ON GEOMETRICAL STRUCTURE AND CONSTRUCTION OF LATIN TRADES

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Abstract. Latin trades can be intrepreted geometrically as oriented surfaces. Those of genus zero are called spherical. The paper contains a construction that shows how all latin trades can be obtained from the spherical ones by a cut-and-paste method that raises the genus.

By latin trade one usually understands a fragment of a latin square for which there exists another fragment that can replace it in the square and has no overlapping cell with the original fragment. In Section 1 we shall introduce an alternative definition that makes it easy to associate with each pair of latin trades a combinatorial oriented surface. In further sections we develop various geometrical properties of these surfaces (sometimes we will call them trading surfaces). We shall show that a trading surface is spherical if and only if each multigon is separating (cf. Section 2 for the definition of the multigon, which is a closed oriented polygonal curve with additional properties). We achieve this result after making a number of auxiliary statements that describe various elementary configurations of surface faces. While such a characterization of trading spherical surfaces is very similar to a well known characterization of combinatorial spheres as combinatorial surfaces where each polygon is separating, the additional structure associated with the notion of multigon seems to make difficult any direct application of standard theorems.

Once we know that every pair of latin trades that yields a surface of higher genus has to contain a simple nonseparating triangle free multigon, we can apply a cut-and-paste construction that builds all trades of higher genus from the spherical ones (i.e. from the trading spheres). The construction can be presented rather simply, but the proof of its correctness is technically quite complicated (cf. Section 8).

The paper should be understood as a part of a broader effort to study latin trades from the geometrical viewpoint. While the association of latin trades with surfaces has been already made available [8, 12], up to now there seems to be no paper published that would develop a framework necessary to embark on a more serious treatement of problems that are naturally associated with geometrical aspects of latin trades (interactions of size and genus, enumerations based on geometrical characteristics, derivations of homogeneous latin trades by cut-and-paste constructions, extremal constructions with respect to the genus, pasting of those latin trades that are obtained by a dissection of an equilateral triangle, and many others). This paper has the ambition to serve such a purpose.

Another paper [10] describes how all trading spheres can be obtained from the most elementary one by a successive application of certain moves, each of them adding from two to six new 2-cells.

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The terminology seems to be standard, with one exception. The term "face" is not used in the generic meaning, but will serve as a synonymous for "2-cell". Unlike some other authors we do not insist upon a (combinatorial) surface being connected. Our surfaces can have more connectivity components (but only finitely many).

The ideas presented in this paper can be found already in [7]. However, I did not see clearly the geometrical intepretation of the results at that time, and I felt that the formal setting in which the results were originally formulated is too complicated to warrant publication. After realizing that combinatorial surfaces provide a more intuitive framework, I tried for some time to find shorter proofs that would utilize known facts of combinatorial geometry. This effort failed. On the other hand, results and observations motivated by the geometrical approach to latin bi-trades have been increasing [12, 2, 3, 4], further papers are in preparation, and so I finally decided to rewrite and expand [7] in a way that would systematically refer to the interpretation by trading surfaces. Parts of the text were circulated in a preliminary form [9].

1. Introduction

By a *latin bi-trade* we shall understand a pair (T°, T^*), where both T° and T^* are subsets of a set $A_1 \times A_2 \times A_3$ such that:

(R1) Sets T° and T^* are disjoint;
(R2) for all $(a_1, a_2, a_3) \in T^\circ$ and all $r, s \in \{1, 2, 3\}$, $r \neq s$, there exists exactly one $(b_1, b_2, b_3) \in T^\circ$ with $a_r = b_r$ and $a_s = b_s$;
(R3) for all $(b_1, b_2, b_3) \in T^*$ and all $r, s \in \{1, 2, 3\}$, $r \neq s$, there exists exactly one $(a_1, a_2, a_3) \in T^\circ$ with $a_r = b_r$ and $a_s = b_s$.

Lemma 1.1. *Suppose that both (a_1, a_2, a_3) and (a'_1, a'_2, a'_3) belong to T° (or T^*). If they agree on two coordinates, then they agree in the third coordinate as well.*

Proof. For example assume that both triples belong to T° and agree in the first two coordinates. Consider $(a_1, a_2, b_3) \in T^*$ that is induced by (R2). We obtain $a_3 = a'_3$ by applying (R3) to (a_1, a_2, b_3) in place of (b_1, b_2, b_3). □

Both T° and T^* can thus be regarded as partial latin squares, where a_3 is the entry value for the cell determined by row a_1 and column a_2. This connects our definition of latin bi-trade to the standard definition of *latin trade*. Latin trades have been studied under various names, and both [11] and [12] are excellent sources for information about the history and origin of the notion. One of the earlier approaches uses the term *exchangeable partial groupoid* [5][6] which reflects an interpretation of T° and T^* by binary operations, with $a_1 \circ a_2 = a_3$ for $(a_1, a_2, a_3) \in T^\circ$ and $b_1 * b_2 = b_3$ for $(b_1, b_2, b_3) \in T^*$.

In addition to (R1)–(R3) we shall always assume, in this paper, that (T°, T^*) satisfies:

(R4) The sets A_1, A_2 and A_3 are pairwise disjoint; and
(R5) for all $\alpha \in \bigcup A_i$, $1 \leq i \leq 3$, there exists $(a_1, a_2, a_3) \in T^\circ$ with $a_i = \alpha$.

These additional assumptions seem to be reasonable here since we shall investigate the intrinsic structure of latin trades, and not their embeddings into latin squares.

Latin bi-trades (T°, T^*), $T^\circ \subseteq A_1 \times A_2 \times A_3$, and (S°, S^*), $S^\circ \subseteq B_1 \times B_2 \times B_3$, are said to be *isotopic* if there exist bijections $\varphi_i : A_i \to B_i$, $1 \le i \le 3$, such that $(a_1, a_2, a_3) \in T^\circ$ (or T^*) if and only if $(\varphi_1(a_1), \varphi_2(a_2), \varphi_3(a_3)) \in S^\circ$ (or S^*).

Consider $r, s \in \{1, 2, 3\}$, $r \ne s$, and denote by t the third element of $\{1, 2, 3\}$. A permutation $\sigma_{s,r}$ of T° is defined in two stages as follows:

(1) For $a = (a_1, a_2, a_3) \in T^\circ$ find $b = (b_1, b_2, b_3) \in T^*$ with $a_s = b_s$ and $a_t = b_t$;
(2) put $\sigma_{s,r}(a) = (a'_1, a'_2, a'_3) \in T^\circ$, where $a'_r = b_r$ and $a'_t = b_t$.

Thus $\sigma_{s,r}$ firstly changes the r-th coordinate, and then the s-th coordinate. Clearly $a'_t = a_t$, $a'_s \ne a_s$ and $a'_r \ne a_r$. Stage (1) expresses a bijection $\beta_r : T^\circ \to T^*$ and stage (2) expresses the inverse bijection $\beta_s^{-1} : T^* \to T^\circ$. The equality $\sigma_{s,r} = \beta_s^{-1}\beta_r$ implies the next lemma in an obvious way.

Lemma 1.2. *Let $r, s, t \in \{1, 2, 3\}$ be pairwise distinct. Then $\sigma_{t,s}\sigma_{s,r} = \sigma_{t,r}$ and $\sigma_{r,s} = \sigma_{s,r}^{-1}$.*

From Lemma 1.2 it follows that using two subscripts in $\sigma_{r,s}$ is superfluous. Instead we shall use notation $\tau_1 = \sigma_{2,3}$, $\tau_2 = \sigma_{3,1}$ and $\tau_3 = \sigma_{1,2}$. These three permutations will be sometimes called the *structural permutations* of (T°, T^*), and (τ_1, τ_2, τ_3) will be known as its *structural triple*. Lemma 1.2 gives

Corollary 1.3. *Let (T°, T^*) be a latin bi-trade. Then $\tau_1\tau_2\tau_3 = \tau_2\tau_3\tau_1 = \tau_3\tau_1\tau_2$ is the identity on T°.*

Each permutation τ_i decomposes to cycles, and the aggregate number of these cycles will be known as the *permutational order* of the latin bi-trade. We shall call it just *order* if there will be no danger of misunderstanding. The number of elements of T° (which is identical with the number of elements of T^*) will be called the *size* of the latin bi-trade.

We see that the triple (τ_1, τ_2, τ_3) fulfils the following two conditions:

(P1) A cycle of τ_r has at most one point in common with a cycle of τ_s, whenever $1 \le r < s \le 3$; and
(P2) all permutations τ_i, $1 \le i \le 3$, are fixed point free and $\tau_1\tau_2\tau_3$ is the identity.

Let τ be a permutation of a set X and let $\rho = (x_1\ x_2\ \dots\ x_k)$ be one of its cycles (with respect to the cyclic decomposition of τ). A point $x \in X$ is said to be *incident* to ρ if and only if $x = x_j$ for some j, $1 \le j \le k$. Thus x is incident to ρ if and only if ρ *moves* x.

If (T°, T^*) is a latin bi-trade with structural triple (τ_1, τ_2, τ_3), then $\tau_i(a_1, a_2, a_3) = (b_1, b_2, b_3)$ implies $a_i = b_i$ for all $a = (a_1, a_2, a_3) \in T^\circ$ and all $i \in \{1, 2, 3\}$. Let a be incident to the cycle ρ of τ_i. Then $a_i = x_i$ for all $x = (x_1, x_2, x_3) \in T^\circ$ that are incident to ρ. This means that a_i is an invariant of ρ; call it the *value* of ρ. The latin bi-trade is said to be *separated* if for each $a_i \in A_i$ there exists only one cycle ρ of τ_i that has a_i as its value, $1 \le i \le 3$.

