COMMUTATIVE SEMIGROUPS THAT ARE NIL OF INDEX 2 AND HAVE NO IRREDUCIBLE ELEMENTS

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Abstract. Every commutative nil-semigroup of index 2 can be imbedded into such a semigroup without irreducible elements.

1. Introduction

Throughout this note, the word *semigroup* will always mean a commutative semigroup, the binary operation of which will be denoted additively.

1.1 An element w of a semigroup S is called *absorbing* if $S + w = w$. There exists at most one absorbing element in S and it will be denoted by the symbol o $(= o_S)$ in the sequel. The fact that S possesses the absorbing element will be denoted by $o \in S$.

1.2 A non-empty subset I of S is an *ideal* if $S + I \subseteq I$.

1.3 Lemma. (i) *A one-element subset $\{w\}$ is an ideal iff $w = o_S$.*
(ii) *If I is an ideal then the relation $r = (I \times I) \cup \mathrm{id}_S$ is a congruence of S and $I = o_T$, where $T = S/r$.*
(iii) *If $o \in S$ and and s is a congruence of S then the set $\{\, a \in S \mid (a, o) \in s \,\}$ is an ideal.* $\square$

1.4 Put $(Q_S(a) =)\ Q(a) = S + a$ and $(P_S(a) =)\ P(a) = Q(a) \cup \{a\}$ for every $a \in S$.

1.5 Lemma. (i) *$Q(a) \subseteq P(a)$ and both these sets are ideals of S.*
(ii) *$P(a)$ is just the (principal) ideal generated by the one-element set $\{a\}$.* $\square$

1.6 Assume that $o \in S$. An element $a \in S$ is said to be *nilpotent (of index at most $m \geq 1$)* if $ma = 0$. We denote by $N(S)$ $(N_m(S))$ the set of nilpotent (of index at most m) elements of S.

The semigroup S is said to be *nil* (*of index at most* m) if $N(S) = S$ $(N_m(S) = S)$ and *reduced* if o_S is the only nilpotent element of S.

1.7 Lemma. (i) *$o = N_1(S) \subseteq N_2(S) \subseteq N_3(S) \subseteq \dots$ and all these sets are ideals.*
(ii) *$N(S) = \bigcup N_m(S)$ is an ideal.*
(iii) *The factor-semigroup $T = S/N(S)$ is reduced.* $\square$

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1.8 Lemma. *The following conditions are equivalent:*

(i) $o \in S$ *and* $2x = o$ *for every* $x \in S$.
(ii) S *is nil of index at most 2.*
(iii) $2x + y = 2z$ *for all* $x, y, z \in S$.
(iv) $2x + y = 2x$ *for all* $x, y \in S$. $\square$

1.9 A semigroup satisfying the equivalent conditions of 1.8 will be called *zeropotent* (or, in a colourless manner, a *zp-semigroup*) in the sequel.

A zp-semigroup without irreducible elements (i.e., when $S+S = S$) will be called a *zs-semigroup.*

1.10 Define a relation $|_S$ on S by $a|_S b$ iff $b = a + u$ for some $u \in S^0$, where S^0 is the least monoid containing S and 0 denotes the neutral element of S^0.

1.11 Lemma. *The following conditions are equivalent:*

(i) $a|_S b$.
(ii) $b \in P(a)$.
(iii) $P(b) \subseteq P(a)$.

Moreover, if $a \neq b$ *then these conditions are equivalent to:*

(iv) $b \in Q(a)$.
(v) $P(b) \subseteq Q(a)$. $\square$

1.12 Lemma. *The relation* $|_S$ *is a fully invariant compatible quasiordering of the semigroup* S *and the equivalence* $||_S = \ker(|_S)$ *is a fully invariant congruence of the semigroup* S. $\square$

1.13 Lemma. *The following conditions are equivalent:*

(i) $a||_S b$.
(ii) $P(a) = P(b)$.

Moreover, if $a \neq b$ *then these conditions are euqivalent to:*

(iii) $Q(a) = Q(b) = P(a) = P(b)$. $\square$

1.14 Lemma. *The following conditions are equivalent:*

(i) S *is a group.*
(ii) $|_S = S \times S$.
(iii) $||_S = S \times S$.
(iv) $P(a) = P(b)$ *for all* $a, b \in S$.
(v) $P(a) = S$ *for every* $a \in S$.
(vi) $Q(a) = S$ *for every* $a \in S$. $\square$

1.15 Lemma. *The relation* $|_S$ *is a (fully invariant compatible) ordering (or, equivalently,* $||_S = \mathrm{id}_S$*), provided that at least one of the following four conditions is satisfied:*

