ON LOOP IDENTITIES THAT CAN BE OBTAINED BY A NUCLEAR IDENTIFICATION

ALEŠ DRÁPAL AND PŘEMYSL JEDLIČKA

Abstract. We start by describing all the varieties of loops Q that can be defined by autotopisms α_x, $x \in Q$, where α_x is a composition of two triples, each of which becomes an autotopism when the element x belongs to one of the nuclei. In this way we obtain a unifying approach to Bol, Moufang, extra, Buchsteiner and conjugacy closed loops. We reprove some classical facts in a new way and show how Buchsteiner loops fit into the traditional context. In Section 6 we describe a new class of loops with coincinding left and right nuclei. These loops have remarkable properties and do not belong to any of the classical classes.

We start this paper by investigating interactions of loop nuclei and loop identities via loop autotopisms. We shall observe that this is a natural way how to obtain nearly all loop varieties that have been studied in the past. To be more exact, we shall get in this way Moufang loops, left Bol loops, right Bol loops, extra loops, left conjugacy closed (LCC) loops, RCC loops and Buchsteiner loops. No other class of loops can be obtained by the method.

In all these varieties the middle nucleus has to coincide with the left or the right nucleus. Furthermore, there exists no finite loop with trivial right nucleus and coinciding left and middle nuclei that would be of index two. However, there exist finite loops with coinciding left and middle nuclei that are of index two, in which the right nucleus is of order two. All such loops have to be left conjugacy closed. In Section 6 we shall construct a class of finite loops in which the left and right nuclei are coinciding and of index two, and the middle nucleus is of order two. Such loops clearly belong to none of the varieties mentioned above. After computing their multiplication groups we shall see that in many cases the inner mapping group is of order pq. Loops of this kind have been studied intensively in the past from the point of solvability of the multiplication group. This paper seems to describe the first known examples of such loops in which the nuclei do not coincide. The construction of Section 6 can be also used to present a new infinite series of finite G-loops.

In Section 1 we explicitly describe the method of nuclear identification. This leads to a dozen of identities (cf. Table 1). By investigating them from the standpoint of the weak inverse property (WIP) we obtain new conceptual proofs for a number of classical results. In Corollary 1.7 we point out that many of the investigated varieties can be described by normalizing properties involving the left

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and right translations. This approach is used, together with that of autotopisms, as the main tool in determining all of the intersections induced by these varieties (cf. Figure 1). In Sections 4 and 5 we study these varieties under the additional assumption that they satisfy WIP. One sees practically immediately that WIP left conjugacy closed loops are right conjugacy closed, and vice versa. We shall explain why the variety of WIP conjugacy closed loops coincides with the variety of WIP Buchsteiner loops.

Sections 1–5 should be regarded as an attempt to lay out a segment of loop theory foundations in a new systematic way. In most of the cases we were able to remove unmotivated symbolic manipulations that, at least in our opinion, often give to the loop theory a feeling of randomness. The new facts mostly pertain to Buchsteiner loops. There is some intersection with [9], which is a paper that was inspired by an early version of this study when it became clear how complicated the variety of Buchsteiner loops really is.

For historical aspects of Bol and Moufang loops consult, e.g., [29]. Standard sources for general information on loop theory are [6], [4] and [28].

1. Nuclear identification

There is no doubt that Moufang loops are the most widely known class of (nonassociative) loops. There are several identies that characterize them, let us choose

$$x(y \cdot xz) = (xy \cdot x)z. \tag{1}$$

In a loop Q let $L_a : x \mapsto ax$ be the *left translation* and $R_a : x \mapsto xa$ the *right translation*. One can express (1) in the form $L_xL_yL_x = L_{xy\cdot x}$, but we shall be more interested in its expression by means of autotopisms. Now, an autotopism of a loop Q is a triple (α, β, γ) consisting of permutations of Q such that

$$\alpha(y) \cdot \beta(z) = \gamma(yz) \text{ for all } y, z \in Q. \tag{2}$$

Since (1) can be also expressed as $x \cdot yz = (xy \cdot x)(x\backslash z)$, we see that Q is a Moufang loop if and only if

$$(R_xL_x, L_x^{-1}, L_x) \text{ is an autotopism for all } x \in Q. \tag{3}$$

It is more common to characterize Moufang loops by autotopisms (L_x, R_x, L_xR_x) which correspond to the identity $xy \cdot zx = x(yz \cdot x)$. We argue below that both these autotopisms can be obtained by a same general procedure.

The *left nucleus* N_λ of a loop Q consists of all $a \in Q$ that satisfy $a \cdot xy = ax \cdot y$, for all $x, y \in Q$. When the position of a is shifted to the right one obtains the *middle nucleus* N_μ and the *right nucleus* N_ρ. Each of the nuclei has to be a subloop, but it is not necessarily a normal subloop. The intersection $N = N_\lambda \cap N_\rho \cap N_\mu$ is known as the *nucleus* of Q. One sees easily that

$$x \in N_\lambda \iff (L_x, \mathrm{id}_Q, L_x) \text{ is an autotopism, and} \tag{4}$$

$$x \in N_\rho \iff (\mathrm{id}_Q, R_x, R_x) \text{ is an autotopism.} \tag{5}$$

Since $x \cdot ay = xa \cdot y$ can be written as $(x/a)(ay) = xy$, we also see that

$$x \in N_\mu \iff (R_x^{-1}, L_x, \mathrm{id}_Q) \text{ is an autotopism.} \tag{6}$$

All autotopisms of Q form a group. In particular, one can always consider the inverse autotopism. Write the autotopism of (3) as $(R_x^{-1}, L_x, \mathrm{id}_Q)^{-1}\,(L_x, \mathrm{id}_Q, L_x)$

TABLE 1. Overview of loop laws induced by nuclear identifications

code	autotopism	law	abbreviation
$(\lambda, \mu, +, +)$	$(L_x R_x^{-1}, L_x, L_x)$	$xy \cdot xz = x(yx \cdot z)$	lE
$(\lambda, \mu, +, -)$	$(L_x R_x, L_x^{-1}, L_x)$	$(x \cdot yx)z = x(y \cdot xz)$	lB
$(\lambda, \rho, +, +)$	$(L_x, R_x, L_x R_x)$	$xy \cdot zx = (x \cdot yz)x$	mM1
$(\lambda, \rho, +, -)$	$(L_x, R_x^{-1}, L_x R_x^{-1})$	$x\backslash(xy \cdot z) = (y \cdot zx)/x$	Buch
$(\lambda, \rho, -, +)$	$(L_x^{-1}, R_x, L_x^{-1} R_x)$	$x(y \cdot zx) = (xy \cdot z)x$	mE
$(\mu, \lambda, +, +)$	$(R_x^{-1} L_x, L_x, L_x)$	$((xy)/x)z = x \cdot y(x\backslash z)$	LCC
$(\mu, \lambda, +, -)$	$(R_x L_x, L_x^{-1}, L_x)$	$(xy \cdot x)z = x(y \cdot xz)$	lM
$(\mu, \rho, +, +)$	$(R_x^{-1}, L_x R_x, R_x)$	$y(x \cdot zx) = (yx \cdot z)x$	rM
$(\mu, \rho, -, +)$	$(R_x, L_x^{-1} R_x, R_x)$	$y(x\backslash(zx)) = (y/x)z \cdot x$	RCC
$(\rho, \lambda, +, +)$	$(L_x, R_x, R_x L_x)$	$xy \cdot zx = x(yz \cdot x)$	mM2
$(\rho, \mu, +, +)$	$(R_x^{-1}, R_x L_x, R_x)$	$y(xz \cdot x) = (yx \cdot z)x$	rB
$(\rho, \mu, -, +)$	$(R_x, R_x L_x^{-1}, R_x)$	$xy \cdot xz = x(yx \cdot z)$	rE

and write $(L_x, R_x, L_x R_x)$ as $(L_x, \mathrm{id}_Q, L_x)\,(\mathrm{id}_Q, R_x, R_x)$. In each of the two composition pairs none of the factors needs to be an autotopism. However, if the resulting triple is an autotopism, and a factor is an autotopism, then the other factor has to be an autotopism as well. We have verified that $N_\mu = N_\lambda = N_\rho$ in every Moufang loop Q. Let us denote the autotopisms of (4), (5) and (6) as $\alpha_\lambda(x)$, $\alpha_\rho(x)$ and $\alpha_\mu(x)$. Say that a loop identity can be obtained by a *nuclear identification* if it can be expressed by autotopisms $\alpha_\xi^\varepsilon(x)\,\alpha_\chi^\eta(x)$, where $\varepsilon, \eta \in \{-1, 1\}$, $\xi, \chi \in \{\lambda, \mu, \rho\}$ and $\xi \neq \chi$. We shall code such an identity as $(\xi, \chi, \varepsilon, \eta)$, and shall replace 1 and -1 by $+$ and $-$ in concrete instances. For example, the identity of (1) and (3) gets coded as $(\mu, \lambda, -, +)$. We clearly have

Lemma 1.1. *Let Q be a loop that satisfies an identity $(\xi, \chi, \varepsilon, \eta)$, where $\varepsilon, \eta \in \{-1, 1\}$, $\xi, \chi \in \{\lambda, \mu, \rho\}$ and $\xi \neq \chi$. Then $N_\xi = N_\chi$.*

If a class of loops Q is characterized by autotopisms $(\alpha_x, \beta_x, \gamma_x)$, $x \in Q$, then the inverse autotopisms characterize the same class. From the point of view of equations there is an effective way how to get from $\alpha_x(y) \cdot \beta_x(z) = \gamma_x(yz)$ to $\alpha_x^{-1}(y) \cdot \beta_x^{-1}(z) = \gamma_x^{-1}(yz)$ by substitutions. Indeed, if the former equality holds, then $\alpha_x^{-1}(\alpha_x^2(y)) \cdot \beta_x^{-1}(\beta_x^2(z)) = \gamma_x(yz) = \gamma_x^{-1}(\gamma_x^2(yz)) = \gamma_x^{-1}(\alpha_x^2(y) \cdot \beta_x^2(z))$. This means that we can identify $(\xi, \chi, \varepsilon, \eta)$ with $(\chi, \xi, -\eta, -\varepsilon)$.