Proposition 1.4. *Let τ_i, $1 \le i \le 3$, be permutations of a set X that satisfy (P1) and (P2). Denote by A_i, $1 \le i \le 3$, the set of all cycles of the permutation τ_i. Define $T^\circ, T^* \subseteq A_1 \times A_2 \times A_3$ by*

$(\rho_1, \rho_2, \rho_3) \in T^\circ \Leftrightarrow$ all ρ_i, $1 \le i \le 3$, are incident to some $x \in X$, and
$(\rho_1, \rho_2, \rho_3) \in T^* \Leftrightarrow$ there exists an element $x \in X$ such that ρ_3 moves x, ρ_2 moves $\rho_3(x)$, and ρ_1 moves $\rho_2(\rho_3(x))$.

Then (T°, T^*) *is a latin bi-trade.*

Proof. If $(\rho_1, \rho_2, \rho_3) \in T^*$, then there exists $y = \rho_3(x)$ such that ρ_2 moves y, ρ_1 moves $\rho_2(y)$ and ρ_3 moves $\rho_1\rho_2(y)$, since $\rho_1\rho_2\rho_3(x) = x$. The definition of T^* in the proposition is hence invariant with respect to the cyclic shift of indices, and so to prove (R2) and (R3) we can assume, say, $r = 1$ and $s = 3$. Note that (R1) follows from the fact that all τ_i are fixed point free.

If $(\rho_1, \rho_2, \rho_3) \in T^*$, then by the definition of T^* there exists $x \in X$ which is incident both to ρ_3 and ρ_1. Such an x is unique, by (P1), and hence there exists unique ρ_2' with $(\rho_1, \rho_2', \rho_3) \in T^\circ$ (this ρ_2' is the cycle of τ_2 that moves x). We have proved (R3).

To prove (R2) consider $(\rho_1, \rho_2, \rho_3) \in T^\circ$, and let x be the common point of ρ_i, $1 \le i \le 3$. By (P1), $\tau_1\tau_2\rho_3(x)$ equals x. Since ρ_1 is the cycle of τ_1 which moves x, we have, in fact, $\rho_1\tau_2\rho_3(x) = x$. Denote by ρ_2' the cycle of τ_2 that moves $\rho_3(x)$. We obtain $(\rho_1, \rho_2', \rho_3) \in T^*$. The cycle ρ_2' is determined uniquely since any such ρ_2' has to move, by the definition of T^*, some $\rho_3(y)$, where y is incident to both ρ_1 and ρ_3. However, such a y has to equal x, by (P1) and by the preceding part of the proof. □

Starting from a latin bi-trade $S = (S^\circ, S^*)$, $S^\circ \subseteq B_1 \times B_2 \times B_3$, one can first consider its structural triple (τ_1, τ_2, τ_3), and then construct the latin bi-trade $T = (T^\circ, T^*)$ by means of Proposition 1.4. The structure of T closely follows the structure of S. For example, if a_i is the value of ρ_i, $1 \le i \le 3$, then $(\rho_1, \rho_2, \rho_3) \in T^\circ$ implies $(a_1, a_2, a_3) \in S^\circ$, and $(\rho_1, \rho_2, \rho_3) \in T^*$ implies $(a_1, a_2, a_3) \in S^*$. One can say that each $a_i \in B_i$ is divided into so many elements of A_i, however many there are cycles ρ of τ_i that have a_i as their value. This means that if S is separated, then there is a natural bijection between B_i and A_i, and these bijections yield an isotopism. In fact, T is isotopic to S if and only if S is separated. These facts are natural and easy. A formal proof appears in [10], together with a formal proof of the (more or less obvious) fact that the construction of Proposition 1.4 always yields a separated latin bi-trade.

Structure of latin bi-trades is determined by structure of separated latin bi-trades and the study of separated latin bi-trades is equivalent to the study of permutations triples (τ_1, τ_2, τ_3) that fulfil (P1) and (P2). These two properties also suffice to describe the connection to (combinatorial) oriented surfaces. We shall obtain such surfaces in a discrete manner from appropriate sets of points, edges and faces.

The set of points, for a given triple (τ_1, τ_2, τ_3), coincides with the set X upon which the permutations τ_i act. The edges are the sets $\{x, \tau_i(x)\}$, $x \in X$ and $i \in \{1, 2, 3\}$. Such an edge appears with multiplicity 2 when $\tau_i^2(x) = x$. An edge can have *positive* orientation $(x, \tau_i(x))$ and *negative* orientation $(\tau_i(x), x)$. In this paper we shall call 2-cells faces, and by a face we shall always understand a 2-cell. The faces fall into two categories: cyclic faces and triangular faces. Every *cyclic face* $(x_1, x_2, \ldots, x_k)$ expresses a cycle of τ_i, $1 \le i \le 3$. Thus $\tau_i(x_j) = x_{j+1}$, $1 \le j < k$ and $\tau_i(x_k) = x_1$. All *triangular faces* are of the form $(x, \tau_2\tau_3(x), \tau_3(x))$, $x \in X$. (Equivalent expressions are $(x, \tau_1\tau_2(x), \tau_2(x))$, $(\tau_1(x), x, \tau_2^{-1}(x))$, etc.) Note that the given presentation of cyclic faces orders vertices by positive orientation of edges, while negative orientation is used for the triangular faces. Each $x \in X$

appears in exactly three different cyclic faces:

$$(\dots,\tau_3^{-1}(x),x,\tau_3(x),\dots),\quad (\dots,\tau_2^{-1}(x),x,\tau_2(x),\dots)$$
$$\text{and}\quad (\dots,\tau_1^{-1}(x),x,\tau_1(x),\dots),$$

and in exactly three different triangular faces:

$$(\tau_2(x),x,\tau_3^{-1}(x)),\quad (\tau_1(x),x,\tau_2^{-1}(x))\quad\text{and}\quad (\tau_3(x),x,\tau_1^{-1}(x)).$$

By interspersing cyclic and triangular faces we clearly get a cyclic ordering of all faces that include x. That induces a cyclic ordering of all edges containing x such that each edge is contained in exactly two faces, and has opposite orientation in each of these two adjacent faces. The existence of such an ordering means that we have obtained a combinatorial surface. The surface is oriented since one obtains a coherent orientation when the triangular faces are oriented in a positive way.

The connection between latin trades and oriented surfaces was first published in [8]. See also [12] for a detailed exposition with examples.

Let $T = (T^\circ, T^*)$ be a latin bi-trade. Elements $a = (a_1, a_2, a_3) \in T^\circ$ and $b = (b_1, b_2, b_3) \in T^\circ$ are said to be *connected* if $a_i = b_i$ for some $i \in \{1,2,3\}$ (we also say that a and b are connected by a_i). The latin bi-trade T is said to be *connected* if for all $x, y \in T^\circ$ there exists a sequence $x = x_0, x_1, \dots, x_k = y$ such that x_j and x_{j+1} are connected for every j, $1 \le j \le k$. If T is separated, then one can easily see that T is connected if and only if the oriented surface of T is connected. Furthermore, in a separated latin bi-trade one has a and b connected if and only there exists a cycle that moves both a and b, and this can also be regarded as the definiton of connectivity when there is given a triple (τ_1, τ_2, τ_3) that satisfies conditions (R1) and (R2).

Proposition 1.5. *Let $T = (T^\circ, T^*)$ be a connected separated latin bi-trade. Then*

$$2\,\mathrm{genus}\,(T) + \mathrm{order}\,(T) = \mathrm{size}\,(T) + 2.$$

Proof. The oriented surface of T has $f = \mathrm{order}\,(T) + \mathrm{size}\,(T)$ faces, $e = 3\,\mathrm{size}\,(T)$ edges and $p = \mathrm{size}\,(T)$ points. The identity of the proposition thus expresses the Euler identity $2(1-g) + e = f + p$, where g is the genus of the surface. □

Call a latin bi-trade T *spherical* if its oriented surface is a combinatorial sphere (i.e., the surface is connected and its genus is equal to 0). In such a latin bi-trade $\mathrm{order}\,(T) > \mathrm{size}\,(T)$. If T is spherical and separated, $T^\circ \subseteq A_1 \times A_2 \times A_3$, then $|A_i| > \mathrm{size}\,(T)/3$ for some $i \in \{1,2,3\}$. The size of T equals the sum of cycle lengths computed over all cycles of τ_i. There are $|A_i|$ such cycles, and the latter inequality could not hold, if each cycle were of the size at least three. We have proved

Lemma 1.6. *Let (T°, T^*) be a spherical latin bi-trade and let (τ_1, τ_2, τ_3) be its structural triple. Then there exists $x \in T^\circ$ and $i \in \{1,2,3\}$ such that $\tau_i^2(x) = x$.*

Lemma 1.6 can be utilized to show the existence of a reduction/generation scheme that yields all spherical latin bi-trades from those of size 4. This is described in [10]. This paper is concerned with a reduction/generation scheme for connected nonspherical separated latin bi-trades. The basic idea is to consider every positively oriented closed nonintersecting path upon the surface of T which never travels twice along the same cyclic face (this will be called a multigon), and to find one with certain additional properties, one of which states that the path does not separate the surface into two components. The surface is then cut along such a path, and two planar substructures are pasted. This simple and standard idea needs quite a lot of technical auxiliary notions, and their precision and clarification is more or less the content of the rest of this paper.

Each latin bi-trade $T = (T^\circ, T^*)$ yields the *dual* latin bi-trade $\tilde{T} = (T^*, T^\circ)$. The structural permutations of $\tilde{T}$ move elements of T^* and will be denoted by $\tilde{\tau}_i$, $1 \le i \le 3$. It is easy to see that T is separated (or connected) if and only if $\tilde{T}$ is separated (or connected).

