LOOPS THAT ARE CONJUGACY CLOSED MODULO THE CENTER

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Abstract. We prove that a loop Q has to be conjugacy closed modulo $Z(Q)$ whenever all mappings $L(x,y)$ and $R(x,y)$ are automorphisms, $N(Q) \trianglelefteq Q$, $Q/N(Q)$ is an abelian group, and $\langle L_x;\, x \in Q\rangle$ is a normal subgroup of the multiplication group.

A binary system $Q = Q(\cdot, 1)$ is called a *loop* if $a \cdot 1 = a = 1 \cdot a$ for all $a \in Q$ and if the equations $ax = b$ and $ya = b$ have unique solutions $x = a\backslash b$ and $y = b/a$ for all $a, b \in Q$.

The permutation group generated by all left translations $L_a : x \mapsto ax$ is called the *left multiplication group* and we shall denote it by $\mathcal{L} = \mathcal{L}(Q)$. Together with they *right multiplication group* $\mathcal{R} = \mathcal{R}(Q)$ they generate the *multiplication group* $\operatorname{Mlt} Q$ of the loop Q.

Call $N_\lambda = N_\lambda(Q) = \{a \in Q;\, a(xy) = (ax)y \text{ for all } x, y \in Q\}$ the *left nucleus* of Q. The *middle* and the *right* nuclei N_μ and N_ρ are obtained by shifting a to the right. Each of them forms a subloop, but this subloop does not have to be normal. This is also true for the *nucleus* $N = N(Q) = N_\lambda \cap N_\mu \cap N_\rho$.

A loop Q is called *left conjugacy closed* (LCC) if $L_a L_b L_a^{-1}$ is equal to some L_c, for all $a, b \in Q$. A loop that is both LCC and RCC is called *conjugacy closed* (CC).

CC loops received a lot of attention recently and the concept of conjugacy closedness emerges as one of fundamental loop-theoretic notions, with important applications. CC loops are known to satisfy these properties:

$$\mathcal{L} \trianglelefteq \operatorname{Mlt} Q,\ \mathcal{R} \trianglelefteq \operatorname{Mlt} Q,\ \mathcal{L}_1 \cup \mathcal{R}_1 \le \operatorname{Aut} Q,\ N \trianglelefteq Q \text{ and}$$
$$Q/N \text{ is an abelian group.}$$

The purpose of this paper is to show that a loop with such properties is conjugacy closed modulo the *center* $Z(Q) = \{a \in N(Q);\, ax = xa \text{ for all } a \in Q\}$.

There are rather few results in loop theory that connect a structural behaviour to an equational law. In fact the only comparable result we are aware of is that of Kinyon, Kunen and Phillips which states that diassoaciative A-loops are Moufang [14].

Section 1 is concerned with various general preliminaries. Some standard facts are reproved in order to make the paper self-contained with respect to its central statements. Section 2 starts with the well known connection of N_λ and $C_{\operatorname{Mlt} Q}\mathcal{R}$,

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and continues by developing a one-sided version of the main result. This section contains the bulk of our arguments. In Section 3 we apply the results of Section 2 to a situation when the assumptions are left-right symmetric and mention some applications.

The concept of conjugacy closedness goes back to Soikis [18] and to Goodaire and Robinson [12] who proved many of the basic facts. Some of them can be transferred to the one sided situation of, say, the LCC loops, see [17, 8, 9]. The recent history of CC loops [7, 15, 16, 10] unwinds from the discovery of Basarab [1] that isotopically invariant LCC loops are abelian groups modulo the left nucleus. Thus Q/N is an abelian group for every CC loop Q, and this is the only difficult property of the conjugacy closed loops among those that are listed above. We shall need very few further facts about conjugacy closedness in this paper, and practically every paper mentioned above can be used as a reference source for the needed basic properties.

The standard books on loop theory are [3] and [2].

1. Preliminaries

We start by listing several consequences of the limited associativity induced by the presence of an element in one of the nuclei.