With this identification in mind we easily compute all equational laws induced by nuclear identification. This is done in Table 1. The first column gives the code of the identity, the second column provides the autotopism involved and the third column gives an explicit form of the law. In some rows this law is not completely identical with the form directly induced by the autotopism, with a small modification done to get a balanced identity. For example the direct interpretation of $(R_x^{-1} L_x, L_x, L_x)$ is $((xy)/x) \cdot xz = x \cdot yz$, which we change to $((xy)/x) \cdot z = x \cdot y(x\backslash z)$ by substitution $z \to x\backslash z$. The fourth column gives the abbreviation of the law. Capital letters B, E and M stand for Bol, extra and Moufang. The starting l, r and m read as left, right and middle. Finally *LCC* (*RCC*) means the left (right) conjugacy closed law, and *Buch* abbreviates the Buchsteiner law.

We shall use I and J to denote the mappings $x \mapsto x\backslash 1$ and $x \mapsto 1/x$, respectively. For an autotopism (α, β, γ) call the triple $(J\gamma I, \alpha, J\beta I)$ the *I-shift* of the autotopism. In the general case the I-shift does not have to be an autotopism. However, it always is an autotopism if $L_x^{-1} = IR_xJ$ for all $x \in Q$. Such loops are said to have the *weak inverse property* and were introduced by Osborn [27]. Other expressions of WIP are in Lemma 1.2. (It is easy to see that I-shifts are autotopisms in WIP loops. From $\alpha(y)\cdot\beta(z) = \gamma(yz)$ we obtain $\beta(y\backslash z) = \alpha(y)\backslash\gamma(z)$. From WIP we have $y\backslash z = I(J(z)y)$, for all $y, z \in Q$, and thus $\beta I(J(z)y) = \alpha(y)\backslash\gamma(z) = I(J\gamma(z)\cdot\alpha(y))$.)

Lemma 1.2. *Let Q be a loop. The following conditions are equivalent:*

(i) $IR_xJ = L_x^{-1}$ *(or $JL_xI = R_x^{-1}$), for all $x \in Q$;*
(ii) $xI(yx) = I(y)$ *(or $J(xy)x = J(y)$), for all $x, y \in Q$;*
(iii) $R_{xy}R_x^{-1}R_y^{-1}(1) = 1$ *(or $L_{yx}L_x^{-1}L_y^{-1}(1) = 1$), for all $x, y \in Q$; and*
(iv) $x \cdot yz = 1 \Rightarrow xy \cdot z = 1$ *(or $xy \cdot z = 1 \Rightarrow x \cdot yz = 1$), for all $x, y, z \in Q$.*

Proof. One can read (i) as $L_xIR_x = I$, which is the same as $xI(yx) = I(y)$. Hence (i) $\Leftrightarrow$ (ii). Now, $R_y^{-1}(1) = R_xR_{xy}^{-1}(1)$ is the same as $J(y) = J(xy)x$, and so (ii) $\Leftrightarrow$ (iii). If $x \cdot yz = 1$, then $z = y\backslash(x\backslash 1)$, and so condition (iv) means, in fact, $y\backslash I(x) = I(xy)$, which is the same as $yI(xy) = I(x)$. □

All of the facts above that involve WIP loops can be traced back to [27]. Some results hold only for WIP loops with $I = J$. This class of loops can be regarded as the most immediate generalization of IP-loops, i.e. of loops in which each element has both the *left* and *right inverse properties*. Elements with LIP are those elements $a \in Q$ for which there exists $b \in Q$ such that $L_a^{-1} = L_b$. Clearly $b = I(a)$, and one easily verifies that IP-loops fulfil $L_x^{-1} = L_{x^{-1}} = IR_xI$ and $R_x^{-1} = R_{x^{-1}} = IR_xI$, for all $x \in Q$. (We write x^{-1} in place of $1/x$ and $x\backslash 1$ if they coincide. Note that $1/x = x\backslash 1$ follows already from LIP. To see this use $a\backslash x = bx$ and set $x = 1$ and $x = a$.)

The I-shift of $(L_xR_x^{-1}, L_x, L_x)$ is $(JL_xI, L_xR_x^{-1}, JL_xI)$ which can be converted into $(R_x^{-1}, L_xR_x^{-1}, R_x^{-1})$ when Q is a WIP loop. Using inversion we get the triple $(R_x, R_xL_x^{-1}, R_x)$, which defines the rE law. We can say that the equality rE is an *I-shift* of the equality lE. By continuing further, we get $(JR_xI, R_x, JR_xL_x^{-1}I)$. This does not offer itself for an immediate simplification if we assume nothing more than that Q is a WIP loop. However, if in addition we assume $I = J$, we can simplify the triple to $(L_x^{-1}, R_x, L_x^{-1}R_x)$, which defines mE. We shall say that mE is the *II-shift* of rE.

By doing similar computations for all laws from Table 1 we obtain:

Proposition 1.3. *The loop laws induced by nuclear identification form four cycles in which each of the laws is the II-shift of the preceding one. These cycles are*

(lE rE mE) (lM rB mM2) (lB rM mM1) (LCC RCC Buch).

In every of these cycles the middle law is the I-shift of the law on the left.

The *flexible* law is defined by the identity $x \cdot yx = xy \cdot x$. By setting $z = 1$ or $y = 1$ one obtains the flexible law from each of the Moufang identities. Hence *mM1* and *mM2* are equivalent and we can use *mM* as a common abbreviation. It is practically immediate to see that each of Moufang laws yields an IP-loop. In this way we obtain from Proposition 1.3 a proof that all Moufang laws are equivalent. Our proof resembles the standard proof of Bruck [5], but is a bit shorter. (One also needs to observe that *lM* is equivalent to *lB* plus flexibility, which is obvious. Similarly for *rM*.)

The extra laws were introduced by Fenyves who proved their equivalence [15]. All of them are in the same cycle. However, only mE yields inverse property in an immediate way. For lE we easily get RIP, but LIP is a bit tricky [15]. We shall show now how to avoid this trick.

Proposition 1.4. *In the variety of loops all extra laws are equivalent.*

Proof. It suffices to prove $lE \Rightarrow mE$, and for that it is enough to show that lE determines a variety of WIP loops with $I = J$. Let Q satisfy $xy \cdot xz = x(yx \cdot z)$. Setting $z = x\backslash 1$ gives RIP. Furthermore,

$$xy \cdot (x \cdot yz) = x(yx \cdot yz) = x \cdot y(xy \cdot z),$$

and so $L_{xy}^{-1} L_x L_y (xy \cdot z) = x \cdot yz$, for all $x, y, z \in Q$. Now, $L_{xy}^{-1} L_x L_y$ fixes 1 for all $x, y \in Q$, and hence Q is WIP by point (iv) of Lemma 1.2. $\square$

It can be observed directly that LCC loops are exactly those loops in which $L_x L_y L_x^{-1}$ is a left translation for all $x, y \in Q$. A loop Q is said to be *conjugacy closed* if it is both LCC and RCC, i.e. if both left and right translations are closed under conjugation. From Proposition 1.3 we see that WIP LCC loops are CC loops. Goodaire and Robinson proved in [18] that WIP CC loops can be characterized by the Wilson [33] identity $xJ(xy) = xz \cdot J(x \cdot yz)$. In Theorem 5.5 we shall show that a WIP loop is conjugacy closed if and only if it satisfies the Buchsteiner law. Within the variety of WIP loops each of the cycles in Proposition 1.3 thus defines exactly one subvariety (extra loops, Moufang loops and Wilson loops, respectively).

The concept of conjugacy closedness seems to have been defined explicitly for the first time by Soikis [32]. He defined a ΛK element as a loop element x for which $(R_x^{-1} L_x, L_x, L_x)$ is an autotopism, and considered loops in which all elements are ΛK. Goodaire and Robinson, working independently, introduced the notion of conjugacy closed loops in [17].

The Buchsteiner law has been introduced for the first time in [7]. H. H. Buchsteiner gave no name to the law. In fact he concentrated his efforts to the law $(xy)\backslash((xy \cdot u)v) = u(v \cdot yx)/(yx)$ which he rightly recognized as the identity that determines all isotopically invariant Buchsteiner loops. However, as it has turned out recently [9], the latter law is equivalent to the Buchsteiner law since all Buchsteiner loops are isotopically invariant.

Lemma 1.5. *Let Q be a Buchsteiner loop. Then $N_\lambda = N_\rho = N_\mu$.*

Proof. By Lemma 1.1, $N_\lambda = N_\rho$. We have $a \in N_\rho$ if and only if $ya = x\backslash(xy \cdot a)$ for all $x, y \in Q$. Elements of N_μ can be characterized by $ya = (y \cdot ax)/x$, and $N_\rho = N_\mu$ follows from $x\backslash(xy \cdot a) = (y \cdot ax)/x$. $\square$

The loop laws induced by a nuclear identification will be sometimes called only *nuclear laws*. There are various ways how to describe them.

Proposition 1.6. *Let Q be a loop. Then Q is a left Bol loop (or an LCC loop, or a Buchsteiner loop) if and only if*

$$L_x^{-1} R_y L_x = R_{xy} R_x^{-1} \text{ (or } L_x R_y L_x^{-1} = R_{xy} R_x^{-1}, \text{ or } L_x^{-1} R_y L_x = R_x^{-1} R_{yx}),$$

for all $x, y \in Q$.