2. Closed oriented polygonal curves

Let τ_i, $1 \le i \le 3$, be permutations of a set X such that the triple (τ_1, τ_2, τ_3) satisfies conditions (P1) and (P2). Use Proposition 1.4 to construct latin bi-trade (T°, T^*), and identify T° with X. The oriented combinatorial surface associated with (T°, T^*) will be denoted by S.

By an *oriented edge* of S we shall mean an edge in its positive orientation, i.e. a pair $(x, \tau_i(x))$, where $x \in X$ and $i \in \{1,2,3\}$. The set of all cycles of τ_i will be denoted by A_i, as in Proposition 1.4.

Sequence $x_1 \dots x_k$ of elements of X is called a *closed oriented polygonal curve* if for each j, $1 \le j \le k$, there exists $\sigma_j \in \bigcup A_i$ such that $\sigma_j(x_j) = x_{j+1}$ (set $x_{k+1} = x_1$ and $x_0 = x_k$; similar conventions will be used elsewhere as well). Pairs (x_j, x_{j+1}) will be called the *(oriented) edges* of the polygonal curve.

Consider the sequence $j_1 < \dots < j_r$ consisting of all j, $1 \le j \le k$, such that $\sigma_j \ne \sigma_{j-1}$. The cyclic sequence $(\sigma_{j_1}, \dots, \sigma_{j_r})$ will be called the *profile* of the curve, and the points $x_{j_1}, \dots, x_{j_r}$ are called the *vertices* of the curve. Every x_j, $1 \le j \le k$, is a *point* of the curve, and so we have vertex points and non-vertex points. It is clear that a curve has no vertex points if and only if its profile is empty, and this takes place if and only if all edges of the curve are induced by a single cycle.

Let C be a closed oriented polygonal curve with profile $(\sigma_1, \dots, \sigma_r)$ and vertices $(v_1, \dots, v_r)$. We shall always keep the convention that σ_j and σ_{j+1} meet in the vertex v_{j+1}. Call C a *multigon* if (1) its profile is nonempty, and (2) $\sigma_i \ne \sigma_j$ and $v_i \ne v_j$, whenever $1 \le i < j \le r$. Call the multigon C *simple* if σ_i meets σ_j, $i \ne j$, if and only if $i = j \pm 1$.

In combinatorial topology one usually calls a polygonal curve $x_1 \dots x_k$ *simple* if $x_i \ne x_j$ when $1 \le i < j \le k$. One of the basic theorems states that a connected combinatorial surface is a combinatorial sphere if and only if each simple closed polygonal curve is separating (e.g., see [1, Theorem 7.7]). In Section 7 we shall obtain a similar characterization of spherical latin bi-trades by means of separating multigons. To get the result we will have to overcome certain technical difficulties that arise from the fact that the multigons in the sense above need not be simple (a vertex $v_j = x_i$ can appear also as $x_{i'}$, $i' \ne i$, on a non-vertex position, and be moved by some $\sigma_{j'}$, $j' \notin \{j-1, j\}$. A non-vertex point can be equal to x_i for up to three indices i, and be moved by up to three cycles σ_j.) Other technical problems have to do with the fact that multigons in the sense of our definition are oriented curves, and so the amount of curves at our disposal is smaller when we wish to show that a latin bi-trade of positive genus yields a nonseparating multigon.

Let us now, for a while, consider a connected combinatorial surface K without multiple edges that need not have any connection to latin bi-trades. Let C be a closed polygonal curve in K. What it means that C is separating? This is clear topologically: the surface becomes disconnected when C is removed. However, here we shall need a combinatorial definition. We shall define (combinatorial) components of K that are induced by C in such a way that there will be at most two components, even in the case when C has repeated points.

Recall that the *star* of the point x is formed by the set of all faces that contain x. Any two edges $\{y_1, x\}$ and $\{y_2, x\}$, $y_1 \neq y_2$, divide the star of x into two connectivity components. We shall call them *components induced* by (y_1, x, y_2).

Let C be equal to $x_1 \dots x_k$. Furthermore, let E_0 be a face with the edge $\{x_k, x_1\}$, and let F_0 be the other face containing $\{x_k, x_1\}$. For each j, $1 \le j \le k+1$, construct faces E_j and F_j containing $\{x_j, x_{j+1}\}$ in such a way that E_{j-1} and E_j are in the same component induced by (x_{j-1}, x_j, x_{j+1}). (Faces F_{j-1} and F_j are then in the other component.)

Let $\sim_C$ be the least equivalence on the set of all faces of K such that

(1) $E_{j-1} \sim_C E_j$ and $F_{j-1} \sim_C F_j$ for every j, $1 \le j \le k+1$; and
(2) $E \sim_C F$ whenever E and F share an edge distinct from $\{x_j, x_{j+1}\}$, $1 \le j \le k$.

Call the curve C *separating* if the equivalence $\sim_C$ has more than one equivalence class.

To see that $\sim_C$ can have at most two equivalence classes, consider the equivalence $\approx$ that contains $\sim_C$, and satisfies $E_0 \approx F_0$. We shall show that $\approx$ has only one equivalence class. For that it suffices to prove $E_j \approx F_j$ for every j, $1 \le j \le k$, since K is assumed to be a connected surface. We can use induction, as $E_j \sim_C E_{j-1} \approx F_{j-1} \sim_C F_j$.

If $E_0 = F_k$, then C is never separating. If $E_0 = E_k$, then also $F_0 = F_k$. In such a case we shall call $(E_1, \dots, E_k)$ and $(F_1, \dots, F_k)$ the *border sequences* of C.

In the definitions above we needed to assume that K is without multiple edges in order to have an edge $\{x, y\}$ uniquely determined by the points x and y. Let us now return to the combinatorial surface S derived from (τ_1, τ_2, τ_3). For any $x, y \in X$ there exists at most one oriented edge (x, y). Hence the above definitions can be applied to any closed polygonal *oriented* curve C of S. In particular, we can speak about *separating* and *nonseparating* multigons.

Every cyclic face gives rise to a simple closed oriented polygonal curve with empty profile. Every triangular face can be uniquely represented as (x_1, x_2, x_3), where x_i is moved by $\sigma_i \in A_i$ to x_{i-1}, and $(\sigma_1, \sigma_2, \sigma_3) \in T^*$. By moving around a triangular face along oriented edges one gets a multigon $x_3 x_2 x_1$ with profile $(\sigma_3, \sigma_2, \sigma_1)$ and vertices (x_3, x_2, x_1). Such multigons will be called *triangles.*

A triangle is clearly a separating multigon. Our aim is to show that S is a sphere if and only if it is connected and *every* multigon is separating.

If $i, i' \in \{1, 2, 3\}$, then there exists a unique $\varepsilon \in \{-1, 0, 1\}$ with $i' - i \equiv \varepsilon \bmod 3$. For $\sigma' \in A_{i'}$ and $\sigma \in A_i$ set $\operatorname{sgn}(\sigma', \sigma) = \varepsilon$ if σ' and σ have a common point, and set $\operatorname{sgn}(\sigma', \sigma) = 0$ if they have no common point.

Let C be the closed polygonal curve $x_1 \dots x_k$, and let $\sigma_j \in \bigcup A_i$ be the cycle that moves x_j to x_{j+1}. For every j, $1 \le j \le k$, put $\operatorname{sgn}_C(x_j) = \operatorname{sgn}(\sigma_{j-1}, \sigma_j)$. Note that if C is a multigon, then x_j is a vertex if and only if $\operatorname{sgn}_C(x_j) = \varepsilon \neq 0$. Call the vertex *acute* if $\varepsilon = 1$, and *obtuse* if $\varepsilon = -1$. Set $\operatorname{sgn}_C(x) = 0$ also for the points of S that are not on C.

Lemma 2.1. *The sum* $\sum \operatorname{sgn}_C x_j$, $1 \le j \le k$, *is divisible by* 3 *for every oriented closed polygonal curve* C.

Proof. Set $s_j = \operatorname{sgn}_C(x_1) + \cdots + \operatorname{sgn}_C(x_j)$, $1 \le j \le k$, and let i_j be the $i \in \{1,2,3\}$ with $\sigma_j \in A_i$. We have $\operatorname{sgn}_C(x_j) = \operatorname{sgn}(\sigma_{j-1}, \sigma_j) \equiv i_{j-1} - i_j \bmod 3$, and so $s_j \equiv i_0 - i_j \bmod 3$. Setting $j = k$ gives the result. □

The integer $\frac{1}{3} \sum \operatorname{sgn}_C(x_j)$ will be denoted by $\operatorname{sgn}(C)$, and called the *sign characteristic* of the curve C.

Note that we have defined in Section 1 the surface S by describing the star of every $x \in X$. For a fixed x there exist cycles $\sigma_i \in A_i$ and triangle t_i, $1 \le i \le 3$, such that the neighboroughing elements of the cyclic sequence

$$(\sigma_1, t_2, \sigma_3, t_1, \sigma_2, t_3)$$

share an edge starting or terminating at x. Each t_i is between σ_{i-1} and σ_{i+1}, the oriented edge shared by t_i and σ_{i-1} is an incoming edge with respect to x, and the edge shared by t_i and σ_{i+1} an outcoming edge.

