = \overline{p_L(H)}^L$. Using ($\star\star$) from the proof of implication (1) $\Rightarrow$ (3) of Theorem 4, we obtain that $f(F)$ is a Borel set of exact class $\mathbf{\Sigma}^0_\alpha$ if $\alpha \ge 2$ (and, Π^0_α if $\alpha \ge 3$, respectively) since $f(F)$ is the union of the set $N \subset D$ and the set $f(F \setminus p_L(H))$, which is K_σ and disjoint with the compact set D. $\square$

3. Remarks.

Remark 1. Our proof of Lemma 5 contains as a substantial part a method of proof of the following strengthened version of a Hurewicz theorem.

Theorem (Kechris, Louveau, and Woodin, [3]). *Let A, B be subsets of a (separable metric) space X (or of a metric, and not necessarily separable, space X) such that A is analytic, $A \cap B = \emptyset$, and A cannot be separated from B by an F_σ subset of X. Then there is a non-empty perfect compact set $C \subset A \cup B$ such that $\overline{C \cap A} = \overline{C \cap B} = C$ and $C \cap B$ is countable.*

Indeed, if we consider a compactification K of $\overline{A}^X$ and put $G = A \times [0,1]$, $H = (B \cap \overline{A}^X) \times [0,1]$, we get the above result from our Lemma 5 if B is analytic. Let us only remark that we used the analyticity of H since we needed to find the closed copies of irrationals in all sections G^y for $y \in Y_0$ in a "measurable" way. When working with one section only, no measurability assumption on B is needed. Our method is quite straightforward. The proof of this theorem in [3] uses the theory of Borel games.

Remark 2. Our Theorem 4, (1) implies (2), improves [11,Theorem 3] saying that the image of a Luzin space by a closed continuous mapping is a Luzin space.

Remark 3. In analogy to our Theorems 4 and 5, we may reformulate Purves' theorem as follows.

Theorem 8. *Let $f : L \to Y$ be Borel measurable, L be a Luzin space, and Y be a metric space. Then the following statements are equivalent.*

(1) For every G_δ set $B \subset L$, the set $f(B)$ is Borel in $f(L)$.
(2) For every Borel set $B \subset L$, the image $f(B)$ is a Luzin space.
(3) The set $S = \{y \in Y; f^{-1}(y)$ is not countable$\}$ is at most countable.

We shall show how the proof can be obtained from the following Lemma 9 that is an analogue of Lemma 5 above. We can proof Lemma 9 modifying the proof of Lemma 5 above. Nevertheless the proof of Lemma 9 is omitted since a much nicer proof can be derived from the results of [7, see Section 4.1]. (Also that proof does not use any measure-theoretic argument in contrary to the Purves', Mauldin's, or Srivastava's proofs.) We think that the method of proof from [7] cannot be used to prove our Lemma 5 easily since we are dealing with the topological property K_σ and not with the set-theoretical property of being countable for the fibers.

Lemma 9. *Let $G \subset L \times Y$ be analytic and $\{y \in Y; G^y$ is uncountable$\}$ be uncountable. Then there is a homeomorphism Φ of $\{0,1\}^{\mathbb{N}} \times C$ onto $E \subset G$, where $C \subset Y$ is a homeomorphic copy of $\{0,1\}^{\mathbb{N}}$, such that $\Phi(\cdot, y)$ is a homeomorphism of $\{0,1\}^{\mathbb{N}}$ onto $E^y \times \{y\}$ for every $y \in C$.*

Proof of Theorem 8. The implication (2) implies (1) is obvious, the implication (3) implies (2) follows immediately from the Novikov theorem mentioned in the introduction. Thus we need to prove the implication (1) implies (3) only. Let (3) does not hold and G be the graph of f. Let Φ, C, and E be as in Lemma 9 above. As E is compact, the projection of it into L is a homeomorphism. We choose a G_δ subset F of $\{0,1\}^{\mathbb{N}} \times Y$ such that its projection P to C is not Borel. Then the projection B of the set $\Phi(F) = H$ to L is G_δ in L and its image by f equals P.

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