Proof. One can rewrite the left Bol law as $R_zL_xR_x = L_xR_{xz}$ which yields $L_x^{-1}R_zL_x = R_{xz}R_x^{-1}$. The *LCC* law can be expressed as $R_zR_x^{-1}L_x = L_xR_{x\backslash z}$, which gives $L_xR_zL_x^{-1} = R_{xz}R_x^{-1}$. Finally, the Buchsteiner law corresponds to $L_x^{-1}R_zL_x = R_x^{-1}R_{zx}$ directly. □

Corollary 1.7. *Let Q be a loop. Then Q is a left Bol loop (or an LCC loop, or a Buchsteiner loop) if and only if*

$$L_x^{-1}R_yR_z^{-1}L_x = R_{xy}R_{xz}^{-1} \text{ (or } L_xR_yR_z^{-1}L_x^{-1} = R_{xy}R_{xz}^{-1},$$
$$\text{or } L_x^{-1}R_y^{-1}R_zL_x = R_{yx}^{-1}R_{zx}), \text{ for all } x, y \in Q.$$

The permutation group generated by all left (or right) translations is called the *left* (or *right*) *multiplication group.* We shall denote it by $\mathcal{L} = \mathcal{L}(Q)$ and $\mathcal{R} = \mathcal{R}(Q)$. Both left and right translations of a loop Q generate the *multiplication group* $\operatorname{Mlt} Q$.

Corollary 1.8. *If Q is a left Bol loop or an LCC loop or an Buchsteiner loop, then $\mathcal{R}(Q) \trianglelefteq \operatorname{Mlt} Q$. If Q is a right Bol loop or an RCC loop or an Buchsteiner loop, then $\mathcal{L}(Q) \trianglelefteq \operatorname{Mlt} Q$.*

If S is a subloop of a loop Q, then one usually cannot define the index $|Q : S|$ in the same way as in groups. However, this can be clearly done if Q is finite with $|Q| = 2|S|$, and also if $S \leq N_\mu$ (see [13]).

Proposition 1.9. *Let Q be a loop such that $N_\lambda = N_\mu$ and $|Q : N_\lambda| = 2$. Then Q is left conjugacy closed and there exists $x \in Q$ with $x\backslash 1 \neq 1/x$.*

Proof. The first part of the statement has appeared as Proposition 1.4 in [13], and was inspired by a similar result for *CC* loops by Goodaire and Robinson [17]. We shall prove only the part about inverses. Set $S = N_\lambda = N_\mu$, and observe that $(ys)\backslash 1 = s^{-1}(y\backslash 1)$ for all $s \in S$ and $y \in Q$ since $(ys)(s^{-1}(y\backslash 1)) = y(y\backslash 1) = 1$. Assume $1/x = x\backslash 1 = x^{-1}$ for all $x \in Q$. Then $(ys)^{-1} = s^{-1}y^{-1}$ for all $s \in S$ and $y \in Q$. We shall use this to show that Q has to be a group. For that it suffices to verify $x(ys) = (xy)s$ and $x(y^{-1}z) = (xy^{-1})z$, where $x, y, z \in Q \setminus S$ and $s \in S$. To get the former equality, express x as $ts^{-1}y^{-1}$. Then $x(ys) = t(ys)^{-1}(ys) = t = ts^{-1}s = (xy)s$. To verify the latter equality set $x = ay$ and $z = yb$. We obtain $(xy^{-1})z = a(yb) = (ay)b = x(y^{-1}z)$. □

It is not difficult to give an explicit description of all loops Q that fulfil the assumptions of Proposition 1.9. All of them can be derived from a triple (S, h, d), where S is a group, $h \in \operatorname{Aut} S$, and $d \in S$ (see [13, Theorem 1.9]). One can construct in this way an infinite loop with $N_\rho = 1$, but all such finite loops must have $N_\rho \neq 1$ (Theorems 1.10 and 1.11 of [13]). It is clear from the construction that for every prime $p \geq 5$ one can have an *LCC* loop Q of order $2p$ with $|N_\lambda| = p$ and $|N_\rho| = 2$.

2. Nuclei and the alternative laws

We start this section by standard results on autotopisms. The proofs are straightforward and we omit them.

Lemma 2.1. *Let Q be a loop and let (α, β, γ) be an autotopism of Q.*

(i) *Assume $\alpha(1) = 1$. Then $\beta = \gamma = R_{\beta(1)}\alpha$.*

(ii) *Assume* $\beta(1) = 1$. *Then* $\alpha = \gamma = L_{\alpha(1)}\beta$.
(iii) *Assume* $\alpha(1) = \beta(1) = 1$. *Then* $\alpha = \beta = \gamma \in \operatorname{Aut} Q$.

Corollary 2.2. *Let* Q *be a loop and let* α *and* β *be permutations of* Q.

(i) *The triple* $(\alpha, \mathrm{id}_Q, \beta)$ *is an autotopism if and only if* $\alpha = \beta = L_{\alpha(1)}$, *where* $\alpha(1) \in N_\lambda$.
(ii) *The triple* $(\mathrm{id}_Q, \alpha, \beta)$ *is an autotopism if and only if* $\alpha = \beta = R_{\alpha(1)}$, *where* $\alpha(1) \in N_\rho$.

The identity $x \cdot xy = x^2y$ is known as the *left alternative law*. It is immediately clear that left Bol loops satisfy this law, while the right Bol loops satisfy the *right alternative law* $yx \cdot x = yx^2$.

Lemma 2.3. *Let* Q *be a left Bol loop (or an LCC loop). Then*

$$Q \text{ is flexible } \Leftrightarrow Q \text{ is right alternative } \Leftrightarrow Q \text{ has } RIP\ .$$

Proof. The flexible law is equivalent to $L_xR_x = R_xL_x$, while the right alternative law to $R_x^2 = R_{x^2}$. The RIP can be expressed as $R_x^{-1} = R_{x\backslash 1}$. From Proposition 1.6 we see that in a left Bol loop (or an LCC loop)) $R_{x^2}R_x^{-1} = R_x$ is the same as $L_x^{-1}R_xL_x = R_x$ (or $L_xR_xL_x^{-1} = R_x$), and $L_x^{-1}R_x^{-1}L_x = L_x^{-1}R_{x\backslash 1}L_x$ (or $L_xR_x^{-1}L_x^{-1} = L_xR_{x\backslash 1}L_x^{-1}$) is the same as $R_xR_{x^2}^{-1} = R_x^{-1}$. □

The part about Bol loops is well known, of course. The idea of the proof carries further to Buchsteiner loops:

Lemma 2.4. *Let* Q *be a Buchsteiner loop. If* Q *fulfils the flexible law, or the left alternative law, or the right alternative law, then* Q *is an* IP *loop that fulfils all these three laws. If* Q *fulfils* LIP *or* RIP, *then it is a flexible* IP *loop as well.*

Proof. Using Proposition 1.6 we express the right alternative law as $L_x^{-1}R_xL_x = R_x$ and the RIP as $R_{x^2}^{-1}R_x = R_x^{-1}$. The Buchsteiner law is left-right symmetric, and so the rest is clear. □

Corollary 2.5. *Let* Q *be a loop that is flexible or right alternative or* RIP. *If* Q *fulfils the left Bol or* LCC *or Buchsteiner law, then it is an extra loop.*

Proof. We can assume that Q is flexible, by Lemmas 2.3 and 2.4. From Table 1 we see that if L_x and R_x commute, then LCC yields *lE*, and *Buch* implies *mE*. □

Let us adopt the convention that for $X \subseteq Q$ we denote by $L_{(X)}$ the set $\{L_x;\ x \in X\}$. Similarly $R_{(X)}$. Denote by $\mathrm{Sym}(Y)$ the symmetric group acting on a set Y. If Q is a group, then the centralizer $C_{\mathrm{Sym}(Q)}(\mathcal{R})$ equals $\mathcal{L}$. This carries to loops in the following form:

Lemma 2.6. *Let* Q *be a loop. Then* $C_{\mathrm{Sym}(Q)}(\mathcal{R}) = L_{(N_\lambda)}$.

Proof. A mapping $\psi \in \mathrm{Sym}(Q)$ centralizes $\mathcal{R}$ if and only if $\psi(xy) = \psi(x)y$ for all $x, y \in Q$. In other words, $(\psi, \mathrm{id}_Q, \psi)$ has to be an autotopism. From Corollary 2.2 we know that this happens exactly when $\psi = L_a$ for some $a \in N_\lambda$. □

Corollary 2.7. *Let* Q *be a loop. If* $\mathcal{L}(Q) \trianglelefteq \operatorname{Mlt} Q$, *then* $N_\rho \trianglelefteq Q$. *If* $\mathcal{R}(Q) \trianglelefteq \operatorname{Mlt} Q$, *then* $N_\lambda \trianglelefteq Q$.

Proof. It is easy to see (and well known) that a subloop S of a loop Q is normal if and only if it is a block of the multiplication group $\mathrm{Mlt}\, Q = \langle L_x, R_x;\ x \in Q\rangle$. Blocks of permutation groups can be often obtained as orbits of a normal subgroup. A centralizer of a normal subgroup is a normal subgroup, and so $C_{\mathrm{Mlt}\, Q}\mathcal{R}$ induces a normal subloop of Q whenever the right multiplication group $\mathcal{R} = \langle R_x;\ x \in Q\rangle$ is a normal subgroup of $\mathrm{Mlt}\, Q$. □

By combining Corollary 1.8 and Corollary 2.7 we obtain:

Proposition 2.8. *Let Q be a loop that satisfies an identity $(\xi, \chi, \varepsilon, \eta)$, where $\varepsilon, \eta \in \{-1, 1\}$, $\xi, \chi \in \{\lambda, \mu, \rho\}$ and $\xi \neq \chi$. Then $N_\xi = N_\chi \trianglelefteq Q$.*

3. Intersections of nuclear varieties

There is a concise way how to characterize elements of N_ρ when Q is an *LCC* loop. Indeed, $(\mathrm{id}_Q, R_x^{-1}, R_x^{-1})(R_x^{-1}L_x, L_x, L_x) = (R_x^{-1}L_x, R_x^{-1}L_x, R_x^{-1}L_x)$, and so we see that

(7) $$x \in N_\rho \ \Leftrightarrow\ R_x^{-1}L_x \in \mathrm{Aut}\, Q, \text{ in every LCC loop } Q.$$

Let us mention that in Buchsteiner loops this characterization pertains to every element of $N = N_\lambda = N_\mu = N_\rho$:

Theorem 3.1. *Let Q be a Buchsteiner loop. Then*

$$L_a R_a^{-1} \in \mathrm{Aut}\, Q \Leftrightarrow L_a^{-1} R_a \in \mathrm{Aut}\, Q \Leftrightarrow a \in N.$$

Proof. If $L_a R_a^{-1} \in \mathrm{Aut}\, Q$, then $a((xy)/a) = a(x/a) \cdot a(y/a)$ for all $x, y \in Q$. Setting $y = a$ yields $ax = (a(x/a))a$, which turns into $a(xa) = (ax)a$. Thus $L_a R_a = R_a L_a$.