Lemma 2.2. *Let* $x \in X$ *be moved by* $\sigma_j \in A_j$, $1 \le j \le 3$. *Consider the components of the star of* x *that are induced by* $(\sigma_i^{-1}(x), x, \sigma_{i'}(x))$, *where* $i, i' \in \{1,2,3\}$. *These components can be denoted by* Γ_0 *and* Γ_1 *in such a way that the following holds.*

(i) *If* $\operatorname{sgn}(\sigma_i, \sigma_{i'}) = 0$, *then* $i = i'$, $\Gamma_0 = \{\sigma_i\}$ *and* $\Gamma_1 = \{t_{i+1}, \sigma_{i-1}, t_i, \sigma_{i+1}, t_{i-1}\}$;
(ii) *if* $\operatorname{sgn}(\sigma_i, \sigma_{i'}) = 1$, *then* $i' \equiv i - 1 \bmod 3$, $\Gamma_0 = \{\sigma_{i-1}, t_i, \sigma_{i+1}, t_{i-1}, \sigma_i\}$ *and* $\Gamma_1 = \{t_{i+1}\}$; *and*
(iii) *if* $\operatorname{sgn}(\sigma_i, \sigma_{i'}) = -1$, *then* $i' \equiv i + 1 \bmod 3$, $\Gamma_0 = \{\sigma_{i+1}, t_{i-1}, \sigma_i\}$ *and* $\Gamma_1 = \{t_{i+1}, \sigma_{i-1}, t_i\}$.

Proof. The cyclic sequence before the lemma can be written in the form

$$(\sigma_i, t_{i+1}, \sigma_{i-1}, t_i, \sigma_{i+1}, t_{i-1}).$$

The incoming edge $(\sigma_i^{-1}(x), x)$ is shared by σ_i and t_{i+1}, while the outcoming edge is shared by either (i) t_{i-1} and σ_i, or (ii) t_{i+1} and σ_{i-1}, or (iii) t_i and σ_{i+1}. □

Corollary 2.3. *Let* S *be the combinatorial surface derived from* (τ_1, τ_2, τ_3). *Every closed polygonal curve has a border sequence consisting only of triangular faces, and a border sequence consisting only of cyclic faces.*

Proof. Elements of both Γ_0 and Γ_1 appear in Lemma 2.2 in the order that is compatible with the cyclic ordering of the faces in the star of x. The starting and terminal faces of Γ_0 are always cyclic faces, while in Γ_1 the corresponding faces are triangular. □

For every closed polygonal curve C we hence have a *triangle border sequence* and a *cycle border sequence*. The elements of the former one will be known also as *border faces*. Note that a cycle appears in the latter one if and only if it belongs to the profile of C (such cycles will be sometimes called *profiled*).

Suppose that C is a separating closed polygonal curve. The component containing the triangle border sequence will be called the *domain* of C. (The domain is, seen formally, a combinatorial complex consisting of faces, edges and points.)

For any multigon C let us denote by $\operatorname{Prof}(C)$ and $\operatorname{Vert}(C)$ the set of its profiled cycles, and the set of its vertices, respectively. If C is separating, then we shall denote by $\mathrm{Cycl}(C)$ the set of cycles that induce a cyclic face in the domain of C. The set of those points in the domain that are not vertices of C will be denoted by $\mathrm{Pnt}(C)$.

Call a separating multigon *planar* if

$$|\,\mathrm{Cycl}(C)| + \mathrm{sgn}(C) = |\,\mathrm{Pnt}(C)| + 1.$$

It will become clear later that a simple multigon is planar if and only if its domain is homeomorphic to a disc. (We cannot start from such a definition, since we have to work with nonsimple multigons as well.)

Let C be a multigon with profile $(\sigma_1, \dots, \sigma_r)$ and vertices $(v_1, \dots, v_r)$. Then there clearly exists a unique multigon C' with profile $(\sigma_r, \dots, \sigma_1)$ and vertices $(v_r, \dots, v_1)$. The multigon C' will be called *opposite*, and will be denoted by C^{op}.

It is easy to describe all points of the multigon C^{op}. Choose j, $1 \le j \le r$, and put $\sigma = \sigma_j$. Let k be the length of σ. There exists a unique h, $1 \le h < k$, such that $v_{j+1} = \sigma^h(v_j)$. The sequence $v_j\,\sigma(v_j)\,\dots\,\sigma^{h-1}(v_j)\,v_{j+1}$ yields the path of C from v_j to v_{j+1}. The path of C^{op} from v_{j+1} to v_j is equal to $v_{j+1}\,\sigma(v_{j+1})\,\dots\,\sigma^{k-h-1}(v_{j+1})\,v_j$.

Consider the sequence $\sigma(v_j)\,\dots\,\sigma^{h-1}(v_j)$. If $\mathrm{sgn}_C(v_j) = -1$, add v_j to the sequence as its initial element. If $\mathrm{sgn}_C(v_{j+1}) = -1$, add v_{j+1} to the sequence as its terminal element. The resulting sequence is called the *arc* of σ in C.

When working with C and C^{op} at the same time, we sometimes call the arc of σ in C the *inner arc*, and the arc of σ in C^{op} the *outer arc*.

The inner arc contains the obtuse vertices, and excludes the acute vertices. By the definition of $\mathrm{sgn}_C(x_j)$ we see that a vertex is obtuse in C if and only if it is acute in C^{op}. We can hence state

Lemma 2.4. *Let C be a multigon in S, and let σ be its profiled cycle, with the inner arc $x_1 \dots x_s$ and the outer arc $y_1, \dots, y_t$. Then $\mathrm{sgn}(C) = -\,\mathrm{sgn}(C^{op})$, and $(x_1, \dots, x_s, y_1, \dots, y_t)$ is the cyclic face of σ. In particular, $s + t = k$, where k is the length of σ.*

Suppose now that the multigon C is separating, and consider the component of C that contains the cyclic border sequence. Denote it by Γ. Note that the cyclic face induced by $\sigma = \sigma_j$ is repeated h times in the sequence. A triangular face that has a common edge induced by σ is either a border face of C, or a border face of C^{op}. Remove from the component Γ all cyclic faces induced by σ_j, $1 \le j \le k$. We obtain a combinatorial complex, in which the edges with only one adjacent face are exactly the edges of C^{op}. Hence the complex coincides with the domain of C^{op}. We have proved

Proposition 2.5. *Let C be a multigon in the combinatorial surface S of (τ_1, τ_2, τ_3). Assume that S is connected. The multigon C is separating if and only if the opposite multigon C^{op} is separating. In such a case each face of S is either in the domain of C, or in the domain of C^{op}, or it is a cyclic face determined by a profiled cycle.*

Proposition 2.6. *Let C be a separating planar multigon in the combinatorial surface S of (τ_1, τ_2, τ_3). Suppose that S is connected. If both C and C^{op} are planar, then S is a combinatorial sphere.*

Proof. Let $T = (T^\circ, T^*)$ be the latin bi-trade determined by (τ_1, τ_2, τ_3). We wish to prove that $\mathrm{order}(T) = \mathrm{size}(T) + 2$ (cf. Proposition 1.5). We have

$$\mathrm{order}(T) = |\,\mathrm{Cycl}(C)| + |\,\mathrm{Cycl}(C^{op})| + |\,\mathrm{Prof}(C)|$$

and

$$\mathrm{size}(T) = |\,\mathrm{Pnt}(C)| + |\,\mathrm{Pnt}(C^{op})| + |\,\mathrm{Vert}(C)|,$$

by Proposition 2.5. The multigons C and C^{op} are assumed to be planar, which means that

$$|\,\mathrm{Pnt}(C)| + |\,\mathrm{Pnt}(C^{op})| + 2 = |\,\mathrm{Cycl}(C)| + |\,\mathrm{Cycl}(C^{op})| + \mathrm{sgn}(C) + \mathrm{sgn}(C^{op}),$$

by the definition. However, $\mathrm{sgn}(C) + \mathrm{sgn}(C^{op}) = 0$, by Lemma 2.4. □

3. Production rules

In this section we shall present two rules that generate, when applied repeatedly, the set $\mathrm{Cycl}(C)$, where C is a separating multigon in S. When the same procedure is used upon a nonseparating multigon, then one gets the set of all cycles in the connectivity component of C.

Lemma 3.1. *Let C be a separating multigon with profile $(\sigma_1, \dots, \sigma_r)$, and let $\sigma = \sigma_j$, $1 \le j \le r$. Any $\rho \in \mathrm{Cycl}(C)$ that meets σ moves a point on the arc of σ. On the other hand, if a cycle ρ moves a point on the arc of σ, then either $\rho \in \mathrm{Cycl}(C)$, or $\rho = \sigma_k$ for some $k \in \{1, \dots, r\}$.*

Proof. Suppose first that v is an acute vertex of C. Then there exists $i \in \{1, 2, 3\}$ and $s \in \{1, \dots, r\}$ such that both σ_s and σ_{s+1} move v, where $\sigma_s \in A_i$ and $\sigma_{s+1} \in A_{i-1}$. Note that the triangles $v\,\tau_i(v)\,\tau_{i+1}^{-1}(v)$ and $\tau_{i-1}^{-1}(v)\,v\,\tau_{i+1}(v)$ are always different. Both of them yield a border triangular face of C^{op}, as $(v, \tau_i(v)) = (v, \sigma_s(v))$ and $(\tau_{i-1}^{-1}(v), v) = (\sigma_{s+1}^{-1}(v), v)$. Hence none of them belongs to the domain of C. We have proved that the domain of C contains exactly one triangular face incident to an acute vertex.

Suppose that $\rho \in \mathrm{Cycl}(C)$ meets σ in a point x. The triangular face with edges that are induced by ρ and σ belongs to the domain of C, and therefore it is a border face of C. Hence x is a point of C. There exists another triangular face with an edge induced by ρ that contains x. This face is also in the domain of C, and hence x cannot be an acute vertex, by the preceding part of the proof. This means that x is incident to the arc of σ.