(1) S *is not a group and* id_S, $S \times S$ *are the only fully invariant congruences of* S*;*
(2) S *is cancellative and* $0 \notin S$*;*
(3) S *is nil;*
(4) S *is idempotent.*

Proof. (1) Combine 1.13 and 1.14.
(2) If $a \neq b$, $b = a+u$ and $a = b+v$, $a, b, u, v \in S$, then $a = a+w$, where $w = u+v$, and hence $w = 0$, a contradiction.
(3) If $a = a + w$, $a, w \in S$, then $a = a + mv$ for every $m \geq 1$, and hence $a = o$.
(4) If $b = a + u$, $a, b, u \in S$, then $a + b = a + a + u = a + u = b$. $\square$

1.16 Define a relation $/_S$ on S by $a/_S b$ iff $Q(b) \subseteq Q(a)$.

1.17 Lemma. *The relation $/_S$ is an invariant compatible quasiordering of the semigroup S and the equivalence $//_S = \ker(/_S)$ is an invariant congruence of the semigroup S.* $\square$

1.18 Lemma. *The following conditions are equivalent:*

(i) $/_S = S \times S$.
(ii) $//_S = S \times S$.
(iii) $S + a = S + b$ *for all* $a, b \in S$.
(iv) $S + S = I$ *is the smallest ideal of* S *and* I *is a subgroup of* S. $\square$

2. The distractibility ordering of zp-semigroups

2.1 In this section, let S be a zp-semigroup. Put $\mathrm{Ann}(S) = \{\, a \in S \mid S + a = o \,\}$.

2.1 Lemma. (i) *The relation $|_S$ is a fully invariant compatible ordering of the semigroup S.*
(ii) *o is the greatest element.*
(iii) $\mathrm{Ann}(S) \setminus \{o\}$ *is the set of maximal elements of* $T = S \setminus \{o\}$.
(iv) *If* $|S| \geq 2$ *then* $S \setminus (S + S)$ *is the set of minimal elements of* S.
(v) *If* $|S| \geq 3$ *then* S *has no smallest element.* $\square$

2.3 Lemma. *If S is a non-trivial zs-semigroup then S has no minimal elements, S is infinite and not finitely generated.*

Proof. Being nil, S is finitely generated iff it is finite. The rest is clear from 2.2(iv). $\square$

2.4 Lemma. *If* $0 \in S$ *then* S *is trivial.* $\square$

3. Every zp-semigroup is a subsemigroup of a zs-semigroup

3.1 Proposition. *Every zp-semigroup is a subsemigroup of a zs-semigroup.*

Proof. Let S be a non-trivial zp-semigroup and $Q = S \setminus (S + S)$. For every $a \in Q$, put $R_a = S \setminus P(a)$; then $o \notin R_a$ and $R_a \neq \emptyset$, provided that $|S| \geq 3$. Further, $0 \notin S$ by 2.4 and we put $R_{a,0} = R_a \cup \{0_a\}$, where the elements 0_a, $a \in Q$, are all distinct, $V_{a,1} = R_{a,0} \times \{1\}$ and $V_{a,2} = R_{a,0} \times \{2\}$. Now, consider the disjoint union

$$T = S \cup \bigcup_{a \in Q} V_{a,1} \cup \bigcup_{a \in Q} V_{a,2}$$

and define an addition on T in the following way:
(1) $x + y$ coincides in $S(+)$ and $T(+)$ for all $x, y \in S$;
(2) $x + (y, i) = (x + y, i) = (y, i) + x$ for all $x \in S$, $(y, i) \in V_{a,i}$, $a \in Q$, $i = 1, 2$, $x + y \in R_a$ (i.e., $x + y \notin P(a)$);

(3) $(x,i)+(y,j)=x+y+a$ for all $x,y\in R_{a,0}$, $a\in Q$, $i\neq j$;
(4) $\alpha+\beta=o$ if $\alpha,\beta\in T$ and the sum $\alpha+\beta$ is not defined by (1), (2) or (3).

Clearly, $\alpha+\beta=\beta+\alpha$, $\alpha+\alpha=o$, $\alpha+o=o$ and $o+\alpha=o$ for every $\alpha\in T$. Next, we check that $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$ for all $\alpha,\beta,\gamma\in T$.

Put $\delta=\alpha+(\beta+\gamma)$, $\epsilon=(\alpha+\beta)+\gamma$ and consider the following cases:
(a) $\alpha,\beta,\gamma\in S$. Then $\delta=\epsilon$ by (1).
(b) $\alpha,\beta\in S$ and $\gamma=(x,i)\in V_{a,i}$. Assume first that $\alpha+\beta+x\in R_a$. Then $\epsilon=(\alpha+\beta+x,i)$ by (2). Moreover, $\beta+x\in R_a$, and hence $\beta+\gamma=(\beta+x,i)$ and $\delta=\alpha+(\beta+x,i)=(\alpha+\beta+x,i)=\epsilon$.

