Numerical simulation of compressible flow in time dependent domains

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Goal and motivation

- Modelling of compressible flow in time dependent domains
- Importance of this problem in several domains of human activity
  - Development of airplanes and turbines
  - Some problems of civil engineering
  - Car industry
  - Medicine, etc.
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Continuous problem

- Bounded domain $\Omega_t \subset R^2$, $\partial \Omega_t = \Gamma_I \cup \Gamma_O \cup \Gamma_W$, $t \subset [0, T]$
- Euler equations:
  \[
  \frac{\partial \rho}{\partial t} + \sum_{s=1}^{N} \frac{\partial (\rho v_s)}{\partial x_s} = 0
  \]
  \[
  \frac{\partial (\rho v_i)}{\partial t} + \sum_{s=1}^{N} \frac{\partial (\rho v_i v_s + \delta_{is} p)}{\partial x_s} = 0, \quad i = 1, \ldots, N
  \]
  \[
  \frac{\partial E}{\partial t} + \sum_{s=1}^{N} \frac{\partial ((E + p)v_s)}{\partial x_s} = 0
  \]
- Thermodynamical relation
  \[
  p = (\gamma - 1)(E - \rho \mid \mathbf{v} \mid^2 / 2)
  \]
Euler equations in the conservative form:

\[
\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(\mathbf{w})}{\partial x_s} = 0 \text{ in } \Omega_t, \quad t \subset [0, T]
\]

State vector \( \mathbf{w} \): \( \mathbf{w} = (\rho, \rho v_1, \rho_2, E)^T \)

Flux of the quantity \( \mathbf{w} \) in the direction \( x_s \):

\[
f_s(\mathbf{w}) = (\rho v_s, \rho v_1 v_s + \delta_1 s p, \rho v_2 v_s + \delta_2 s p, (E + p)v_s)^T, \quad s = 1, 2
\]

Transformation of the equation of the pressure:

\[
p = (\gamma - 1) \left( w_4 - \frac{1}{2} \left( \frac{w_2^2}{w_1} + \frac{w_3^2}{w_1} \right) \right)
\]
Euler equations written in the form of first order quasilinear system of partial differential equations:

\[
\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{N} \mathbf{A}_s(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_s} = 0,
\]

where \( \mathbf{A}_s(\mathbf{w}) = \frac{Df_s(\mathbf{w})}{D\mathbf{w}} = \left( \frac{\partial f_{si}(\mathbf{w})}{\partial w_j} \right)_{i,j=1}^4 \)
Regular one-to-one ALE mapping:

\[ A_t : \Omega_{ref} \longrightarrow \Omega_t \]
\[ X \subset \Omega_{ref} \longrightarrow x = x(X, t) = A_t(X) \subset \Omega_t \]
ALE velocity

\[ \tilde{z} : \bar{\Omega}_{\text{ref}} \times (0, T) \rightarrow \mathbb{R}^2 \]

\[ \tilde{z}(X, t) = \frac{\partial}{\partial t} x(X, t) = \frac{\partial}{\partial t} A_t(X) \]

\[ z(x, t) = \tilde{z}(A_t^{-1}(x), t), \quad t \in (0, T), \quad x \in \bar{\Omega}_t \]
ALE derivative

- ALE derivative of a function $f = f(x, t)$, $x \in \Omega_t$, $t \in (0, T)$:

\[
\frac{D^A}{Dt} f(x, t) = \frac{\partial \tilde{f}}{\partial t}(X, t), \quad X = A_t^{-1}(x)
\]

\[
\tilde{f}(X, t) = f(A_t(X), t), \quad X \in \Omega_{ref}, \quad t \in (0, T)
\]

- Two forms of the time derivative of the function $f$

\[
\frac{\partial f}{\partial t} = \frac{D^A}{Dt} f - z \cdot \nabla f,
\]

\[
\frac{\partial f}{\partial t} = \frac{D^A}{Dt} f + f \text{div}(z) - \text{div}(f z)
\]
ALE derivative