Any $\rho \in \mathrm{Cycl}(C^{op})$ therefore meets σ only in the points of the outer arc, and so the rest follows from Proposition 2.5. □

Lemma 3.2. *Let C be a separating multigon. Cycles σ and ρ never meet when $\sigma \in \mathrm{Cycl}(C)$ and $\rho \in \mathrm{Cycl}(C^{op})$. If both σ and ρ belong to the profile of C, then the inner arc of σ never meets the outer arc of ρ.*

Proof. If $\sigma \in \mathrm{Cycl}(C)$ meets $\rho \in \mathrm{Cycl}(C^{op})$, then there exists a triangular face with one edge induced by σ and another edge induced by ρ. The cyclic face of σ is in the domain of C, and hence the considered triangular face is in the domain as well. However, it is also in the domain of C^{op}, a contradiction.

Suppose now that the inner arc of $\sigma \in \mathrm{Prof}(C)$ meets the outer arc of $\rho \in \mathrm{Prof}(C)$ in a point x. We have $\rho \ne \sigma$, by Lemma 2.4, and hence x is determined uniquely. Suppose first that x is a vertex, and let ρ' and ρ'' be the two consecutive cycles of the profile that meet in x. The vertex x is incident either to the inner arcs, or to the outer arcs of both ρ' and ρ''. Therefore we cannot have $\{\rho, \sigma\} = \{\rho', \rho''\}$. However, the two sets clearly have to intersect. We can assume $\rho \in \{\rho', \rho''\}$, since the outer arc of $\bar{\rho} = \sigma$ meets the inner arc of $\bar{\sigma} = \rho$ when C is replaced by C^{op}. The vertex x belongs to the outer arc of $\rho \in \{\rho', \rho''\}$, and so $\operatorname{sgn}_C(x) = -1 = \operatorname{sgn}(\rho', \rho'')$. The points $\sigma^{-1}(x)$, x and $\sigma(x)$ are consecutive points of C^{op}, as $\sigma \notin \{\rho', \rho''\}$ and x is on the outer arc of σ. We also have $\operatorname{sgn}(\sigma, \rho') = -1 = \operatorname{sgn}(\rho'', \sigma)$. This means that both $(\sigma(x), x, (\rho')^{-1}(x))$ and $(\rho''(x), x, \sigma^{-1}(x))$ are border faces of C^{op}. If $\rho = \rho'$, then the former triangular face is a border face of C, and the same holds for the latter face if $\rho = \rho''$. A face cannot be in the triangle border sequences of both C and C^{op}, and hence we have obtained a contradiction.

Suppose now that x is not a vertex. Since we can consider C^{op} in place of C, we see that $\operatorname{sgn}(\rho, \sigma) = 1$ can be assumed. Thus $(\sigma(x), x, \rho^{-1}(x))$ is a triangular face that is a border face of both C and C^{op}. □

To capture formally the relations appearing in Lemma 3.1, we shall define sets $\operatorname{In}(C)$ and $\operatorname{Out}(C)$, C a multigon in S, as sets of pairs of cycles that intersect in a predescribed way. Let $(\sigma_1, \dots, \sigma_r)$ be the profile of C. Set

$$\operatorname{In}(C) = \{(\rho, \sigma_j);\ \rho \text{ meets the inner arc of } \sigma_j, \text{ and } \rho \neq \sigma_{j\pm\varepsilon}, \text{ where } -1 \leq \varepsilon \leq 1\},$$

and

$$\operatorname{Out}(C) = \{(\rho, \sigma_j);\ \rho \text{ meets the outer arc of } \sigma_j, \text{ and } \rho \neq \sigma_{j\pm\varepsilon}, \text{ where } -1 \leq \varepsilon \leq 1\}.$$

Inner arcs of C become outer arcs of C^{op}, and vice versa. The definition of $\operatorname{Out}(C)$ could be thus replaced by setting $\operatorname{Out}(C) = \operatorname{In}(C^{op})$.

For a set of cycles $\mathcal{N}$ consider rules:

(N1) If $\sigma \in \operatorname{Prof}(C)$, $(\rho, \sigma) \in \operatorname{In}(C)$, $(\sigma, \rho) \notin \operatorname{In}(C)$, then $\rho \in \mathcal{N}$.
(N2) If $\rho \in \mathcal{N}$ meets such a cycle σ that $(\rho, \sigma) \notin \operatorname{In}(C)$, then $\sigma \in \mathcal{N}$.

There follow several comments on the meaning of these rules. Assume $(\rho, \sigma) \in \operatorname{In}(C)$. If $(\sigma, \rho) \notin \operatorname{In}(C)$, and $\sigma, \rho \in \operatorname{Prof}(C)$, $\sigma \neq \rho$, then either $\rho = \sigma_{j\pm 1}$, or $(\sigma, \rho) \in \operatorname{Out}(C)$. The former option is excluded by the assumption $(\rho, \sigma) \in \operatorname{In}(C)$. Hence ρ and σ meet in the inner arc of σ, and in the outer arc of ρ. Inner and outer aracs do not meet when the multigon is separating, by Lemma 3.2. Furthermore, if C is separating, then $(\rho, \sigma) \in \operatorname{In}(C)$ implies $\rho \in \operatorname{Cycl}(C) \cup \operatorname{Prof}(C)$, by Lemma 3.1. We have proved

Lemma 3.3. *Suppose that $(\rho, \sigma) \in \operatorname{In}(C)$ fulfils the assumptions of (N1). If C is separating, then $\rho \in \operatorname{Cycl}(C)$. If $\rho \in \operatorname{Prof}(C)$, then the inner arc of σ meets the outer arc of ρ, and C is not separating.*

Let us now turn to rule (N2). Assume that C is separating, $\rho \in \operatorname{Cycl}(C)$ meets σ, and $(\rho, \sigma) \notin \operatorname{In}(C)$. Lemma 3.1 states that if $\rho \in \operatorname{Cycl}(C)$ meets $\sigma \in \operatorname{Prof}(C)$, then $(\rho, \sigma) \in \operatorname{In}(C)$. Thus $\sigma \notin \operatorname{Prof}(C)$ and we easily get $\sigma \in \operatorname{Cycl}(C)$. (To see this consider the triangular face induced by ρ and σ. The face belongs to the domain of C, and hence any of its edges is induced by a cycle from $\operatorname{Cycl}(C) \cup \operatorname{Prof}(C)$.) Hence we can state

Lemma 3.4. *If C is a separating multigon, then the set $\mathcal{N} = \operatorname{Cycl}(C)$ is closed under rules (N1) and (N2).*

We also have

Lemma 3.5. *Suppose that a set of cycles $\mathcal{N}$ is closed under rules (N1) and (N2). If $\mathcal{N} \cap \operatorname{Prof}(C) \neq \emptyset$, then $\mathcal{N}$ contains all cycles in the connectivity component of S.*

Proof. Consider $\rho \in \mathcal{N} \cap \mathrm{Prof}(C)$, and let σ be the cycle immediately following σ in the profile of C. Then $(\rho, \sigma) \notin \mathrm{In}(C)$, and $\sigma \in \mathcal{N}$ by (N2). Hence $\mathrm{Prof}(C) \subseteq \mathcal{N}$, and it remains to show that a cycle σ belongs to $\mathcal{N}$ whenever it meets some $\rho \in \mathcal{N}$. This follows from (N2) immediately since $\sigma \notin \mathrm{Prof}(C)$ can be assumed. □

Let us have a nonempty collection of sets $(\mathcal{N}_j;\ j \in J)$, and suppose that each set $\mathcal{N}_j$ is closed under rules (N1) and (N2), for a given multigon C. Then $\bigcap(\mathcal{N}_j;\ j \in J)$ clearly preserves the rules as well, and so there exists the least set of cycles that is closed under (N1) and (N2).

Proposition 3.6. *Let C be a multigon in S, where the combinatorial surface S is derived from (τ_1, τ_2, τ_3). Then there exists the least set of cycles $\mathcal{N}$ that is closed under rules (N1) and (N2). If C is separating, then this set $\mathcal{N}$ coincides with* $\mathrm{Cycl}(C)$*, and if it is not separating, then it is equal to the set of all cycles in the connectivity component of S.*

Define $\mathcal{F}$ as a set of faces that consists of all

(F1) *border faces of C;*
(F2) *cyclic faces induced by $\rho \in \mathcal{N}$; and*
(F3) *triangular faces that have an edge induced by $\rho \in \mathcal{N}$.*

Then $\mathcal{F}$ is formed by all faces in the domain of C when C is a separating multigon. If it is not, then $\mathcal{F}$ coincides with the set of all faces in the connectivity component.

Proof. If C is separating, then $\mathrm{Cycl}(C) \supseteq \mathcal{N}$, by Lemma 3.4, and thus all the faces of $\mathcal{F}$ are in the domain of C. We see that $\mathcal{F}$ is contained in exactly one block of the equivalence $\sim_C$ defined in Section 2. To prove the statement it suffices to show that $\mathcal{F}$ coincides with this block, by (F2). For that we need to verify the implication $F \in \mathcal{F} \Rightarrow G \in \mathcal{F}$ for all faces F and G that share an edge which is not an edge of C.

If F is a cyclic face induced by ρ, then $\rho \in \mathcal{N}$, by (F2), and $G \in \mathcal{F}$, by (F3). Hence we can assume that F is a triangular face. Note that to prove the case of C nonseparating it is enough to find any $\rho \in \mathrm{Prof}(C) \cap \mathcal{N}$, by Lemma 3.5.

Let first F be a border face. Suppose that σ induces a common edge of F and C, and that ρ induces the common edge of F and G. The latter edge is not an edge of C, and hence $(\rho, \sigma) \in \mathrm{In}(C)$. Assume first $(\sigma, \rho) \in \mathrm{In}(C)$, and denote by x the common point of σ and ρ. Then x is on the inner arc of ρ, and hence the other edge of ρ that moves x must be an edge of C. Thus $x \in \mathrm{Vert}(C)$ and both edges $(x, \sigma(x))$ and $(\sigma^{-1}(x), x)$ are edges of C (recall that the cycles ρ and σ are not neighbours in the profile of C when $(\sigma, \rho) \in \mathrm{In}(C)$). It follows that x is an acute vertex of C, and so $(\sigma, \rho) \notin \mathrm{In}(C)$. Thus $\rho \in \mathcal{N}$, by (N1), and $G \in \mathcal{F}$, by (F2).