Similarly, $L_a^{-1} R_a \in \mathrm{Aut}\, Q$ means $a\backslash(xy \cdot a) = a\backslash(xa) \cdot a\backslash(ya)$ for all $x, y \in Q$. Again, $x = a$ implies $ay \cdot a = a \cdot ya$, i.e. $L_a R_a = R_a L_a$. Thus $L_a R_a^{-1} \in \mathrm{Aut}\, Q \Leftrightarrow L_a^{-1} R_a \in \mathrm{Aut}\, Q$.

If $L_a R_a^{-1} = R_a^{-1} L_a \in \mathrm{Aut}\, Q$, then

(8) $$(R_a, L_a^{-1}, \mathrm{id}_Q) = (R_a, R_a^{-1} L_a^{-1} R_a, \mathrm{id}_Q) = \\ (L_a, R_a^{-1}, R_a^{-1} L_a)(L_a^{-1} R_a, L_a^{-1} R_a, L_a^{-1} R_a),$$

and so $a \in N_\mu = N$, by (6) and Lemma 1.5. If $a \in N_\mu$, then R_a and L_a commute, and hence (8) can be used to show that $L_a^{-1} R_a \in \mathrm{Aut}\, Q$. □

From Proposition 1.6 and Corollary 1.7 we easily compute that every element x of an *LCC* loop Q fulfils

(9) $$[L_x, R_x] = R_{x\backslash 1} R_x \quad \text{and} \quad L_x R_x R_{x\backslash 1}^{-1} L_x^{-1} = R_{x^2}.$$

If (α, β, β) is an autotopism of a loop Q and $\beta(1) = 1$, then $\alpha = \beta$ and $\alpha \in \mathrm{Aut}\, Q$, by Lemma 2.1. Hence $L_{xy}^{-1} L_x L_y \in \mathrm{Aut}\, Q$ in every *LCC* loop Q. Mappings of this form generate the *left inner multiplication group* $\mathcal{L}_1 = \{\psi \in \mathcal{L};\ \psi(1) = 1\}$, and so $\mathcal{L}_1 \leq \mathrm{Aut}\, Q$. Such loops are called *A_ℓ-loops.* Similarly, $R_{yx}^{-1} R_x R_y \in \mathrm{Aut}\, Q$ in every *RCC* loop, and *RCC* loops are *A_r-loops.* Note that for $y = x$ we get $R_{x^2}^{-1} R_x^2 \in \mathrm{Aut}\, Q$.

Lemma 3.2. *Let Q be a conjugacy closed loop, and let x be an element of Q. Then $x^2 \in N(Q)$ if and only if $L_{x^2}^{-1} R_x^2 \in \mathrm{Aut}\, Q$. In such a case $[L_x, R_x] = R_{x^2}^{-1} R_x^2$.*

Proof. Since $R_{x^2}^{-1}R_x^2 \in \operatorname{Aut} Q$, we get from (7) that

$$x^2 \in N(Q) \;\Leftrightarrow\; L_{x^2}^{-1}R_{x^2} \in \operatorname{Aut} Q \;\Leftrightarrow\; L_{x^2}^{-1}R_x^2 \in \operatorname{Aut} Q.$$

Assume now $x^2 \in N(Q)$. We know from (9) that $[L_x, R_x] = R_{x\backslash 1}R_x$ and that $R_xR_{x\backslash 1}^{-1} = L_x^{-1}R_{x^2}L_x = R_{x^2}$ (the last equality holds because we are assuming $x^2 \in N(Q)$). Thus $R_{x^2}^{-1}R_x^2 = R_{x\backslash 1}R_x^{-1}R_x^2 = [L_x, R_x]$. □

Theorem 3.3. *Let Q be a conjugacy closed loop. Then Q is a Buchsteiner loop if and only if $x^2 \in N(Q)$ for every $x \in Q$.*

Proof. From Table 1 we see that a CC loop Q is a Buchsteiner loop if and only if

$$\begin{aligned}(L_{x^2}^{-1}R_{x^2}, L_{x^2}^{-1}, L_{x^2}^{-1})\,(L_x^{-1}, R_x, R_xL_x^{-1})\,(R_x^{-1}L_x, L_x, L_x)\,(R_x, L_x^{-1}R_x, R_x)&\\ = (L_{x^2}^{-1}R_{x^2}[L_x, R_x], L_{x^2}^{-1}R_x^2, L_{x^2}^{-1}R_x^2)&\end{aligned}$$

is an autotopism for each $x \in Q$. By Lemma 2.1 this can happen only if both $[L_x, R_x] = R_{x^2}^{-1}R_x^2$ and $L_{x^2}^{-1}R_x^2 \in \operatorname{Aut} Q$ are true, for every $x \in Q$. From Lemma 3.1 we see that this is equivalent to $x^2 \in N(Q)$, for all $x \in Q$. □

One of the most intriguing binary structures are the *commutative Moufang loops*, shortly *CML* (cf. [6, 31]). A commutative Bol loop is necessarily Moufang since commutative loops are flexible. It is worth mentioning that all other nuclear laws offer no new commutative structures:

Lemma 3.4. *Let Q be a commutative loop. If Q is left (or right) conjugacy closed, or if it is a Buchsteiner loop, then Q is an abelian group.*

Proof. From (7) and from Theorem 3.1 we see that Q coincides with one of its nuclei, and so Q is a group. (We have used the fact that commutative loops are characterized by $L_x = R_x$; there are many other different ways how to prove this lemma.) □

Our goal now is to show that Figure 1 rightly captures the intersections of nuclear varieties. We start by explaning the abbreviations in the figure. Call a group G *boolean* if $x^2 = 1$ for all $x \in G$. All boolean groups are abelian. Call a loop Q *boolean* if it is a boolean group. By *boolCC* we understand all CC loops such that Q/N is boolean. Can we really claim, on the basis of Theorem 3.3, that Buchsteiner CC loops belong to this class? Basarab proved in [2] that Q/N is an abelian group if Q is conjugacy closed. The answer is hence affirmative. We shall give a direct proof.

Lemma 3.5. *Let Q be a conjugacy closed loop with $x^2 = 1$ for all $x \in Q$. Then Q is a boolean group.*

Proof. From Theorem 3.3 we know that Q satisfies the Buchsteiner law. Hence $x = x\backslash 1 = x\backslash(xy \cdot xy) = (y(xy \cdot x))/x$ for all $x, y \in Q$, and so $1 = y(xy \cdot x)$, which implies $y/x = xy$. Therefore $R_x^{-1}L_x = L_x^2 \in \mathcal{L}_1 \in \operatorname{Aut} Q$, and $x \in N$, by (7). □

Lemma 3.6. *A Buchsteiner loop is an LCC loop if and only if it is an RCC loop.*

Proof. Write the LCC law as $(y(xz))/z = (x((x\backslash(yx))z))/z$. In Buchsteiner loops this equality can be turned into $z\backslash((zy)x) = z\backslash((zx)(x\backslash(yx)))$, which is the same as $zy \cdot x = zx \cdot (x\backslash(yx))$. From that we can obtain the RCC law by replacing y with $(xy)/x$. □

The first systematic treatment of LCC loops was done by P. T. Nagy and Strambach in [26]. They proved a number of basic properties of LCC loops, including the relevant part of Lemma 3.4. They showed that several examples of right Bol loops constructed by Burn in [8] are right conjugacy closed, and they decided to call LCC left Bol loops Burn loops. For the purposes of this paper let us speak about *left Burn* loops and *right Burn* loops. We shall now reproduce their characterization of left Burn loops. We have:

$$(L_xR_x, L_x^{-1}, L_x)\,(R_x^{-1}L_x, L_x, L_x) = (L_x^2, \mathrm{id}_Q, L_x^2). \tag{10}$$

The latter triple is an autotopism if and only if $x^2 \in N_\lambda$ and $L_x^2 = L_{x^2}$, by Corollary 2.2. Left Bol loops are left alternative, and so

$$\text{Left Bol loop } Q \text{ is LCC } \Leftrightarrow\ x^2 \in N_\lambda \text{ for all } x \in Q. \tag{11}$$

Left Bol loops satisfy LIP, and hence have two-sided inverse elements. For the converse direction we cannot ignore the left alternative law since the LCC loops mentioned in Proposition 1.9 always possess an element x with $1/x \neq x\backslash 1$. Thus

$$\text{LCC loop } Q \text{ is left Bol } \Leftrightarrow x^2 \in N_\lambda \text{ and } x \cdot xy = x^2y \text{ for all } x, y \in Q. \tag{12}$$

Note that $x^2 \in N_\lambda$ does not imply that Q/N_λ has to be boolean. Every left Bol loop which satisfies $x^2 = 1$, for all $x \in Q$, is LCC. One speaks about involutory Bol loops. They have been studied in [19], and there are myriads of them (however, there is no finite example known of an order different from a power of two).