Lemma 1.1. *Let a and b be elements of a loop Q. Then:*

(i) *if $a\backslash b \in N_\mu$, then $L_a^{-1}L_b = L_{a\backslash b} = L_{b\backslash a}^{-1}$;*
(ii) *if $b/a \in N_\mu$, then $R_a^{-1}R_b = R_{b/a} = R_{a/b}^{-1}$;*
(iii) *if $b/a \in N_\lambda$, then $L_bL_a^{-1} = L_{b/a} = L_{a/b}^{-1}$; and*
(iv) *if $a\backslash b \in N_\rho$, then $R_bR_a^{-1} = R_{a\backslash b} = R_{b\backslash a}^{-1}$.*

Proof. Assume $a\backslash b \in N_\mu$. Then $a((a\backslash b)x) = (a(a\backslash b))x = bx$ for all $x \in Q$, and so $L_aL_{a\backslash b} = L_b$ and $L_{a\backslash b} = L_a^{-1}L_b$. Set $c = a\backslash b$. From $c \in N_\mu$ we obtain that $L_c^{-1} = L_{c^{-1}}$, and so it suffices to prove $c(b\backslash a) = 1$. However, that follows from $a = a(a\backslash b)(b\backslash a)$. The other cases are similar. □

Lemma 1.2. *Assume $\varphi \in \operatorname{Aut} Q$. Then $\varphi L_a = L_{\varphi(a)}\varphi$ and $\varphi R_a = R_{\varphi(a)}\varphi$ for all $a \in Q$.*

Proof. Indeed, $\varphi L_a(x) = \varphi(ax) = \varphi(a)\varphi(x) = L_{\varphi(a)}\varphi(x)$ for all $x \in Q$. □

Corollary 1.3. *Assume $\varphi \in \operatorname{Aut} Q$ and $a \in Q$. Then*

(i) $[\varphi, L_a] = L_{\varphi^{-1}(a)\backslash a}$ *if $a\backslash\varphi(a) \in N_\mu$;*
(ii) $[\varphi, R_a] = R_{a/\varphi^{-1}(a)}$ *if $\varphi(a)/a \in N_\mu$;*
(iii) $[\varphi, L_a^{-1}] = L_{\varphi^{-1}(a)/a}$ *if $a/\varphi(a) \in N_\lambda$; and*
(iv) $[\varphi, R_a^{-1}] = R_{a\backslash\varphi^{-1}(a)}$ *if $\varphi(a)\backslash a \in N_\rho$.*

Proof. To get point (i) we need to show $L_{\varphi^{-1}(a)}^{-1}L_a = L_{\varphi^{-1}(a)\backslash a}$. However, that follows from point (i) of Lemma 1.1 since $\varphi^{-1}(a)\backslash a$ belongs to N_μ if and only if $a\backslash\varphi(a) = \varphi(\varphi^{-1}(a)\backslash a)$ does. The rest is similar. □

A loop Q is said to be an *A_ℓ-loop* if all elements of $\mathcal{L}_1 = \{\varphi \in \mathcal{L};\ \varphi(1) = 1\}$ are automorphisms. The group $\mathcal{L}_1$ is generated by mappings $L(x, y) = L_{xy}^{-1}L_xL_y$, and hence Q is an A_ℓ-loop if and only if $L(x, y) \in \operatorname{Aut} Q$ for all $x, y \in Q$. Similarly one defines A_r-loops and sets $R(x, y) = R_{yx}^{-1}R_xR_y$.

Lemma 1.4. *Let Q be a loop with elements x and y. Then:*

(i) $L(ax, y) = L(x, y)$ *and* $[L_x, R_y] = [L_{ax}, R_y]$ *for all* $a \in N_\lambda$*; and*
(ii) $R(x, yb) = R(x, y)$ *and* $[L_x, R_y] = [L_x, R_{yb}]$ *for all* $b \in N_\rho$.

Proof. If $x(yu) = (xy)v$, then $(ax)(yu) = (ax \cdot y)v$ for all $a \in N_\lambda$, and so $L(x, y) = L(ax, y)$ for every $a \in N_\lambda$. The other cases are similar. □

Lemma 1.5. *Let Q be a loop with elements x and y. Suppose that $a \in Q$ belongs to a normal subloop $S \leq N_\lambda \cap N_\mu$. Then $L(x, y) = L(ax, y) = L(x, ay)$.*

Proof. We only need to show $L(x, ay) = L(x, y)$, which means to prove that $x(yu) = (xy)v$ implies $x(ay \cdot u) = (x \cdot ay)v$. By our assumption there exists $a' \in S$ with $xa = a'x$, and hence $x((ay)u) = (xa)(yu) = (a'x)(yu) = a'(x(yu)) = a'(xy \cdot v) = ((a'x)y)v = (x(ay))v$. □

Denote by $A(Q)$ the least normal subloop A such that Q/A is a group. Both $\mathcal{L}_1$ and $\mathcal{R}_1$ act within the cosets of $A(Q)$. Denote by $\operatorname{Inn} Q$ the group $\{\varphi \in \operatorname{Mlt} Q;\ \varphi(1) = 1\}$, and call it the *inner mapping group*. Note that each $[L_x, R_y]$ belongs to $\operatorname{Inn} Q$ and acts within the cosets of $A(Q)$ as well.