Assume now that the triangular face F has an edge induced by some $\rho \in \mathcal{N}$, by (F3). Let σ be the cycle that induces the common edge of F and G, and let x be the common point of σ and ρ. We know that we can assume $\rho \notin \mathrm{Prof}(C)$. If $(\rho, \sigma) \notin \mathrm{In}(C)$, then $\sigma \in \mathcal{N}$, by (N2), and so $G \in \mathcal{F}$, by (F2). To finish the proof it hence suffices to assume $(\rho, \sigma) \in \mathrm{In}(C)$ and to find a contradiction. This will be achieved by proving that the common edge of F and G is an edge of the multigon C.

If both edges $(\sigma^{-1}(x), x)$ and $(x, \sigma(x))$ are edges of C, then this is clear, since x is a point of F. Assume that one of the edges is not an edge of C. Then the other edge is an edge of C, since $(\rho, \sigma) \in \mathrm{In}(C)$. Hence $x \in \mathrm{Vert}(C)$. The vertex x is obtuse, since it is on the inner arc of σ. Denote by μ the third cycle that moves the point x. Then μ has to be a cycle that immediately precedes or follows σ in the profile of C since $\rho \notin \mathrm{Prof}(C)$. If it is preceding, then $\mathrm{sgn}(\mu,\sigma) = -1$, $\mathrm{sgn}(\rho,\sigma) = 1$, and we see that the common edge of F and G is equal to $(x, \sigma(x))$, and that this edge is an edge of C. In the case when μ consecutes σ in the profile of C one gets $\mathrm{sgn}(\sigma,\rho) = 1$, and the common edge is equal to $(\sigma^{-1}(x), x)$, and that is an edge of C again. □

In the following we shall often use rules (N1) and (N2) in a way that implicitly refers to Proposition 3.6. The multigon C will be understood from the context or will be explicitly declared. If C will be separating, then by saying that a cycle belongs to $\mathrm{Cycl}(C)$ because of a rule (N1) or (N2) we shall mean that the rule can be applied to the case $\mathcal{N} = \mathrm{Cycl}(C)$.

4. Triangle free multigons

Let C be a multigon with profile $(\sigma_1, \dots, \sigma_r)$. Call C *triangle free* if there exists no j, $1 \le j \le r$, such that $(\sigma_{j-1}, \sigma_j, \sigma_{j+1})$ or $(\sigma_{j+1}, \sigma_j, \sigma_{j-1})$ is a profile of a triangle. Ultimately we are interested only in triangle free multigons. However to establish that there are enough of them in nonspherical trading surfaces we shall also need to work with multigons that are not triangle free. In this section we shall prove that each multigon either can be made triangle free by cutting and inserting stepwise the triangles that hinder the triangle freeness, or can be reduced by the same process to a single triangle.

In Section 3 we have observed that the set $\mathrm{In}(C)$ is useful for description of $\mathrm{Cycl}(C)$ when C is separating. Sometimes it is technically more suitable to work with the set $\overline{\mathrm{In}}(C)$, which is defined as

$$\mathrm{In}(C) \bigcup \{(\sigma_j, \sigma_{j+1});\ 1 \le j \le r\} \bigcup \{(\sigma_{j+1}, \sigma_j);\ 1 \le j \le r\}.$$

Furthermore, for every cycle σ define sets $\mathrm{In}(C,\sigma)$ and $\overline{\mathrm{In}}(C,\sigma)$ by

$$\rho \in \mathrm{In}(C,\sigma) \Leftrightarrow (\rho,\sigma) \in \mathrm{In}(C) \quad \text{and} \quad \rho \in \overline{\mathrm{In}}(C,\sigma) \Leftrightarrow (\rho,\sigma) \in \overline{\mathrm{In}}(C).$$

Finally, let $\overline{\mathrm{Out}}(C)$, $\mathrm{Out}(C,\sigma)$ and $\overline{\mathrm{Out}}(C,\sigma)$ be equal to $\overline{\mathrm{In}}(C^{op})$, $\mathrm{In}(C^{op},\sigma)$ and $\overline{\mathrm{In}}(C^{op},\sigma)$, respectively.

Lemma 4.1. *Let C be a multigon with profile $(\sigma_1, \dots, \sigma_r)$. Put $\sigma = \sigma_j$, $1 \le j \le r$, and define Σ as the set of all cycles ρ that meet σ. Order Σ cyclically:*

Let $\rho, \rho' \in \Sigma$ meet σ in x and x', respectively. Then ρ' is an immediate succesor to ρ if and only if

(a) either $x = x'$ and $\mathrm{sgn}(\sigma,\rho) > \mathrm{sgn}(\sigma,\rho')$;

(b) or $x' = \sigma(x)$ and $\mathrm{sgn}(\sigma,\rho) < \mathrm{sgn}(\sigma,\rho')$.

Both $\mathrm{In}(C,\sigma)$ and $\overline{\mathrm{In}}(C,\sigma)$ are intervals in this cyclic ordering,

$$\overline{\mathrm{In}}(C,\sigma) = \mathrm{In}(C,\sigma) \cup \{\sigma_{j-1}, \sigma_{j+1}\},$$

and exactly one of the following cases holds:

1. *$\mathrm{In}(C,\sigma) = \emptyset$, and $(\sigma_{j-1}, \sigma_j, \sigma_{j+1})$ is a profile of a triangle. The arc of σ in C is empty.*
2. *$\overline{\mathrm{In}}(C,\sigma) = \Sigma$ and $(\sigma_{j+1}, \sigma_j, \sigma_{j-1})$ is a profile of a triangle. The arc of σ in C consists of all points moved by σ.*

(3) *Both* $\mathrm{In}(C,\sigma)$ *and* $\overline{\mathrm{In}}(C,\sigma)$ *are proper subsets of* Σ*, and none of the triples* $(\sigma_{j-1},\sigma_j,\sigma_{j+1})$ *and* $(\sigma_{j+1},\sigma_j,\sigma_{j-1})$ *is a triangle profile. Both the inner and outer arcs of* σ *in* C *are nonempty.*

Proof. The lemma is of descriptive nature and practically all of its statements immediately follow from definitions of $\mathrm{In}(C)$ and $\overline{\mathrm{In}}(C)$. Case (2) can be reduced to case (1) since the extreme cases are mirror images of each other when C is replaced by C^{op}. We shall prove that

$$\mathrm{In}(C,\sigma)=\emptyset \Leftrightarrow (\sigma_{j-1},\sigma_j,\sigma_{j+1}) \text{ is a triangle profile.}$$

Let $(v_1,\dots,v_r)$ be the vertices of C. If $(\sigma_{j-1},\sigma_j,\sigma_{j+1})$ profiles a triangle, then v_j and v_{j+1} are vertices of this triangle. Hence $\sigma_j(v_j)=v_{j+1}$ and $\mathrm{sgn}(\sigma_{j-1},\sigma_j)=1=\mathrm{sgn}(\sigma_j,\sigma_{j+1})$. However, that is clearly equivalent to $\mathrm{In}(C,\sigma_j)=\emptyset$. Thus $\mathrm{In}(C,\sigma_j)=\emptyset$ implies $\sigma_{j-1}^{-1}(v_j)=\sigma_{j+1}(v_{j+1})$. Denote this point by x, and note that $(\sigma_{j-1},\sigma_j,\sigma_{j+1})$ is a profile of triangle $v_j\,v_{j+1}\,x$. □

Corollary 4.2. *Let* C *be a multigon. Then* $\mathrm{In}(C)=\emptyset$ *if and only if* C *is a triangle.*

Proof. Suppose $\mathrm{In}(C)=\emptyset$ and consider case (1) of Lemma 4.1. Since $\mathrm{In}(C,\sigma_{j+1})=\emptyset$ as well, there must be $\sigma_{j+1}(v_{j+1})=v_{j+2}$. Similarly we obtain $\sigma_j^{-1}(v_j)=v_{j-1}$, and $v_{j-1}=v_{j+2}$ follows since cycles σ_{j-1} and σ_{j+2} have the vertices v_{j-1} and v_{j+2} as their common point. □

Lemma 4.3. *Let* C *be a multigon of profile* $(\sigma_1,\dots,\sigma_r)$ *and vertices* $(v_1,\dots,v_r)$*. Suppose that* C *is not a triangle, and that* $\mathrm{In}(C,\sigma)$ *is empty for some* $\sigma=\sigma_j$*,* $1\le j\le r$*. Then* σ_{j-1} *and* σ_{j+1} *have a common point, say* x*, and* $v_{j-1}\ne v_{j+2}$*. There exists a unique polygon* C' *with a profile that is obtained from* $(\sigma_1,\dots,\sigma_r)$ *by omitting*

(1) σ_j *if* $x\notin\{v_{j-1},v_{j+2}\}$*,*
(2) σ_{j-1} *and* σ_j*, if* $x=v_{j-1}$*, and*
(3) σ_j *and* σ_{j+1}*, if* $x=v_{j+2}$*.*

If C *is separating, then* C' *is separating as well. In such a case the domain of* C' *is included in the domain of* C*, and* (x,v_{j+1},v_j) *is the only face in the domain of* C *that does not belong to the domain of* C'*.*

Proof. A cyclic sequence of pairwise distinct cycles forms a profile if the points common to every two consecutive cycles are pairwise distince as well. The corresponding intersection points yield cyclic sequence $(\dots,v_{j-1},x,v_{j+2},\dots)$ in case (1), and cyclic sequence $(\dots,v_{j-1},v_{j+2},\dots)$ in cases (2) and (3). The point x in case (1) equals no v_i, $1\le i\le r$, since otherwise there would have to be $\{\sigma_i,\sigma_{i+1}\}\cap\{\sigma_{j-1},\sigma_{j+2}\}\ne\emptyset$, which would mean that x is incident to σ_{j-1} or σ_{j+2}. If this were true, then $x\in\{v_{j-1},v_{j+2}\}$, and one of the cases (2) or (3) would apply.