Assume next that $\alpha+\beta+x\notin R_a$. Then $\epsilon=o$ by (4). Moreover, either $\beta+x\notin R_a$, $\beta+\gamma=o$ and $\delta=\alpha+o=o=\epsilon$, or $\beta+x\in R_a$, $\beta+\gamma=(\beta+x,i)$ and $\delta=\alpha+(\beta+x,i)=o=\epsilon$.
(c) $\alpha,\gamma\in S$, $\beta\in V_{a,i}$ (or $\beta,\gamma\in S$, $\alpha\in V_{a,i}$). These cases are similar and/or dual to (b).
(d) $\alpha=(x,i)\in V_{a,i}$, $\beta=(y,i)\in V_{a,i}$ and $\gamma\in S$. Then $\alpha+\beta=o$ by (4), and so $\epsilon=o+\gamma=o$. Assume first that $y+\gamma\in R_a$. Then $\beta+\gamma=(y+\gamma,i)$ by (2) and $\delta=(x,i)+(y+\gamma,i)=o$ by (4). Thus $\epsilon=\delta$.

Assume next that $y+\gamma\notin R_a$. Then $\beta+\gamma=o$ by (4) and $\delta=(x,i)+o=o=\epsilon$.
(e) $\alpha,\gamma\in V_{a,i}$, $\beta\in S$ (or $\beta,\gamma\in V_{a,i}$, $\alpha\in S$). These cases are similar to (d).
(f) $\alpha=(x,i)\in V_{a,i}$, $\beta=(y,j)\in V_{a,j}$, $i\neq j$, $\gamma\in S$. Then $\alpha+\beta=x+y+a$ by (3), and hence $\epsilon=x+y+a+\gamma$ by (1). Assume first that $y+\gamma\in R_a$. Then $\beta+\gamma=(y+\gamma,j)$ by (2) and $\delta=(x,i)+(y+\gamma,j)=x+y+\gamma+a=\epsilon$.

Assume next that $y+\gamma\notin R_a$. Then $\beta+\gamma=o$ by (4), and hence $\delta=(x,i)+o=o$. However, $y+\gamma\notin R_a$ means $y+\gamma\in P(a)$ and then $a+y+\gamma=o$, since S is nil of index at most 2. Thus $\epsilon=x+a+y+\gamma=x+o=o=\delta$.
(g) $\alpha\in V_{a,i}$, $\gamma\in V_{a,j}$, $\beta\in S$ (or $\beta\in V_{a,i}$, $\gamma\in V_{a,j}$, $\alpha\in S$). These cases are similar to (f).
(h) $\alpha,\beta,\gamma\in V_{a,i}$. Then $\beta+\gamma=o=\alpha+\beta$, and hence $\delta=a+o=o=o+\gamma=\epsilon$.
(i) $\alpha=(x,i)\in V_{a,i}$, $\beta=(y,i)\in V_{a,i}$ and $\gamma=(z,j)\in V_{a,j}$, $i\neq j$. Then $\alpha+\beta=o$ by (4), and hence $\epsilon=o+(z,j)=o$. Further, $\beta+\gamma=y+z+a$ by (3). Now, $x+y+z+a\in P(a)$ and $\delta=(x,i)+y+z+a=o$ by (4). Thus $\delta=\epsilon$.
(j) $\alpha,\gamma\in V_{a,i}$, $\beta\in V_{a,j}$ (or $\beta,\gamma\in V_{a,i}$, $\alpha\in V_{a,j}$). These cases are similar to (i).
(k) In all the remaining cases we get $\delta=o=\epsilon$ due to (4).

We have shown that $T=T(+)$ is a zp-semigroup and S is a subsemigroup of T. Clearly,

$$T+T=S\cup\bigcup_{a\in Q}(R_a\times\{1\})\cup\bigcup_{a\in Q}(R_a\times\{2\})\,.$$

Thus $S\subseteq T+T$ and

$$T\setminus(T+T)=\bigcup_{a\in Q}\{(0_a,1),(0_a,2)\}\,.$$

Finally, put $T_0=S$, $T_1=T$ and consider a sequence

$$T_0\subseteq T_1\subseteq T_2\subseteq\dots$$

of zp-semigroups such that T_i is a subsemigroup of T_{i+1} and $T_i\subseteq T_{i+1}+T_{i+1}$. Then $\bigcup T_i$ is a zs-semigroup. $\square$

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