Proposition 1.6. *Let Q be an A_ℓ-loop such that $A(Q) \leq N_\lambda \cap N_\mu$. Then $\mathcal{L}_1$ is commutative. If in addition $A(Q) \leq N_\rho$ and $[L_x, R_y] \in \operatorname{Aut} Q$ for all $x, y \in Q$, then the set $\{L(x, y), [L_x, R_y];\ x, y \in Q\}$ generates a commutative subroup of* $\operatorname{Inn} Q$.

Proof. If $\varphi \in \operatorname{Aut} Q$, then $\varphi L(x, y)\varphi^{-1} = L(\varphi(x), \varphi(y))$ and $\varphi[L_x, R_y]\varphi^{-1} = [L_{\varphi(x)}, R_{\varphi(y)}]$, by Lemma 1.2. The statement thus follows from Lemmas 1.4 and 1.5, with $S = A(Q)$. □

LCC loops are known to be A_ℓ-loops, they satisfy $N_\lambda = N_\mu$, and N_λ is always a normal subloop. Let Q be an LCC loop. If all loop isotopes of Q are also LCC, then Q/N_λ is an abelian group. As we have already mentioned, this fundamental result belongs to Basarab [1], and the converse assertion (which is much easier) is also true [8]. We say that Q is an *isotopically invariant* LCC loop (or universal LCC loop) if all loop isotopes of Q are LCC.

If U is a subset of Q, then by $L_{(U)}$ we denote the set $\{L_x;\ x \in U\}$. We now turn to one of our key observations:

Lemma 1.7. *Let Q be a loop such that $[L_x, L_y] \in L_{(N_\mu)}$ for all $x, y \in Q$. Then Q is an isotopically invariant LCC loop.*

Proof. We assume that each $L_x^{-1}L_y^{-1}L_xL_y$ is equal to some L_a, $a \in N_\mu$. But that means $L_y^{-1}L_xL_y = L_xL_a = L_{xa}$. □

We shall be using T_x as a shortcut for $R_x^{-1}L_x$. A loop Q is clearly LCC if and only if $L_xL_yL_x^{-1} = L_{T_x(y)}$, which can be equivalently expressed as $L_x^{-1}L_yL_x = L_{T_x^{-1}(y)}$, for all $x, y \in Q$.

Lemma 1.8. *Let Q be a loop with a normal subloop $S \leq N_\lambda \cap N_\mu$. Then $L_xL_aL_x^{-1} = L_{T_x(a)}$ for all $a \in S$ and $x \in Q$.*

Proof. Clearly, $L_xL_a(y) = (xa)y = (T_x(a)x)y = T_x(a)(xy) = L_{T_x(a)}L_x(y)$. □

Proposition 1.9. *For a loop Q the following conditions are equivalent:*

(i) $[L_x, L_y] \in L_{(N_\lambda \cap N_\mu)}$ *for all* $x, y \in Q$*;*
(ii) $\mathcal{L}' \le L_{(N_\lambda \cap N_\mu)}$*; and*
(iii) Q *is an isotopically invariant left conjugacy closed loop.*

Proof. Point (i) implies point (iii) by Lemma 1.7, and (ii) $\Rightarrow$ (i) trivially. If Q is LCC, then $N_\lambda = N_\mu \trianglelefteq Q$, and thus $L_{(N_\lambda)} \trianglelefteq \mathcal{L}$, by Lemma 1.8. To get a proof of (iii) $\Rightarrow$ (ii) it hence suffices to prove (iii) $\Rightarrow$ (i). If point (iii) holds, then Q/N_μ is an abelian group, and therefore $x\backslash T_y^{-1}(x) \in N_\mu$ for all $x, y \in Q$. Thus $[L_x, L_y] = L_x^{-1} L_{T_y^{-1}(x)} = L_{x\backslash T_y^{-1}(x)}$, by Lemma 1.1. □

We finish this section by a well known observation that will be needed later.