It is clear that the border sequence of C' is obtained from the border sequence of C by omission of the triangular face (x,v_{j+1},v_j). Denote this face by F and assume that C is a separating multigon. If we consider any other face G in the domain of C together with a face $H\ne G$ adjacent to G such that the common edge is not the edge of C, then $H\ne F$ and the common edge is not an edge of C'. This is because all edges of F are also edges of C. The block of $\sim_C$ that contains G thus becomes a block of $\sim_{C'}$ when F is removed. □

Lemma 4.4. *Let C and C' be multigons from Lemma 4.3. If C (or C^{op}) is planar, then C' (or $(C')^{op}$) is planar as well.*

Proof. If C is planar, then $|\operatorname{Cycl}(C)| + \operatorname{sgn}(C) = |\operatorname{Pnt}(C)| + 1$. The domain of C' differs only by one missing triangular face, and so $|\operatorname{Cycl}(C)| = |\operatorname{Cycl}(C')|$. Clearly $\operatorname{Pnt}(C) = \operatorname{Pnt}(C')$ in cases (2) and (3), and $\operatorname{Pnt}(C') = \operatorname{Pnt}(C) \setminus \{x\}$ in case (1). To compute the change in the sign characteristic note first that $\operatorname{sgn}_C(v_j) = \operatorname{sgn}_C(v_{j+1}) = 1$. In case (1) we have $\operatorname{sgn}_C(v_{j-1}) = \operatorname{sgn}_{C'}(v_{j-1})$, $\operatorname{sgn}_C(v_{j+2}) = \operatorname{sgn}_{C'}(v_{j+2})$ and $\operatorname{sgn}_{C'}(x) = -1$. Thus $\operatorname{sgn}(C) - \operatorname{sgn}(C') = \frac{1}{3}(2-(-1)) = 1$, which compensates the loss of x.

In case (2) the point $x = v_{j-1}$ is moved by σ_{j-2}, σ_{j-1} and σ_{j+1}. Clearly, $\operatorname{sgn}_C(v_{j+1}) = 1 = \operatorname{sgn}_{C'}(v_{j+1})$. Hence $\operatorname{sgn}_C(x) = \operatorname{sgn}(\sigma_{j-2}, \sigma_{j-1}) = -1$ and $\operatorname{sgn}_{C'}(x) = \operatorname{sgn}(\sigma_{j-2}, \sigma_{j+1}) = 1$, as $\operatorname{sgn}(\sigma_{j-1}, \sigma_{j+1}) = -1$. Thus $\operatorname{sgn}_C(v_{j-1}) + \operatorname{sgn}_C(v_j) + \operatorname{sgn}_C(v_{j+1}) = \operatorname{sgn}_{C'}(v_{j-1})$, and the sign characteristic does not change.

The case (3) is similar, and the sign characteristic does not change as well.

Suppose now that C^{op} is planar. From the preceding part of proof we know that the sign characteristic increases by 1 in case (1) and remains stable in the other two cases when passing from C^{op} to $(C')^{op}$. Furthermore, $\operatorname{Cycl}((C')^{op})$ is obtained by uniting $\operatorname{Cycl}(C^{op})$ with the profiled cycles that are being omitted. The left hand side of the planarity equation thus increase by two in all cases. The new nonvertex points of C^{op} are v_j and v_{j+1}, in all cases again. □

The procedure described in Lemma 4.3 can be repeated until there is no j with $\operatorname{In}(C, \sigma_j) = \emptyset$. If C is separating, then each of the modifications keeps $\operatorname{Cycl}(C)$ stable and increases $\operatorname{Cycl}(C^{op})$ by a cycle from $\operatorname{Prof}(C)$. We can apply the same procedure upon C^{op}, whenever $\operatorname{Out}(C, \sigma_j) = \emptyset$ for some j. Ultimately we obtain a multigon that is either triangle free, or is itself a triangle. We record this as

Proposition 4.5. *Let C be a polygon with profile $(\sigma_1, \dots, \sigma_r)$. There exists a subsequence $1 \le j_r < \cdots < j_2 \le r$ such that $(\sigma_{j_1}, \dots, \sigma_{j_r})$ is a profile of such a multigon C' that is either a triangle, or a triangle free multigon. Furthermore, if C is separating or planar, then C' has the property as well. If C and C' are separating, then $\operatorname{Cycl}(C') \subseteq \operatorname{Cycl}(C) \cup \operatorname{Prof}(C)$ and $\operatorname{Cycl}((C')^{op}) \subseteq \operatorname{Cycl}(C^{op}) \cup \operatorname{Prof}(C)$.*

5. Standard situations

We shall be often dealing with two or three multigons that have common cycles which are positioned in a specific way. The purpose of this section is to describe several such positions for a further reference. This will be done mostly in the language of $\operatorname{In}(C)$, $\operatorname{Out}(C)$, $\overline{\operatorname{In}}(C)$ and $\overline{\operatorname{Out}}(C)$.

Lemma 5.1. *Let C be a multigon with profile $(\sigma_1, \dots, \sigma_r)$ and vertices $(v_1, \dots, v_r)$. Assume $\sigma = \sigma_j$, $1 \le j \le r$, and $\sigma^h(v_j) = v_{j+1}$, where h is less than the length of σ. Then $(\sigma^i(v_j), \sigma^{i+1}(v_j))$, $1 \le i < h$, are all edges of C induced by σ. Each of the pairs $(\operatorname{In}(C,\sigma), \operatorname{Out}(C,\sigma))$ and $(\overline{\operatorname{In}}(C,\sigma), \overline{\operatorname{Out}}(C,\sigma))$ determines the integer h and the points $\sigma_{j-1}^{-1}(v_j)$ and $\sigma_{j+1}(v_{j+1})$. If C is triangle free, then to determine these points and the integer h it suffices to know only one of $\overline{\operatorname{In}}(C,\sigma)$, $\operatorname{In}(C,\sigma)$, $\operatorname{Out}(C,\sigma)$, and $\overline{\operatorname{Out}}(C,\sigma)$.*

Proof. The lemma is descriptive and one can derive all of the statements from definitions in an immediate way. One can also use Lemma 4.1. □

Lemma 5.2. *Let U and V be multigons, and $\sigma \in \operatorname{Prof}(U) \cap \operatorname{Prof}(V)$. Consider the conditions*

(a) $\operatorname{In}(U, \sigma) \supseteq \operatorname{In}(V, \sigma)$ *and* $\operatorname{Out}(V, \sigma) \supseteq \operatorname{Out}(U, \sigma)$, *and*
(b) $\operatorname{Out}(U, \sigma) \supseteq \operatorname{In}(V, \sigma)$ *and* $\operatorname{Out}(V, \sigma) \supseteq \operatorname{In}(U, \sigma)$.

If U and V are triangle free, then the inclusions in both conditions are mutually equivalent. Each of the conditions is also equivalent to the condition obtained by replacing In *and* Out *by* $\overline{\operatorname{In}}$ *and* $\overline{\operatorname{Out}}$, *respectively.*

Proof. Condition (a) turns into condition (b) when U is replaced by U^{op}. We shall hence consider only the condition (a). If the assertion holds for U and V, then it holds for U^{op} and V^{op} as well. Hence we do not have to consider the extreme case $\operatorname{Out}(V, \sigma) = \emptyset$ (cf. Lemma 4.1). Assume $\operatorname{In}(V, \sigma) \neq \emptyset$. Then $\operatorname{In}(U, \sigma)$ and $\operatorname{In}(V, \sigma)$ determine $\operatorname{Out}(U, \sigma)$ and $\operatorname{Out}(V, \sigma)$, respectively, and we also get $\overline{\operatorname{In}}(U, \sigma) \supseteq \overline{\operatorname{In}}(V, \sigma)$ (one can use Lemma 4.1 or the description of Lemma 5.1). The other inclusion $\operatorname{Out}(V, \sigma) \supseteq \operatorname{Out}(U, \sigma)$ follows from the fact that $\overline{\operatorname{In}}(V, \sigma) \cup \operatorname{Out}(V, \sigma)$ and $\overline{\operatorname{In}}(U, \sigma) \cup \operatorname{Out}(V, \sigma)$ are disjoint unions which yield the set Σ of all cycles $\rho \neq \sigma$ that meet σ. If $\operatorname{In}(V, \sigma) \neq \emptyset$, then $\operatorname{Out}(V, \sigma) \neq \Sigma$, and so the converse implication follows again by resorting to U^{op} and V^{op}.

Let us have $\operatorname{In}(V, \sigma) = \emptyset$. Then the data of Lemma 5.1 can be obtained from $\operatorname{Out}(U, \sigma)$ and $\operatorname{Out}(V, \sigma)$ for both U and V. We see that there is only one edge of V that is induced by σ, and this edge is an edge of U. Hence $\overline{\operatorname{In}}(V, \sigma) \subseteq \overline{\operatorname{In}}(U, \sigma)$. Note that $\overline{\operatorname{Out}}(V, \sigma) = \Sigma$.