By considering (11) together with its mirror version we can reprove Fenyves's result [16]:

$$\text{Moufang loop } Q \text{ is CC } \Leftrightarrow\ x^2 \in N \text{ for all } x \in Q. \tag{13}$$

Conjugacy closed Moufang loops are extra, by Corollary 2.5. Every extra loop is flexible, and so one can see directly from Table 1 that every extra loop is an LCC RCC Buchsteiner loop. From Theorem 3.3 we obtain that each extra loop satisfies $x^2 \in N$. Since it also satisfies the left and right alternative laws, it has to be a Moufang loop, by (12). We have reproved a classical result of Goodaire and Robinson [17], by which extra loops are exactly conjugacy closed Moufang loops.

Up to now we gave no example of a Buchsteiner loop that is not conjugacy closed. Such examples really exist, but their construction is not completely easy. We refer to [7].

Theorem 3.7. *Figure 1 shows the Hasse diagram for all varieties of loops that can be obtained through intersections of all nuclear varieties.*

Proof. We have mentioned above that extra loops are exactly the conjugacy closed Moufang loops. Buchsteiner loops that are LCC or RCC give *boolCC*, by Theorem 3.3 and Lemma 3.6. A right Bol loop that is Buchsteiner or LCC is an extra loop, by Corollary 2.5, and a mirror argument can be used for left Bol loops that are Buchsteiner loops or LCC loops. We have considered all pairs of varieties the intersection of which gives in Figure 1 the variety of extra loops or the variety *boolCC*. All other intersections are clear. □

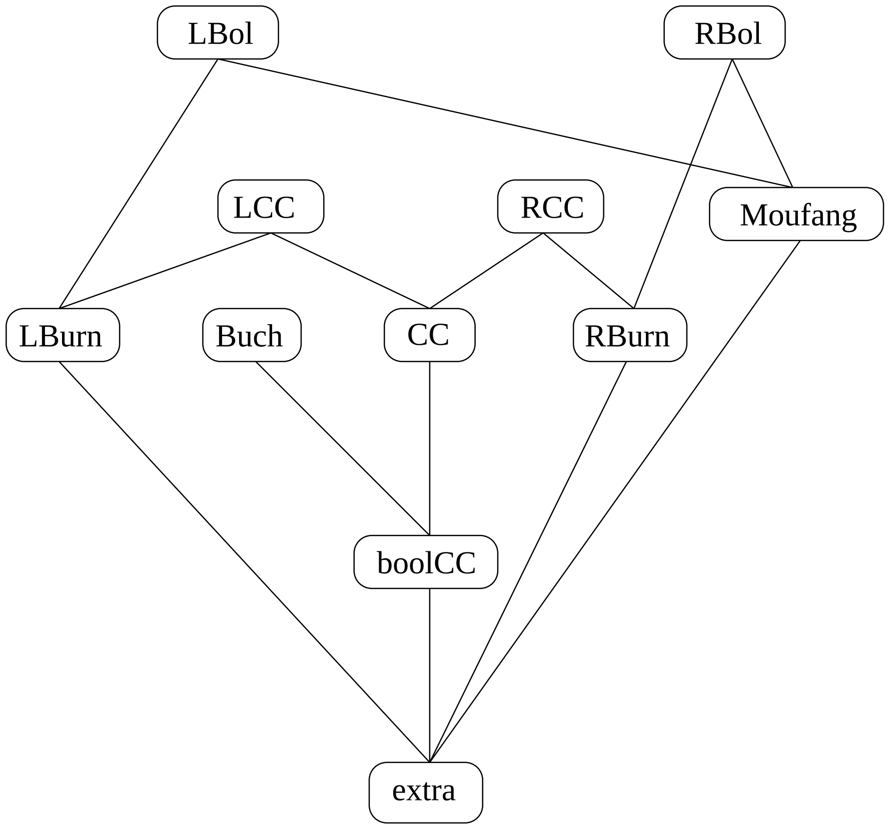

FIGURE 1. Intersections of nuclear varieties

Lemma 3.8. *Let Q be a commutative loop with an autotopism (α, β, γ). Then (β, α, γ) is an autotopism of Q as well.*

Proof. Clearly $\beta(x) \cdot \alpha(y) = \alpha(y) \cdot \beta(x) = \gamma(yx) = \gamma(xy)$. $\square$

In Lemma 3.4 we observed that the variety of commutative Moufang loops encompasses all commutative loops that fulfil a nuclear law. This observation can be extended when we consider laws obtained by exchanging the first and the second component of the autotopisms that determine a law $(\xi, \chi, \varepsilon, \eta)$. Let $(\alpha_x, \beta_x, \gamma_x)$ be the autotopisms of the latter variety as given in Table 1. Let $(\xi, \chi, \varepsilon, \eta)^*$ be the variety determined by $(\beta_x, \alpha_x, \gamma_x)$, for all $x \in Q$. Table 2 exhibits the laws obtained for these varieties (Ab stands for the variety of abelian groups). It can be easily verified, by substituting 1 for one of the variables, or by identifying two variables, that each of these laws describes a variety of commutative loops. In light of Lemma 3.8 we can hence conclude this section by

Proposition 3.9. *The variety $(\xi, \chi, \varepsilon, \eta)^*$ consists of all commutative loops in the variety $(\xi, \chi, \varepsilon, \eta)$, whenever $\varepsilon, \eta \in \{-1, 1\}$ and $\xi, \chi \in \{\lambda, \rho, \mu\}$, $\chi \neq \xi$.*

Table 2. Loop laws obtained by twisted nuclear identifications

code	autotopism	law	variety
$(\lambda,\mu,+,+)^*$	$(L_x, L_xR_x^{-1}, L_x)$	$xy\cdot xz = x(y\cdot zx)$	Ab
$(\lambda,\mu,+,-)^*$	(L_x^{-1}, L_xR_x, L_x)	$y(x\cdot zx) = x(xy\cdot z)$	CML
$(\lambda,\rho,+,+)^*$	(R_x, L_x, L_xR_x)	$yx\cdot xz = x(yz\cdot x)$	CML
$(\lambda,\rho,+,-)^*$	$(R_x^{-1}, L_x, L_xR_x^{-1})$	$(y/x)\cdot xz = x((yz)/x)$	Ab
$(\lambda,\rho,-,+)^*$	$(R_x, L_x^{-1}, L_x^{-1}R_x)$	$x(yx\cdot z) = (y\cdot xz)x$	Ab
$(\mu,\lambda,+,+)^*$	$(L_x, R_x^{-1}L_x, L_x)$	$y((xz)/x) = x((x\backslash y)z)$	Ab
$(\mu,\lambda,+,-)^*$	(L_x^{-1}, R_xL_x, L_x)	$y(xz\cdot x) = x(xy\cdot z)$	CML
$(\mu,\rho,+,+)^*$	(L_xR_x, R_x^{-1}, R_x)	$(x\cdot yx)z = (y\cdot zx)x$	CML
$(\mu,\rho,-,+)^*$	$(L_x^{-1}R_x, R_x, R_x)$	$(x\backslash(yx))z = (y(z/x))x$	Ab
$(\rho,\lambda,+,+)^*$	(R_x, L_x, R_xL_x)	$yx\cdot xz = x(yz\cdot x)$	CML
$(\rho,\mu,+,+)^*$	(R_xL_x, R_x^{-1}, R_x)	$(xy\cdot x)z = (y\cdot zx)x$	CML
$(\rho,\mu,-,+)^*$	$(R_xL_x^{-1}, R_x, R_x)$	$yx\cdot zx = (xy\cdot z)x$	Ab

4. Structural homomorphisms and the weak inverse property

Let Q be a loop. Recall that the left and right multiplication groups are denoted by $\mathcal{L}$ and $\mathcal{R}$, respectively. The loop $(\operatorname{Mlt} Q)_1 = \{\varphi \in \operatorname{Mlt} Q;\ \varphi(1) = 1\}$ is known as the *inner mapping group* of Q. We shall denote it by $\operatorname{Inn} Q$.

We shall use the notion of a structural homomorphism rather loosely, denoting so any homomorphism that can be obtained in a canonic way and involves the above groups. For example $L_x \mapsto R_x^{-1}$, for all $x \in Q$, induces a structural isomorphism $\mathcal{L} \cong \mathcal{R}$ in every WIP loop Q since $R_x^{-1} = JL_xI$.

It is well known that loops $\mathcal{L}_1$ and $\mathcal{R}_1$ are generated by mappings $L_{xy}^{-1}L_xL_y$ and $R_{yx}^{-1}R_xR_y$, respectively.

Proposition 4.1. *Let Q be a Buchsteiner WIP loop. Then there exists a unique isomorphism $\mathcal{L} \cong \mathcal{R}$ that sends each L_x to R_x^{-1}. This isomorphism is identical upon $\mathcal{L}_1 = \mathcal{R}_1$. Furthermore, the loop Q is an A_ℓ-loop and an A_r-loop.*

Proof. It suffices to show that the isomorphism is identical on the generators of $\mathcal{L}_1$. For that consider the autotopism

$$(L_{xy}^{-1}L_xL_y, R_{xy}R_x^{-1}R_y^{-1}, R_{xy}L_{xy}^{-1}L_xR_x^{-1}L_yR_y^{-1})$$

and note that both $L_{xy}^{-1}L_xL_y \in \operatorname{Aut} Q$ and $L_{xy}^{-1}L_xL_y = R_{xy}R_x^{-1}R_y^{-1}$ follow from Lemma 2.1, by the point (iii) of Lemma 1.2. □

Let us now turn to LCC loops. If $\alpha = L_{x_1}^{\varepsilon_1}\dots L_{x_n}^{\varepsilon_n}$, where $x_i \in Q$ and $\varepsilon_i \in \{-1,1\}$, for all i, $1 \le i \le n$, then we have an autotopism (β,α,α), where β is obtained from α in such a way that each $L_{x_i}^{\varepsilon_i}$ is replaced by $(R_{x_i}^{-1}L_{x_i})^{\varepsilon_i}$. If $\alpha = \mathrm{id}_Q$, then $\beta = \mathrm{id}_Q$ as well, by Lemma 2.1. However, that means that we can construct a structural homomorphism $\mathcal{L} \mapsto \operatorname{Inn} Q$ that sends each L_x to $R_x^{-1}L_x$. The construction of this homomorphims and the discussion of the consequences is the main content of [12].