- ALE derivative of a function $f = f(x, t), \ x \in \Omega_t, \ t \in (0, T)$:

  $$\frac{D^A}{Dt} f(x, t) = \frac{\partial \tilde{f}}{\partial t}(X, t), \ X = A_t^{-1}(x)$$

  $$\tilde{f}(X, t) = f(A_t(X), t), \ X \in \Omega_{ref}, \ t \in (0, T)$$

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ALE formulations

1. formulation

\[
\frac{DA\mathbf{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial f_s(w)}{\partial x_s} - \sum_{s=1}^{2} z_s \frac{\partial w}{\partial x_s} = 0
\]

2. formulation

\[
\frac{DA\mathbf{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial g_s(w)}{\partial x_s} + w \text{div} z = 0,
\]

where \( g_s(\mathbf{w}) = f_s(\mathbf{w}) - z_s \mathbf{w}, \) \( s = 1, 2 \)
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\]
Discontinuous Galerkin finite element method (DGFEM)

- Partition $T_{ht}$ of $\Omega_{ht}$ consisting of triangles $K_i$, $i \in I$, $\Gamma_{ij} = \partial K_i \cup \partial K_j$
- Finite element space

$$S_{ht} = S^{r,-1}(\Omega_{ht}, T_{ht}) = \{ \varphi_h; \varphi_h|_K \in P^r(K) \quad \forall K \in T_{ht} \}^4$$

- $n_{ij}$ ... unit outer normal to $\partial K_i$ on $\Gamma_{ij}$
- $\varphi_{ij} = \varphi|_{\Gamma_{ij}}$ ... values of $\varphi$ on $\Gamma_{ij}$ from the side of $K_i$
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- $S(i)$ ... set of all indexes $j$ such that $\Gamma_{ij}$ is a face of $K_i$
2. formulation

\[
\sum_{K_i \in T_{ht}} \int_{K_i} \frac{DA}{Dt} \mathbf{w}(t) \cdot \varphi \, dx + b_h(\mathbf{w}, \varphi) + \sum_{K_i \in T_{ht}} \int_{K_i} \text{div} \mathbf{z} (\mathbf{w} \cdot \varphi) \, dx = 0
\]

\[
b_h(\mathbf{w}, \varphi) = - \sum_{K_i \in T_{ht}} \int_{K_i} \sum_{s=1}^{2} g_s(w(t)) \cdot \frac{\partial \varphi}{\partial x_s} \, dx
\]

\[
+ \sum_{K_i \in T_{ht}} \sum_{j \in S_t(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} g_s(w(t)) (n_{ij})_s \cdot \varphi \, dS
\]

Approximation of fluxes

\[
\int_{\Gamma_{ij}} \sum_{s=1}^{2} g_s(w) (n_{ij})_s \cdot \varphi \, dS \approx \int_{\Gamma_{ij}} H_g(w_h(t)|_{\Gamma_{ij}}, w_h(t)|_{\Gamma_{ji}}, n_{ij}) \cdot \varphi \, dS
\]
Semi-implicit time discretization

- Partition 0 = $t_0 < t_1 < t_2 < \ldots$ of the interval $(0, T)$, $\tau_k = t_{k+1} - t_k$
- ALE derivative - the first order backward difference

$$\frac{D^A w_h}{Dt}(x, t_{k+1}) \approx \frac{w^{k+1}(x) - \hat{w}_h^k(x)}{\tau_k}, \quad x \in \Omega_{ht_{k+1}}$$

$$\hat{w}_h^j(x) = w^j \left( A_{t_j} \left( A_{t_k+1}^{-1} \right)(x) \right), \quad x \in \Omega_{t_{k+1}}$$
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\]
Linearization of the term $b_h(w, \varphi)$:

$$\sum_{K_i \in T_{ht}} \int_{K_i} \sum_{s=1}^{2} g_s(w(t)) \cdot \frac{\partial \varphi}{\partial x_s} dx \approx$$

$$\approx \sum_{K_i \in T_{htk+1}} \int_{K_i} \sum_{s=1}^{2} \left( A_s (\hat{w}_{h}^k(x)) - z_s I \right) w_{h}^{k+1}(x) \cdot \frac{\partial \varphi_{h}(x)}{\partial x_s} dx,$$
\[
\sum_{K_i \in T_{ht}} \sum_{j \in S_t(i)} \sum_{s=1}^{2} \int_{\Gamma_{ij}} f_s(w(t))(n_{ij})_s \cdot \varphi \, dS \approx \\
\approx \sum_{K_i \in T_{ht_k+1}} \sum_{j \in S_{t_k+1}(i)} \int_{\Gamma_{ij}} \left[ \mathbb{P}^+ \left( \frac{\hat{w}_h^k|_{\Gamma_{ij}} + \hat{w}_h^k|_{\Gamma_{ji}}}{2}, n_{ij} \right) w_h^{k+1}|_{\Gamma_{ij}} \right. \\
\left. + \mathbb{P}^- \left( \frac{\hat{w}_h^k|_{\Gamma_{ij}} + \hat{w}_h^k|_{\Gamma_{ji}}}{2}, n_{ij} \right) w_h^{k+1}|_{\Gamma_{ji}} \right] \cdot \varphi_h \, dS,
\]

where \( \mathbb{P}^+ \) and \( \mathbb{P}^- \) are positive and negative part of the matrix

\[
\sum_{s=1}^{2} \left( \mathbb{A}_s(\hat{w}_h^k(x)) - z_s I \right) n_s
\]
Remaining terms are treated implicitly.
Boundary conditions

- **Impermeability condition** $\mathbf{v} \cdot \mathbf{n} = \mathbf{z} \cdot \mathbf{n}$

- Choice of the boundary conditions is governed by the sign of the eigenvalues of the matrix

$$\mathbb{P}(\mathbf{w}, \mathbf{n}) = \sum_{s=1}^{2} \frac{Dg_s(\mathbf{w})}{D\mathbf{w}} n_s = \sum_{s=1}^{2} (A_s n_s - z_s n_s I)$$

- The number of prescribed quantities on the boundary is equal to the number of negative eigenvalues
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Numerical experiment I

- Rectangular channel $[-2, 2] \times [0, 1]$ with the moving lower wall
- ALE mapping
  - Identity in the sets $[-2, -1] \times [0, 1]$ and $[1, 2] \times [0, 1]$
  - The lower wall - graph of the function
    \[
    \alpha \sin 0.4t \left( \cos(\pi X_1) + 1 \right), \quad X_1 \in (-1, 1)
    \]
- Interpolation to the rest of the domain
- Coefficient $\alpha = 0.45$
Velocity isolines
Numerical experiment II

- Rectangular channel $[-4, 4] \times [0, 1]$ with the moving lower wall
- ALE mapping
  - Identity in the sets $[-4, -1] \times [0, 1]$ and $[1, 4] \times [0, 1]$
  - The lower wall - graph of the function
    \[
    \alpha \sin 0.4t \left( \cos(\pi X_1) + 1 \right), \quad X_1 \in (-1, 1)
    \]
- Interpolation to the rest of the domain
- Coefficient $\alpha = 0.45$
Pressure isolines

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Results

- The use of the first formulation is limited by the height of the closure of the channel
- The second formulation is more suitable because it allows higher closure of the channel
- Complete closure is not possible because of degeneration of elements of the computational mesh
- Presented results show the robustness of the second ALE formulation
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Future work

- Extension to viscous flow - the solution of the compressible Navier-Stokes equations
- Simulation of the flow in a complicated domain modelling human vocal folds
- Fluid-structure interaction
Thank you for your attention!