Lemma 1.10. *Let* Q *be a loop and put* $C(Q) = \{a \in Q;\ xa = ax \text{ for all } x \in Q\}$*. Then* $Z(Q) = C(Q) \cap N_\lambda \cap N_\mu = C(Q) \cap N_\mu \cap N_\rho = C(Q) \cap N_\lambda \cap N_\rho$*.*

Proof. Let $a \in Q$ belong to $C(Q) \cap N_\lambda \cap N_\mu$. Then $(xy)a = a(xy) = (ax)y = (xa)y = x(ay) = x(ya)$ for all $x, y \in Q$, and so $a \in N_\rho$. The rest is similar. □

2. Left conjugacy closedness

Denote by $\mathrm{Sym}\, Y$ the symmetric group acting on a set Y. If Q is a group, then the centralizer $C_{\mathrm{Sym}\, Q}(\mathcal{R})$ equals $\mathcal{L}$. This carries to loops in the following form:

Lemma 2.1. *Let* Q *be a loop. Then* $C_{\mathrm{Sym}\, Q}(\mathcal{R}) = C_{\mathrm{Mlt}\, Q}(\mathcal{R}) = L_{(N_\lambda)}$*.*

Proof. A mapping $\psi \in \mathrm{Sym}\, Q$ centralizes $\mathcal{R}$ if and only if $\psi(xy) = \psi(x)y$ for all $x, y \in Q$. Setting $a = \psi(1)$ yields $\psi(y) = L_a(y)$ and $a(xy) = (ax)y$. The rest is clear. □

Corollary 2.2. *Let* Q *be a loop. If* $\mathcal{L}(Q) \trianglelefteq \mathrm{Mlt}\, Q$*, then* $N_\rho \trianglelefteq Q$*. If* $\mathcal{R}(Q) \trianglelefteq \mathrm{Mlt}\, Q$*, then* $N_\lambda \trianglelefteq Q$*.*

Proof. It is easy to see (and well known) that a subloop S of a loop Q is normal if and only if it is a block of $\mathrm{Mlt}\, Q$. Blocks of permutation groups can be often obtained as orbits of a normal subgroup. A centralizer of a normal subgroup is a normal subgroup, and so $C_{\mathrm{Mlt}\, Q}\mathcal{R}$ induces a normal subloop of Q whenever the right multiplication group $\mathcal{R} = \langle R_x;\ x \in Q\rangle$ is a normal subgroup of $\mathrm{Mlt}\, Q$. □

It is well known (and can be easily obtained from Lemmas 2.1 and 1.10) that $Z(\mathrm{Mlt}\, Q)$ is equal to $L_{(Z(Q))} = R_{(Z(Q))}$.

Proposition 2.3. *Let* Q *be a loop with* $N_\lambda \cap N_\mu \trianglelefteq Q$ *such that* $Q/(N_\lambda \cap N_\mu)$ *is an abelian group. If* $[L_x, L_y]$ *centralizes* R_z *for all* $x, y, z \in Q$*, then* Q *is LCC. If* $[[L_x, L_y], R_z] \in Z(\mathrm{Mlt}\, Q)$ *for all* $x, y, z \in Q$*, then* $Q/Z(Q)$ *is an LCC loop.*

Proof. By assuming $[L_x, L_y] \in C_{\mathrm{Mlt}\, Q}(\mathcal{R})$ we in fact assume that $[L_x, L_y]$ is equal to some L_a, $a \in N_\lambda$, by Lemma 2.1. Then also $a \in N_\mu$, by the assumptions of the proposition. Hence our first statement follows from Proposition 1.9, and the application to the quotient loop $Q/Z(Q)$ yields the second one. □

Lemma 2.4. *Let Q be an A_ℓ-loop with $N_\lambda \cap N_\mu \trianglelefteq Q$ such that $Q/(N_\lambda \cap N_\mu)$ is an abelian group. Then for all $x, y \in Q$ there exist $a, b \in N_\lambda \cap N_\mu$ and $\varphi \in \mathcal{L}_1$ such that*

$$L_y L_x L_y^{-1} = L_x L_a \varphi \text{ and } L_y^{-1} L_x L_y = L_x L_b \varphi^{-1}.$$