For WIP Buchsteiner loops we can take a similar approach. Let α be as above. Then we get an autotopism $(\alpha, J\alpha I, \beta)$ where β is obtained from α by replacing each L_{x_i} with $L_{x_i}R_{x_i}^{-1}$. By using Lemma 2.1 we thus get a homomorphism $\mathcal{L} \to \mathcal{D}$, $L_x \mapsto L_xR_x^{-1}$, where $\mathcal{D}(Q) = \langle R_x^{-1}L_x;\ x \in Q\rangle$. The kernel of this homomorphism can be extracted from Corollary 2.2. Indeed, if $(\alpha, J\alpha I, \mathrm{id}_Q)$ is an autotopism, then the I-shift $(\mathrm{id}_Q, \alpha, J^2\alpha I^2)$ is also an autotopism, and so $\alpha = R_x$ for some $x \in N$. However, R_x also belongs to $\mathcal{L}$, and so we see that $\alpha \in Z(\mathcal{L})$, by the mirror version of (8). If $R_x \in \mathcal{L}$, then $x \in N$, by Theorem 3.1 and Proposition 4.1. Hence $Z(\mathcal{L}) = R_{(M)}$, where $M = \{x \in Q;\ R_x \in \mathcal{L}\}$. Our homomorphism is identical on $\mathcal{L}_1$, by Lemma 2.1. An element $R_x = L_x L_x^{-1} R_x \in Z(\mathcal{L})$ is hence mapped upon $L_x R_x^{-1} L_x^{-1} R_x$, which equals id_Q since $R_x L_x = L_x R_x$, by $x \in N$. We see that $Z(\mathcal{L})$ is indeed the kernel of the homomorphism. We have proved:

Proposition 4.2. *Let Q be a WIP Buchsteiner loop. Denote by $\mathcal{D}$ the group generated by all $L_x R_x^{-1}$, $x \in Q$. Then there exists a unique homomorphism $\mathcal{L} \to \mathcal{D}$ that sends each L_x to $L_x R_x^{-1}$. This homomorphism is identical on $\mathcal{L}_1$ and its kernel is equal to $Z(\mathcal{L}) = R_{(M)}$, where $M = \{a \in Q;\ R_a \in \mathcal{L}\} \le N$.*

We have $Z(\mathcal{L}) \trianglelefteq \mathrm{Mlt}\, Q$ in every Buchsteiner loop since $\mathcal{L} \trianglelefteq \mathrm{Mlt}\, Q$, by Corollary 1.8. This means that M, which is an orbit of $Z(\mathcal{L})$, is a normal subloop of every WIP Buchsteiner loop Q.

Lemma 4.3. *Let Q be a WIP Buchsteiner loop. Then $\mathcal{D} \cap \mathcal{L} = \mathcal{L}_1 \trianglelefteq \mathcal{D}$.*

Proof. Every element of $\mathcal{D}$ has the form $L_x R_x^{-1}\varphi$, $\varphi \in \mathcal{L}_1$, as it is an image of some $L_x\varphi$. We need to prove that $L_x R_x^{-1}\varphi$ fixes 1 whenever it belongs to $\mathcal{L}$. For that it suffices to show $x \in N$, since then $L_x R_x^{-1} = R_x^{-1} L_x \in \mathcal{L}_1$. We have $M \subseteq N$, and $L_x R_x^{-1}\varphi \in \mathcal{L} \Leftrightarrow R_x \in \mathcal{L} \Leftrightarrow x \in M$. □

Proposition 4.4. *Let Q be a WIP Buchsteiner loop. Then Q/M is a group.*

Proof. Consider again the homomorphism $\mathcal{L} \to \mathcal{D}$, $L_x \mapsto L_x R_x^{-1}$. The preimage of $\mathcal{L}_1$ is $R_{(M)}\mathcal{L}_1$, by Proposition 4.2. By Lemma 4.3 this is a normal subgroup of $\mathcal{L}$, and it coincides with $L_{(M)}\mathcal{L}_1$, as $L_a^{-1} R_a \in \mathcal{L}_1$ for all $a \in M$. Consider the left multiplication group of the loop Q/M. The stabilizer of $1_{Q/M} = M$ can be obtained as the image of $L_{(M)}\mathcal{L}_1 \trianglelefteq \mathcal{L}$ under the projection $\mathcal{L}(Q) \to \mathcal{L}(Q/M)$. The stabilizer is hence normal, and thus trivial, which means that Q/M is a group. □

Corollary 4.5. *Let Q be a WIP Buchsteiner loop. Then Q/N is a group.*

In [7] a much stronger result is proved. It turns out that Q/N is an abelian group for every Buchsteiner loop Q. The proof of that result is quite complicated.

5. Weak inverse property and nuclear identities

Let Q be a WIP conjugacy closed loop. Kinyon, Kunen and Phillips remark in [20] at the end of Section 6 that one can prove that Q/N is boolean by coupling the results of Basarab [3] on "generalized Moufang loops" with the characterization of WIP CC loops as Wilson loops [18]. We shall give a direct proof (see also Remark 5.4).

Lemma 5.1. *Let Q be a WIP CC loop. Then Q is boolean modulo N.*

Proof. It suffices to prove $x\backslash(1/x) \in N$, by Lemma 3.5. Start by composing the autotopism $(L_x^{-1}R_x, L_x^{-1}, L_x^{-1})$ (the LCC law) with (JR_xI, R_x, R_xJR_xI) (which is the I-shift of $(R_x, L_x^{-1}R_x, R_x)$). We get $(L_x^{-1}R_xM_x, L_x^{-1}R_x, L_x^{-1}R_xM_x)$, where $M_x = JR_xI$. Since $L_x^{-1}R_x(1) = 1$, we obtain $L_x^{-1}R_xM_x = L_{x\backslash 1}L_x^{-1}R_x$, by Lemma 2.1. The loop Q is LCC, and thus $L_xL_{x\backslash 1}L_x^{-1} = L_{1/x}$. Therefore

$$R_xM_x = L_xL_{x\backslash 1}L_x^{-1}R_x = L_{1/x}R_x \quad \text{and} \quad M_x = R_x^{-1}L_{1/x}R_x.$$

By substituting for M_x we get an autotopism

$$(L_x^{-1}L_{1/x}R_x,\ L_x^{-1}R_x,\ L_x^{-1}L_{1/x}R_x).$$

The composition with $(R_x^{-1}, R_x^{-1}L_x, R_x^{-1})$ gives $(L_x^{-1}L_{1/x}, \mathrm{id}_Q, L_x^{-1}L_{1/x})$, and it remains to employ Corollary 2.2. □

Corollary 5.2. *Every WIP CC loop is a Buchsteiner loop.*

Proof. Use Lemma 5.1 and Theorem 3.3. □

We wish to show the converse of Corollary 5.2, i.e. to prove that WIP Buchsteiner loops are conjugacy closed. For that we shall need some further basic facts about WIP loops, as established by Osborn [27]. Let us use the equality $J(xy)x = J(y)$ (cf. Lemma 1.2). From that one derives $J^2(x)J(yx) = J(J(yx) \cdot y)J(yx) = J(y)$, and thus

$$J^2(x)J^2(y) = J(J^2(y)J(xy)) \cdot J^2(y) = J^2(xy).$$

The mappings J^2 and I^2 are hence automorphims of Q. Therefore

$$JR_xI = J^2IR_xJI^2 = J^2L_x^{-1}I^2 = L_{J^2(x)}^{-1}. \tag{14}$$

Let x be an element of a WIP Buchsteiner loop Q. The I-shift of $(L_x, R_x^{-1}, L_xR_x^{-1})$ can be written as $(R_x^{-1}L_{J^2(x)}, L_x, L_{J^2(x)})$, by (14), and $J^2(x) = xa$ for some $a \in N$, by Corollary 4.5. Thus $L_{J^2(x)} = L_xL_a$, and $(L_a^{-1}, \mathrm{id}_Q, L_a^{-1})$ is an autotopism by (4). That means that $(R_x^{-1}L_x, L_x, L_x)$ is an autotopism for every $x \in Q$. We have proved:

Lemma 5.3. *Let Q be a WIP Buchsteiner loop. Then Q is conjugacy closed.*

This result was first announced by Kinyon in a personal communication with one of the authors. His original proof was very long and relied upon properties of Osborn loops [1]. At about the time the above proof was conceived Kinyon found another short proof that does not need Corollary 4.5.

Remark 5.4. This paper is intended to be self-contained, and for that reason we gave a direct proof of Lemma 5.1. If there is assumed knowledge of the fact that Q/N is a group whenever Q is a CC loop, then Corollary 5.2 can be proved directly, in a similar way as Lemma 5.3. Indeed, (JR_xI, R_x, R_xJR_xI), the I-shift of RCC, is equal to $(L_{J^2(x)}^{-1}, R_x, R_xL_{J^2(x)}^{-1})$, and $L_{J^2(x)}^{-1}$ has to be equal to some $L_x^{-1}L_a^{-1}$, where a is nuclear, since $J^2(x) \equiv x \bmod N$. Now, $(L_a, \mathrm{id}_Q, L_a)$ is an autotopism, by (4), and so $(L_x^{-1}, R_x, R_xL_x^{-1})$ is an autotopism for every $x \in Q$.

Theorem 5.5. *Let Q be a WIP loop. If Q satisfies any of the LCC, RCC and Buchsteiner laws, then it fulfils all three of them.*

Proof. From Proposition 1.3 we know that a WIP loop is LCC if and only if it is RCC. The rest follows from Corollary 5.2 and Lemma 5.3. □

Let us repeat again that the loops of Theorem 5.5 can be also characterized as Wilson loops, by Goodaire and Robinson [18].