Proof. Every element of $\mathcal{L}'$ is clearly of the form $L_a\varphi$, for some $a \in N_\lambda \cap N_\mu$ and $\varphi \in \mathcal{L}_1$ (for that one does not have to assume that $\mathcal{L}_1 \leq \operatorname{Aut} Q$). From $L_x L_a \varphi = L_y L_x L_y^{-1}$ we get $L_x = L_y^{-1} L_x L_a \varphi L_y = (L_y^{-1} L_x L_y)(L_y^{-1} L_a L_y)\varphi[\varphi, L_y]$, which is equal to $L_x^{L_y} L_{T_y^{-1}(a)} \varphi L_{\varphi^{-1}(y)\backslash y} = L_x^{L_y} \varphi L_{\varphi^{-1} T_y^{-1}(a)} L_{\varphi^{-1}(y)\backslash y}$, by Corollary 1.3 and Lemmas 1.2 and 1.8. We see that one can get b as the inverse of $\varphi^{-1} T_y^{-1}(a) \cdot (\varphi^{-1}(y)\backslash y)$. □

Lemma 2.5. *Let Q be an A_ℓ-loop with $N_\lambda \cap N_\mu \trianglelefteq Q$ such that $Q/(N_\lambda \cap N_\mu)$ is an abelian group. Let $x, y \in Q$ and $b \in N_\lambda \cap N_\rho$ be such that $L_y^{-1} L_x L_y = L_x L_b \varphi^{-1}$ for some $\varphi \in \mathcal{L}_1$. Then $[[L_x, L_y], R_z] = R_{z/\varphi(z)}$ and $[[L_y, L_x], R_z] = R_{z/\varphi^{-1}(z)} = R^{\varphi}_{\varphi(z)/z}$, for all $z \in Q$.*

Proof. From Corollary 1.3 and Lemma 2.1 we obtain $[[L_x, L_y], R_z] = [L_b\varphi^{-1}, R_z] = \varphi L_b^{-1} R_z^{-1} L_b \varphi^{-1} R_z = \varphi R_z^{-1} \varphi^{-1} R_z = [\varphi^{-1}, R_z] = R_{z/\varphi(z)}$. In a similar manner, $[[L_y, L_x], R_z] = [\varphi L_b^{-1}, R_z] = L_b \varphi^{-1} R_z^{-1} \varphi L_b^{-1} R_z = L_b[\varphi, R_z] L_b^{-1} = R_{z/\varphi^{-1}(z)}$. □

Lemma 2.6. *Let Q be an A_ℓ-loop with $N_\lambda \cap N_\mu \trianglelefteq Q$ such that $Q/(N_\lambda \cap N_\mu)$ is an abelian group. Suppose that $\{L(x,y), [L_x, R_y];\ x, y \in Q\}$ generates a commutative subgroup of $\operatorname{Aut} Q$. Choose $x, y \in Q$ and put $\varphi = [L_y, L_x] L_{x\backslash(y\backslash(xy))}$. Then $\varphi(z)/z \in Z(Q)$ for all $z \in Q$.*

Proof. Put $b = x\backslash(y\backslash(xy))$. Then $[L_x, L_y] = L_b \varphi^{-1}$, and so $L_y L_x L_y^{-1} = L_x L_a \varphi$ for some $a \in N_\lambda \cap N_\mu$, by Lemma 2.4. We know that $L_x^{[R_z, L_y]}$ is equal to $L_{[L_y, R_z](x)}$, by Lemma 1.2. Our strategy is to compute $L_x^{[R_z, L_y]}$ in another way, and then to compare both results.

By Corollary 1.3, $\varphi^{R_z} = \varphi[\varphi, R_z] = \varphi R_m$, where $m = z/\varphi^{-1}(z)$, and therefore we easily get $L_x^{[R_z, L_y]} = (L_x [R_z, L_x]^{R_z^{-1}})^{L_y^{-1} R_z L_y} = (L_y L_x L_y^{-1})^{R_z L_y} [R_z, L_x] = (L_x L_a \varphi)^{R_z L_y} [R_z, L_x] = (L_x [L_x, R_z] L_a \varphi R_m)^{L_y} [R_z, L_x]$. Define $c, d \in N_\lambda \cap N_\mu$ by $[L_x, R_z](a) = ac$ and $d = \varphi^{-1}(c)$. Then $[L_x, R_z] L_a \varphi$ is equal to $L_{ac}[L_x, R_z]\varphi = L_a L_c \varphi [L_x, R_z] = L_a \varphi L_d [L_x, R_z]$. We can thus express $L_x^{[R_z, L_y]}$ as

$$(L_x L_a \varphi)^{L_y} L_d^{L_y} ([L_x, R_z] R_m [R_z, L_x])^{L_y} [L_y, [R_z, L_x]].$$

Now, $(L_x L_a \varphi)^{L_y} = L_x$ as $L_x L_a \varphi = L_y L_x L_y^{-1}$. The terms $L_d^{L_y}$, $[L_y, [R_z, L_x]]$ and $L_x^{-1} L_x^{[R_z, L_y]} = [L_x, [R_z, L_y]]$ belong to $L_{(N_\lambda \cap N_\mu)}$, by Corollary 1.3 and Lemma 1.8. The permutation $R_m^{[R_z, L_x] L_y}$ therefore belongs to this set as well, and so do $R_m^{[R_z, L_x]}$ and R_m, by Lemmas 1.2 and 1.8. That means that R_m is equal to some left translation, and hence it has to be equal to L_m. Since m belongs to $N_\lambda \cap N_\mu$, it has to belong to $Z(Q)$, by Lemma 1.10, and thus $\varphi(m) = \varphi(z)/z \in Z(Q)$, as required. □

Theorem 2.7. *Let Q be a loop with $N_\lambda \cap N_\mu \trianglelefteq Q$ such that $Q/(N_\lambda \cap N_\mu)$ is an abelian group. Suppose that $\{L(x,y), [L_x, R_y];\ x,y \in Q\}$ generates a commutative subgroup of $\operatorname{Aut} Q$. For all $x,y,z \in Q$ define $a = a(x,y,z)$ by $z = a \cdot [L_y, L_x]L_{x\backslash(y\backslash(xy))}(z)$. Then $a \in Z(Q)$ and $[[L_x, L_y], R_z] = R_a$. Furthermore, $Q/Z(Q)$ is left conjugacy closed.*

Proof. Put $\varphi = [L_y, L_x]L_{x\backslash(y\backslash(xy))}$. Then $\varphi(z)/z \in Z(Q)$, by Lemma 2.6, and $a = z/\varphi(z) \in Z(Q)$ too, by Lemma 1.1. Therefore $[[L_x, L_y], R_z] = R_a \in Z(\operatorname{Mlt} Q)$, by Lemma 2.5, and $Q/Z(Q)$ is left conjugacy closed by Proposition 2.3. □

Corollary 2.8. *Let Q be an A_ℓ-loop in which the nucleus N is normal, and Q/N is an abelian group. If $[L_x, R_y] \in \operatorname{Aut} Q$ for all $x, y \in Q$, then $Q/Z(Q)$ is left conjugacy closed.*

Proof. Use Theorem 2.7 and Proposition 1.6. □

Corollary 2.9. *Let Q be an A_ℓ-loop such that $\mathcal{L}(Q) \trianglelefteq \operatorname{Mlt} Q$. If $N_\lambda \cap N_\mu \trianglelefteq Q$ and if $Q/(N_\lambda \cap N_\mu)$ is an abelian group, then $Q/Z(Q)$ is left conjugacy closed.*

Proof. If $\mathcal{L}(Q) \trianglelefteq \operatorname{Mlt} Q$, then $[L_x, R_y] \in \mathcal{L}_1$ for all $x, y \in Q$, and $\mathcal{L}_1$ is commutative, by Proposition 1.6. □

3. The main result and applications

Theorem 3.1. *Let Q be a loop such that $N(Q) \trianglelefteq Q$, $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$ and $\mathcal{L}_1 \cup \mathcal{R}_1 \subseteq \operatorname{Aut} Q$. If $Q/N(Q)$ is an abelian group, then $Q/Z(Q)$ is conjugacy closed.*

Proof. Corollary 2.9 provides the left conjugacy closedness, and the rest follows by the mirror argument. □

Say that a loop Q is an $A_{r,\ell}$*-loop* if it is both an A_ℓ-loop and an A_r-loop. Q is thus an $A_{r,\ell}$-loop if and only if $\mathcal{L}_1 \cup \mathcal{R}_1 \subseteq \operatorname{Aut} Q$, and that takes place if and only if $L(x,y)$ and $R(x,y)$ are automorphims for all $x, y \in Q$. From Corollary 2.8 we directly obtain:

Proposition 3.2. *Let Q be an $A_{r,\ell}$-loop in which the nucleus N is normal, and Q/N is an abelian group. If $[L_x, R_y] \in \operatorname{Aut} Q$ for all $x, y \in Q$, then $Q/Z(Q)$ is conjugacy closed.*

This proposition offers an immediate consequence that is in fact slightly stronger than Theorem 3.1.