Lemma 5.6. *An A_ℓ-loop Q is a WIP loop if and only if*

$$y(x(yx)) = (yx)(yx) \text{ for all } x, y \in Q.$$

Proof. Choose $x, y \in Q$ and put $\psi = L_{yx}L_x^{-1}L_y^{-1}$ and $\varphi = L_x^{-1}L_y^{-1}L_{yx}$. We assume $\varphi \in \operatorname{Aut} Q$, and the WIP is equivalent to $\psi(1) = 1$, by Lemma 1.2. Now, $\psi = L_{yx}\varphi L_{yx}^{-1} = L_{yx}L_{\varphi(yx)}^{-1}\varphi$, and so $\psi(1) = 1$ if and only if $\varphi(yx) = yx$. □

We have observed that an A_ℓ-loop is WIP if and only if every triple (x, y, xy) is associative.

Proposition 5.7. *Let Q be a loop in which $N_\lambda = N_\mu$ is of index two. Then Q is a WIP loop if and only if it is a CC loop. This takes place if and only if $N_\lambda = N_\rho$.*

Proof. If Q is a WIP loop, then it is a CC loop, by Proposition 1.9 and Theorem 5.5. If Q is an CC loop, then $N_\lambda = N_\rho$. Assume $N_\lambda = N_\rho$. The equality of Lemma 5.6 clearly holds if one of x, y and yx belongs to N. However, that is always true if $|Q : N| = 2$. □

There exist Buchsteiner CC loops Q (i.e. CC loops that are boolean modulo N) which are not WIP loops. The most natural small example we know can be defined upon F^3, where F is a field of four elements, by formula

$$(x, y, z) \cdot (u, v, w) = (x + u + zv, y + w + zw, z + w). \tag{15}$$

6. Loops with coinciding left and right nuclei

Denote by $\oplus$ the operation on $\{0, 1\}$ such that $a + b \equiv a \oplus b \bmod 2$, for all $a, b \in \{0, 1\}$. We need a separate notation since we shall be mixing $\oplus$ and $+$ when operating with integers. Clearly:

$$a(-1)^b + b = a \oplus b \text{ for all } a, b \in \{0, 1\}. \tag{16}$$

Let F be a field and let $\gamma \in F^*$ be a noninvolutory nontrivial element (thus $\rho \notin \{0, 1, -1\}$). Define a loop Q_γ on $F \times \{0, 1\}$ by

$$(x, a) \cdot (y, b) = (x + \gamma^{a(-1)^b}y, a \oplus b). \tag{17}$$

Sometimes we shall write just $a + b$ for the second coordinate. Note that $(0, 0)$ gives the unit and that

$$\begin{aligned} (x, a)\backslash(y, b) &= (\gamma^{-a(-1)^{a+b}}(y - x), a \oplus b) \text{ and} \\ (y, b)/(x, a) &= (y - \gamma^{(a\oplus b)(-1)^a}x, a \oplus b), \end{aligned} \tag{18}$$

for all $x, y \in F$ and $a, b \in \{0, 1\}$.

The element $(0, 1)$ is of the special importance in the loop and hence the following equalities are worth recording:

$$(0, 1) \cdot (x, a) = (\gamma^{(-1)^a}x, a + 1) \text{ and } (x, a) \cdot (0, 1) = (x, a + 1), \tag{19}$$

for all $x \in F$ and $a \in \{0, 1\}$.

Lemma 6.1. *Let (x, a) be an element of Q_γ. Then*

$$(0, 1) \cdot (x, a)(0, 1) = (0, 1)(x, a) \cdot (0, 1) \tag{20}$$

if and only if $x = 0$. Similarly, each of $(x, 1) \cdot (x, 1)(0, 1) = (x, 1)^2(0, 1)$ and $(0, 1)(x, 1) \cdot (x, 1) = (0, 1)(x, 1)^2$ is true if and only if $x = 0$.

Proof. If $a = 0$, then the left hand side of (20) is equal to $(\gamma^{-1}x, 0)$, while the right hand side evaluates to $(\gamma x, 0)$. For $a = 1$ we get $(\gamma x, 1)$ and $(\gamma^{-1}x, 1)$. The rest is also easy. $\square$

Proposition 6.2. *Let F be a field. For $\gamma \in F^*$, $\gamma \neq \pm 1$, define a loop Q on $F \times \{0,1\}$ by $(x,a)\cdot(y,b) = (x + \gamma^{a(-1)^b} y, a+b)$. Then $N_\lambda = N_\rho = F \times \{0\}$ and $N_\mu = \{(0,0),(0,1)\}$.*

Proof. We have $(x,a)\cdot(0,1)(y,b) = (x,a)(\gamma^{(-1)^b} y, b+1) = (x+\gamma^{a(-1)^{b\oplus 1}+(-1)^b} y, a+b+1)$ and $(x,a)(0,1)\cdot(y,b) = (x,a+1)\cdot(y,b) = (x+\gamma^{(a\oplus 1)(-1)^b} y, a+b+1)$. From (16) we first get $1 \oplus b = (-1)^b + b$, and then $a(-1)^{b\oplus 1} + (-1)^b = (a \oplus b \oplus 1) + (-1)^b - (1 \oplus b) = (a \oplus b \oplus 1) - b = (a \oplus 1)(-1)^b + (b - b)$. This shows that $(0,1)$ is in the middle nucleus. From Lemma 6.1 we see that N_μ contains no further nontrivial element. From Lemma 6.1 we also see that none of the nuclei is equal to Q. Hence it is enough to prove that each $(x,0)$ associates both at the left and the right. We have

$$(x,0)\cdot(y,a) = (x+y,a) \text{ for all } x,y \in F \text{ and } a \in \{0,1\},$$

and from that one gets $(x,0) \in N_\lambda$ in an immediate way.

For the right nucleus we compute $(x+\gamma^{a(-1)^b} y, a+b)\cdot(z,0) = (x+\gamma^{a(-1)^b} y + \gamma^{a\oplus b} z, a+b)$, while $(x,a)\cdot(y,b)(z,0) = (x,a)(y+\gamma^b z, b) = (x+\gamma^{a(-1)^b} y + \gamma^{a(-1)^b + b} z, a+b)$, and we again get the equality by means of (16). $\square$

Proposition 6.3. *Let F be a field. For $\gamma \in F^*$, $\gamma \neq \pm 1$, define a loop Q on $F \times \{0,1\}$ by $(u,a)\cdot(v,b) = (u+\gamma^{a(-1)^b} v, a+b)$. Then $\mathcal{L}_1 = \mathcal{R}_1$ consists of all mappings that are of the form $(u,0) \mapsto (u,0)$, $(u,1) \mapsto (u+x,1)$, where x runs through F. The inner mapping group* $\operatorname{Inn} Q$ *consists of all mappings $(u,0) \mapsto (\gamma^{-i}u, 0)$, $(u,1) \mapsto (\gamma^i u + x, 1)$, where $x \in F$ and $i \in \mathbb{Z}$.*

Proof. The mappings $L_{xy}^{-1} L_x L_y$ fix all elements of N_ρ, and $L_{xy}^{-1} L_x L_y = L_{x'y}^{-1} L_{x'} L_y$ if $x' = ax$ for some $a \in N_\lambda$, in every loop Q. Furthermore, $L_{xy}^{-1} L_x L_y$ is trivial when $x \in N_\lambda$. The left multiplication group $\mathcal{L}$ is generated by the mappings $L_{xy}^{-1} L_x L_y$. To describe them all, it is thus in our case enough, by Proposition 6.2, to consider their action upon $F \times \{1\}$ with the assumption that $x = (0,1)$. Using (16) we see that $(u,1)$ is sent to $(u + (1-\gamma^2)v, 1)$ when $y = (v,0)$, and to $(u + (\gamma - \gamma^{-1})v, 1)$ when $y = (v,1)$. This verifies the structure of $\mathcal{L}$.

Similarly, to get $\mathcal{R}$ we need only to investigate the action of $R_{yx}^{-1} R_x R_y$ upon $Q \setminus N_\lambda$ in the case $x = (0,1)$. For $y = (v,0)$ we obtain $(u,1) \mapsto (u + (\gamma - \gamma^{-1})v, 1)$, while $(u,1) \mapsto (u + (\gamma^{-1} - \gamma)v, 1)$ when $y = (v,1)$.

If $y = (v,0)$, then $L_y^{-1} R_y$ fixes every $(u,0)$, and $(u,1)$ is mapped unto $(u + (\gamma - 1)v, 1)$. If $y = (v,1)$, then $L_y^{-1} R_y : (u,0) \mapsto (\gamma^{-1}u, 0)$, $(u,1) \mapsto (\gamma u + (1-\gamma)v, 1)$. $\square$

Corollary 6.4. *Let p be a prime and let m be a proper divisor of $p-1$. Then there exists a loop Q on $2p$ elements such that* $\operatorname{Inn} Q$ *is of order mp and has a trivial centre.*

Let us turn again to the structure of Q_γ in the general case. We shall state two lemmas that can be verified in a straightforward way.