Corollary 3.3. *Let Q be an $A_{r,\ell}$-loop in which the nucleus N is normal, and Q/N is an abelian group. If at least one of groups $\mathcal{L}(Q)$ and $\mathcal{R}(Q)$ is normal in $\operatorname{Mlt} Q$, then $Q/Z(Q)$ is conjugacy closed.*

LCC loops are always A_ℓ-loops. The structure of LCC A_r-loops is not known, but some constructions [5] suggest that they are not far from CC loops. This surmise gets a support from the following fact:

Proposition 3.4. *Let Q be an isotopically invariant LCC loop. If Q is an A_r-loop with $N_\rho \trianglelefteq Q$, such that Q/N_ρ is an abelian group, then $Q/Z(Q)$ is a conjugacy closed loop.*

Proof. By our assumptions both $Q/(N_\lambda \cap N_\mu)$ and Q/N_ρ are abelian groups. Hence Q/N is an abelian group too. Furthermore, $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$ in every LCC loop, and so Corollary 3.3 can be used. □

Buchsteiner loops [4] are those loops that satisfy the *Buchsteiner law* $x\backslash(xy\cdot z) = (y\cdot zx)/x$. Every Buchsteiner loop Q is an $A_{r,\ell}$ loop with $N \trianglelefteq Q$, $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$ and Q/N an abelian group [6]. Hence we have:

Theorem 3.5. *Let Q be a Buchsteiner loop. Then $Q/Z(Q)$ is a conjugacy closed loop.*

For some time it was not clear if there exist any Buchsteiner loops that are not conjugacy closed, and Theorem 3.5 was obtained while solving the problem of their existence. The arguments used in the original proof of Theorem 3.5 were later generalized, and that is how this paper originated. (An example of a non-CC Buchsteiner loops can be found in [6].)

Left Bol loops are those loops that satisfy the *left Bol law* $x(y(xz)) = (x(yx))z$. If Q is a left Bol loop, then $N_\lambda = N_\mu \trianglelefteq Q$ and $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$.

A triple of permutations (α, β, γ) is said to be an *autotopism* of Q if $\alpha(x)\beta(y) = \gamma(xy)$ for all $x, y \in Q$. If $\beta(1) = 1$, then $\alpha = \gamma = L_{\alpha(1)}\beta$, and one can easily verify the well known fact that in such a case $\beta \in \operatorname{Aut} Q$ if and only if and $\alpha(1) \in N_\lambda$.

Lemma 3.6. *Let Q be a left Bol loop. If Q/N_λ is an abelian group, then Q is an A_l-loop.*

Proof. A loop Q is a left Bol loop if and only if (L_xR_x, L_x^{-1}, L_x) is an autotopism for all $x \in Q$. Therefore $(\alpha, L_{xy}^{-1}L_xL_y, L_{xy}L_x^{-1}L_y^{-1})$ is an autotopism too, where $\alpha = L_{xy}R_{xy}R_x^{-1}L_x^{-1}R_y^{-1}L_y^{-1} = L_{xy}L_x^{-1}L_y^{-1}$. Thus $L(x,y) \in \operatorname{Aut} Q$ if and only if $(xy)(x\backslash(y\backslash 1)) \in N_\lambda$. However, that follows from our assumptions. □

Proposition 3.7. *Let Q be a left Bol loop such that Q/N_λ is an abelian group. If $\mathcal{L}(Q) \trianglelefteq \operatorname{Mlt} Q$, then $x^2(yz) \equiv (x^2y)z \bmod Z(Q)$ for all $x, y, z \in Q$. This is also true when Q is an A_r-loop, with $N_\rho \trianglelefteq Q$ and Q/N_ρ an abelian group.*

Proof. A left Bol loop is LCC if and only if $x^2 \in N_\lambda$ for all $x \in Q$ [17]. If $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, use Lemma 3.6 and Corollary 2.9. In the other case one can use Lemma 3.6 and Corollary 3.3 since $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$ in every left Bol loop. □

Theorem 3.8. *Let Q be a Moufang loop such that $Q/N(Q)$ is an abelian group. Then $Q/Z(Q)$ is an extra loop.*

Proof. Such a loop Q is an $A_{r,\ell}$-loop, by Lemma 3.6 and by its mirror version. Hence $Q/Z(Q)$ is a CC loop, by Theorem 3.1. A Moufang loop is a CC-loop if and only if it is an extra loop [13]. □

Extra loops are those Moufang loops in which each square belongs to the nucleus [11]. Extra loops of odd order are necessarilly groups, and so Moufang loops of odd order are groups modulo the center whenever they are abelian groups modulo the nucleus.

These facts do not seem to have been noted before. It looks plausible that there will be found alternative proofs which will use standard tools of Moufang loops, i. e. the calculus of commutators and associators. Let us remark that the nucleus was used rather rarely as a tool for investigating Moufang loops. We do not know, for example, if there exists a Moufang loop of odd order with a trivial nucleus.

A loop Q is said to be an *A-loop* if $\operatorname{Inn} Q \le \operatorname{Aut} Q$. If Q is an LCC loop, then $L_x^{-1}R_x \in \operatorname{Aut} Q$ if and only if $x \in N_\rho$. LCC A-loops thus coincide with groups. We can hence conclude by

Theorem 3.9. *Let Q be an A-loop such that $Q/N(Q)$ is an abelian group. Then $Q/Z(Q)$ is a group.*

References

[1] A. S. Basarab: Klass LK-lup, *Mat. Issled.* **120** (1991), 3–7, 118.

[2] V. D. Belousov, *Osnovy teorii kvazigrupp i lup*, Nauka, Moskva, 1967.

[3] R. H. Bruck: *A survey of binary systems*, Springer Verlag, 1958.

[4] H. H. Buchsteiner: O nekotorom klasse binarnych lup, *Mat. Issled.* **39** (1976), 54–66.

[5] P. Csörgő and A. Drápal: On left conjugacy closed loops in which the left multiplication group is normal, *Abh. Math. Sem. Univ. Hamburg* **76** (2006) (in press).

[6] P. Csörgő, A. Drápal and M. Kinyon: Buchsteiner loops (in preparation).

[7] A. Drápal: Conjugacy closed loops and their multiplication groups, *J. Algebra* **272** (2004), 838–850.

[8] A. Drápal: On multiplication groups of left conjugacy closed loops, *Comment. Math. Univ. Carolin.* **45** (2004), 223–236.

[9] A. Drápal: On left conjugacy closed loops with a nucleus of index two, *Abh. Math. Sem. Univ. Hamburg* **74** (2004), 205–221.

[10] A. Drápal: On extraspecial left conjugacy closed loops, *J. Algebra* **302** (2006), 771-792.

[11] F. Fenyves: Extra loops. II. On loops with identities of Bol-Moufang type, *Publ. Math. Debrecen* **16** (1969), 187–192.

[12] E. G. Goodaire and D. A. Robinson: A class of loops which are isomorphic to all loop isotopes, *Canad. J. Math.* **34** (1982), 662–672.

[13] E. G. Goodaire and D. A. Robinson: Some special conjugacy closed loops, *Canad. Math. Bull.* **33** (1990), 73–78.

[14] M. K. Kinyon, K. Kunen and J. D. Phillips: Every Diassociative A-loop is Moufang, *Proc. Amer. Math. Soc.* **130** (2002), 619–624.

[15] M. K. Kinyon, K. Kunen and J. D. Phillips: Diassociativity in conjugacy closed loops, *Comm. Algebra* **32** (2004), 767–786.

[16] M. K. Kinyon and K. Kunen: Power-associative, conjugacy closed loops, *J. Algebra* **304** (2006), 679–711.

[17] P. T. Nagy and K. Strambach: Loops as invariant sections in groups, and their geometry, *Canad. J. Math.* **46** (1994), 1027–1056.

[18] L. R. Soikis, O specialnych lupach, in: *Voprosy teorii kvazigrupp i lup* (V. D. Belousov, ed.), Akademia Nauk Moldav. SSR, Kishinev, 1970, pp. 122–131.

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