Lemma 6.5. *Let U be the subgroup of* $\mathrm{Mlt}\, Q_\gamma$ *generated by $L_{(0,1)}$ and $R_{(0,1)}$. Then U is isomorphic to a dihedral group and contains a cyclic subgroup of index two that is formed by all mappings $(u,0) \mapsto (\gamma^i u, 0)$, $(u,1) \mapsto (\gamma^{-i}u, 1)$.*

Lemma 6.6. *Let $Q = Q_\gamma$. Then $\mathcal{L} \cap \mathcal{R} \trianglelefteq \mathrm{Mlt}\, Q$, and $\mathcal{L} \cap \mathcal{R} \cong F(+) \times F(+)$ consists of mappings $(u,0) \mapsto (u+x, 0)$, $(u,1) \mapsto (u+y, 1)$, where x and y run through F. Furthermore, $|\mathcal{L} : \mathcal{L} \cap \mathcal{R}| = |\mathcal{R} : \mathcal{L} \cap \mathcal{R}| = 2$, and the dihedral group U generated by $L_{(0,1)}$ and $R_{(0,1)}$ forms in* $\mathrm{Mlt}\, Q$ *a complement to $\mathcal{L} \cap \mathcal{R}$.*

Corollary 6.7. *Let $Q = Q_\gamma$. Then neither $\mathcal{L}$ nor $\mathcal{R}$ is a normal subloop of* $\mathrm{Mlt}\, Q$. *The group* $\mathrm{Mlt}\, Q$ *is solvable, and its centre is trivial. The middle nucleus is not a normal subloop of Q.*

Proof. If a dihedral group normalizes an involution, then this involution has to be central. From that we see that neither $\mathcal{L}/(\mathcal{L} \cap \mathcal{R})$ nor $\mathcal{R}/(\mathcal{L} \cap \mathcal{R})$ is normal in $(\mathrm{Mlt}\, Q)/(\mathcal{L} \cap \mathcal{R})$, by Lemma 6.6. That lemma makes also clear that $\mathrm{Mlt}\, Q$ is a solvable group. The centre of $\mathrm{Mlt}\, Q$ is determined by $Z(Q)$, and $Z(Q) = 1$ follows from $N_\lambda \cap N_\mu = 1$. The middle nucleus cannot be a normal subloop because it is not retained by the inner mappings, by Proposition 6.3. □

For some time it has been an open question if $\mathrm{Mlt}\, Q$ has to be solvable whenever $\mathrm{Inn}\, Q$ is a nonabelian group of order pq. Partial solutions appeared in [10, 22, 23, 24, 25], and the final (positive) answer can be found in [11]. There are reasons to believe that one will be evenutally able to characterize all loops Q with $|\mathrm{Inn}\, Q| = pq$. In this section we described one of the several possible constructions (cf. Corollary 6.4). Another construction is that of nonassociative conjugacy closed loops of order pq [14].

Let us discuss the naturally arising isomorphism problems. Note that some information about the element γ can be recovered from the abstract structure of Q_γ, as $(0,1)$ is determined by the middle nucleus, and $(0,1)(u,0) \cdot (0,1) = (\gamma u, 0)$.

Lemma 6.8. *Loops Q_γ and $Q_{\gamma'}$ are isomorphic if and only if there exists $\alpha \in$* $\mathrm{Aut}\, F(+)$ *such that $\alpha(\gamma x) = \gamma' \alpha(x)$ for all $x \in F$.*

Proof. Assume $Q_\gamma \cong Q_{\gamma'}$. The isomorphism has to map the left nucleus upon the left nucleus, and hence the isomorphim must extend some $\alpha \in \mathrm{Aut}\, F(+)$. The isomorphism also has to map $(0,1)$ upon $(0,1)$, and therefore it sends $(u,1) = (u,0)(0,1)$ onto $(\alpha(u), 1)$. The equality $\alpha(\gamma x) = \gamma' \alpha(x)$ follows from the above expression of $(\gamma u, 0)$. The converse direction can be easily verified from (17) in a direct way. □

We shall now construct a loop $Q(*)$ on $F \times \{0,1\}$ such that $\varphi : Q_{\gamma^{-1}} \cong Q(*)$, $(u,a) \mapsto (\gamma^{2a}u, a)$.

Using (16) we can verify immediately that

$$-a(-1)^b - 2b + 2(a \oplus b) = a(-1)^b \text{ and } -a + (a \oplus b) = b(-1)^a, \tag{21}$$

for all $a, b \in \{0,1\}$. Our goal is to show that

$$(u,a) * (v,b) = (\gamma^{2b(-1)^a} u + \gamma^{a(-1)^b} v, b) = (0,1)((0,1)(u,a) \cdot (v,b)), \tag{22}$$

for all $a, b \in \{0,1\}$ and $u, v \in F$. From (21) we get the first equality since $(u,a) * (v,b) = \varphi((\gamma^{-2a}u, a)(\gamma^{-2b}v, b)) = \varphi(\gamma^{-2a}u + \gamma^{-a(-1)^b - 2b}v, a \oplus b)$. To obtain the second equality we use (17) and (19). We have $(0,1)\cdot(\gamma^{(-1)^a}u, a+1)(v,b) = (0,1)(\gamma^{(-1)^a}u + \gamma^{(a\oplus 1)(-1)^b}v, a+b+1)$. To verify the equality it suffices, by (17), to see that $(-1)^a + (-1)^{a+b+1} = (-1)^a(1+(-1)^{b+1}) = 2b(-1)^a$ and $(-1)^{a+b+1} + (a\oplus 1)(-1)^b = (-1)^b((-1)^{a+1} + (a\oplus 1)) = (-1)^b a$.

Proposition 6.9. *Let F be a field. For $\gamma \in F^*$, $\gamma \neq \pm 1$, let Q_γ be the loop on $F\times\{0,1\}$ where $(u,a)\cdot(v,b) = (u+\gamma^{a(-1)^b}v, a+b)$ for all $u,v\in F$ and $a,b\in\{0,1\}$. Then each loop isotope of Q_γ is isomorphic either to Q_γ or to $Q_{\gamma^{-1}}$.*

Proof. Every loop isotope of a loop Q is isomorphic to a loop with multiplication $(x/e)\cdot(f\backslash y)$, for some $e, f\in Q$. One can introduce e and f sequentially, and hence it will suffice to show that each of operations $x/e\cdot y$ and $x\cdot f\backslash y$ gives a loop isomorphic to Q_γ or $Q_{\gamma^{-1}}$. Since $x\mapsto xe$ yields an isomorphism between $(x\cdot ye)/e$ and $x/e\cdot y$, we see that one can, in fact, investigate only the operations $(x\cdot ye)/e$ and $e\backslash(ex\cdot y)$. If $e\in N_\lambda\cap N_\rho$, then both operations yield xy, and so the isotopes coincide with the original loop. If $a\in N_\rho$, then $(x\cdot ye)/e = (x\cdot yea)/(ea)$, and so only the case $e=(0,1)$ needs to be discussed, by Proposition 6.2.

Now, $e^2=1$ and $e\in N_\mu$ imply that $e\backslash(ex\cdot y) = e(ex\cdot y)$. The formula for this operation is given in (22), and so the isotope coincides with $Q(*)\cong Q_{\gamma^{-1}}$.

Let us now compute $(x\cdot ye)e$, where $x=(u,a)$, $y=(v,b)$ and $e=(0,1)$. We obtain

$$(u,a)(v,b+1)\cdot(0,1) = (u+\gamma^{a(-1)^{b+1}}, a+b) = (u, (\gamma^{-1})^{a(-1)^b}, a+b),$$

which means that this isotope coincides with $Q_{\gamma^{-1}}$. □

Recall that by a *G-loop* one understands any loop Q that is isomorphic to all of its loop isotopes.

Corollary 6.10. *Let p be a prime and let $1\le r<k$ be such that p^r+1 divides p^k-1 (for example, $k=2r$). Then there exists a G-loop of order $2p^k$ which satisfies no nuclear identity.*

Proof. Let F be a field of order p^k. By Proposition 6.9 it suffices to find $\gamma\in F^*$, $\gamma\neq\pm1$, such that $Q_\gamma\cong Q_{\gamma^{-1}}$. From Lemma 6.8 we know that this will be satisfied when $\gamma^{-1}=\alpha(\gamma)$ for some $\alpha\in\operatorname{Aut}F$. It is hence enough to find $\gamma\in F^*$ of order p^r+1 since then $\gamma^{p^r}=\gamma^{-1}$. The condition $p^r+1\mid p^k-1$ guarantees the existence of such γ because $F^*(\cdot)$ is a cyclic group of order p^k-1. □

All conjugacy closed loops are G-loops [17] and Buchsteiner loops are G-loops as well [9]. By Kunen [21] the class of G-loops has to be very rich, and cannot be captured by any set of first order formulas that would involve only the loop operations. Nevertheless it seems that prior to our construction no other infinite series of nonnuclear (finite) G-loops was described.

7. Conclusions and prospects

The variety of loops is too large to show structural behaviour similar to that of variety of groups. Intuitively said, the nature of general loops is rather combinatorial than algebraic. The question arises which loops are of algebraic interest, or, in other words, what are the qualities that can set some loops apart. One way is to look for identities that are easy to express. Because the Bol and Moufang identities are the most classical, one has often considered their closest generalizations, i.e. loops that can be described by an identity $t(x, y, z) = s(x, y, z)$, where y and z appear exactly once in both t and s, the variable x appears twice on the each side, and both s and t are terms that use only multiplication. Such identities are said to be of Bol-Moufang type, and the program to classify them was started by Fenyves in [16] and completed by Phillips and Vojtěchovský in [30]. When one abstracts from identities that can be also expressed by terms of length three, and from the identities that express the presence of squares in some of the nuclei, there remain only few Bol-Moufang varieties: extra loops, left and right Bol loops, Moufang loops and the so called LC, RC and C loops (identities $(xx)(yz) = (x(xy))z$, $x((yz)z) = (xy)(zz)$ and $x(y(yz)) = ((zy)y)x$, respectively).

We expect that in the near future there will be classified all varieties $t(x, y, z) = s(x, y, z)$ that involve any of the three binary operations $\cdot$, $/$ and $\backslash$ (assuming that y and z appear once in s and t, and x appears twice in y and z). This paper can be regarded as a first step towards such classification. However, we took more limited and more structurally guided approach that allowed us to reconstruct a number of earlier results in a coherent way. We have observed that none of the varieties obtained by a nuclear identification contains a loop Q such that $N_\lambda = N_\rho \neq N_\mu$. Hence the construction of Section 6 can be also regarded as a challenge to find a sufficiently general variety in which the constructed loops could live.

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Dept. of Mathematics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Rep.

E-mail address: drapal@karlin.mff.cuni.cz

Dept. of Mathematics, Czech University of Agriculture, Kamýcká 129, 165 21 Praha 6, Czech Rep.

E-mail address: jedlickap@tf.czu.cz