To get from $\overline{\operatorname{Out}}$ and $\overline{\operatorname{In}}$ to Out and In one uses reverse arguments. The critical cases are $\overline{\operatorname{In}}(U, \sigma) = \Sigma$ and $\overline{\operatorname{Out}}(V, \sigma) = \Sigma$, of course. $\square$

Say that a multigon U *wraps* a multigon V *on* σ if $\sigma \notin \operatorname{Prof}(U) \cap \operatorname{Prof}(V)$ or condition (a) holds. Say that U *wraps* V if U wraps V on every $\sigma \in \operatorname{Prof}(U)$ and $\operatorname{Prof}(U) \cap \operatorname{Prof}(V) \neq \emptyset$.

Say that a multigon U *abuts* a polygon V *on* σ if $\sigma \notin \operatorname{Prof}(U) \cap \operatorname{Prof}(V)$ or condition (b) holds. Say that U *abuts* V if U abuts V on every $\sigma \in \operatorname{Prof}(U)$ and $\operatorname{Prof}(U) \cap \operatorname{Prof}(V) \neq \emptyset$.

Lemma 5.3. *Let U and V be multigons and $\sigma \in \operatorname{Prof}(U) \cap \operatorname{Prof}(V)$. The condition*

$$\overline{\operatorname{In}}(U, \sigma) \cap \operatorname{In}(V, \sigma) = \emptyset \quad \textit{and} \quad \operatorname{In}(U, \sigma) \cap \overline{\operatorname{In}}(V, \sigma) = \emptyset$$

is equivalent the the condition (b) of Lemma 5.2. If both U and V are triangle free, then each of the equalities implies the other one.

Proof. Just note that $\operatorname{Out}(U, \sigma) \cup \overline{\operatorname{In}}(U, \sigma)$ is equal to the set Σ of all $\rho \neq \sigma$ that meet σ. $\square$

Corollary 5.4. *Let U and V be multigons. Then U abuts V $\Leftrightarrow$ V abuts U $\Leftrightarrow$ U^{op} wraps V $\Leftrightarrow$ V^{op} wraps U.*

Say that multigons U and V *complement* each other on a cycle σ if

(1) U abuts V on σ; and
(2) every edge induced by σ is an edge of U or V.

Note that an edge induced by σ is never an edgte of U and V simultaneously when U abuts V on σ. Note also that $\sigma \in \operatorname{Prof}(U) \cap \operatorname{Prof}(V)$ when U complements V on σ.

Say that a multigon U *divides into* multigons V and W on σ if

(1) U wraps both V and W on σ;
(2) V abuts W on σ; and
(3) every edge of U induced by σ is an edge of V or W.

Suppose that U divides into V and W on σ. If $\sigma \notin \mathrm{Prof}(U)$, then the conditions just say that $\sigma \notin \mathrm{Prof}(V) \cup \mathrm{Prof}(W)$. Assume $\sigma \in \mathrm{Prof}(U)$. Then no edge of U induced by σ can be an edge of both V and W. Note that if $\sigma \in \mathrm{Prof}(U)$, then $\sigma \in \mathrm{Prof}(V) \cup \mathrm{Prof}(W)$, but not necessarilly $\sigma \in \mathrm{Prof}(V) \cap \mathrm{Prof}(W)$. If $\sigma \in \mathrm{Prof}(U) \cap \mathrm{Prof}(V)$ and $\sigma \notin \mathrm{Prof}(W)$, then the edges of U and V that are induced by σ overlap. Hence U^{op} and V have complementary edges, and we can use Lemma 5.4 to obtain

Lemma 5.5. *Let U, V and W be multigons and suppose that $\sigma \in \mathrm{Prof}(U)\cap\mathrm{Prof}(V)$ is not a profiled cycle of W. Then U divides into V and W on σ if and only if U^{op} complements V on σ.*

If U divides into V and W on σ, then clearly

$$\mathrm{In}(U,\sigma) \subseteq \overline{\mathrm{In}}(V,\sigma) \cup \overline{\mathrm{In}}(W,\sigma) \subseteq \overline{\mathrm{In}}(U,\sigma).$$

Lemma 5.6. *Let σ' and σ'' be neighbours of σ in the profile of a multigon U. Suppose that U divides into multigons V and W on σ, σ' and σ''. Suppose that $\rho \in \mathrm{In}(U,\sigma)$ does not belong to $\mathrm{In}(V,\sigma) \cup \mathrm{In}(W,\sigma)$. Let x, v' and v'' be the points where σ meets ρ, σ' and σ'', respectively. Then ρ neighbours σ in the profile of V or W, x belongs to $\mathrm{Vert}(V) \cap \mathrm{Vert}(W)$, and*

(1) *either $x \in \{v', v''\}$ and $\mathrm{sgn}_V(x) = 1 = \mathrm{sgn}_W(x)$;*
(2) *or $x \notin \{v', v''\}$ and $\mathrm{sgn}_V(x) + \mathrm{sgn}_W(x) = 0$;*
(3) *or $x \notin \{v', v''\}$ and $\mathrm{sgn}_V(x) = 1 = \mathrm{sgn}_W(x)$.*

In cases (1) and (2) the cycles ρ and σ are neighbours in profiles of both V and W.

Proof. The inclusions before the lemma immediately imply $\rho \in \mathrm{Prof}(V)\cup\mathrm{Prof}(W)$, since $\rho \notin \mathrm{In}(V,\sigma) \cup \mathrm{In}(W,\sigma)$. If $\rho \in \overline{\mathrm{In}}(V,\sigma) \setminus \mathrm{In}(V,\sigma)$, then ρ neighbours σ in the profile of V, by the definition of $\overline{\mathrm{In}}(V,\sigma)$. We see that ρ neighbours σ in one of the profiles.

Assume $x \notin \{v', v''\}$, and suppose that ρ neighbours σ in the profile of V. There are two edges of U that are induced by σ and contain x. Only one of them is an edge of V, as $x \in \mathrm{Vert}(V)$. The other one is hence an edge of W, and so one of cases (2) and (3) clearly applies.

Assume now $x \in \{v', v''\}$, say $x = v'$. The edge of U that contains x and is induced by σ is also an edge of V or W. Let it be an edge of V. The other edge of U that contains x is induced by σ'. It cannot be an edge of V, since then x would appear twice as a vertex of V. Hence it is an edge of W, and x is a vertex of W since U divides into V and W on σ. The cycle ρ thus neighbours σ' in the profile of W, and we see that case (1) applies. □

Let us now turn to applications that involve separated multigons.

Proposition 5.7. *Let U and V be separated multigons. If U wraps V, then the domain of U contains the domain of V.*

Proof. All border faces of V are clearly in the domain of U if $\mathrm{Prof}(V) \subseteq \mathrm{Prof}(U) \cup \mathrm{Cycl}(U)$. Hence it suffices to prove this inclusion and the inclusion $\mathrm{Cycl}(V) \subseteq \mathrm{Cycl}(U)$, by Proposition 3.6.

Let $(\mu_1,\dots,\mu_m)$ be the profile of U, and $(\nu_1,\dots,\nu_n)$ the profile of V. If $\nu_j = \mu_i \in \mathrm{Prof}(U)$, then $\nu_{j+1} = \mu_{i+1}$ or $\nu_{j+1} \in \mathrm{In}(U,\mu_i)$, as $\overline{\mathrm{In}}(V,\mu_i) \subseteq \overline{\mathrm{In}}(U,\mu_i)$ by Lemma 5.2. In the latter case one can use for example Lemma 3.1 to obtain $\nu_{j+1} \in \mathrm{Prof}(U)\cup\mathrm{Cycl}(U)$. If $\nu_j \in \mathrm{Cycl}(U)$, then clearly $\nu_{j+1} \in \mathrm{Prof}(U)\cup\mathrm{Cycl}(U)$, say by rule (N2). Hence $\mathrm{Prof}(V) \subseteq \mathrm{Prof}(U)\cup\mathrm{Cycl}(U)$ follows by induction on j. This also yields

$$\mathrm{Prof}(U) \subseteq \mathrm{Prof}(V)\cup\mathrm{Cycl}(V^{op}),$$

by Corollary 5.4. To finish we need to show that $\mathcal{N} = \mathrm{Cycl}(U)\cap\mathrm{Cycl}(V)$ is closed under rules (N1) and (N2) when $C = V$, since then $\mathrm{Cycl}(U)\cap\mathrm{Cycl}(V) = \mathrm{Cycl}(V)$, by Proposition 3.6.

Each of the rules yields an element of $\mathrm{Cycl}(V)$, by Proposition 3.6, and so our task rests in verifying that this element belongs to $\mathrm{Cycl}(U)$. This will be done again by means of rules (N1) and (N2), but this time with respect to $\mathcal{N} = \mathrm{Cycl}(U)$ and $C = U$.

Since $(\sigma,\rho) \in \mathrm{In}(U)$ implies $\rho \in \mathrm{Prof}(U)$, we have in such a case either $\rho \in \mathrm{Prof}(V)$, or $\rho \in \mathrm{Cycl}(V^{op})$, by the inclusion above.

There are three situations to consider, the first two of which pertain to rule (N1). They distinguish if $\sigma \in \mathrm{Prof}(V)$ belongs to $\mathrm{Prof}(U)$ or $\mathrm{Cycl}(U)$ (we have already proved that one of these cases holds).

(1) Assume $\sigma \in \mathrm{Prof}(V)\cap\mathrm{Prof}(U)$, $\rho \in \mathrm{In}(V,\sigma)$ and $(\sigma,\rho) \notin \mathrm{In}(V)$. Since $\rho \in \mathrm{Cycl}(V)$, there cannot be $(\sigma,\rho) \in \mathrm{In}(U)$. We have $\mathrm{In}(V,\sigma) \subseteq \mathrm{In}(U,\sigma)$, and so $\rho \in \mathrm{Cycl}(U)$, by rule (N1).

(2) Assume $\sigma \in \mathrm{Prof}(V)\cap\mathrm{Cycl}(U)$, $\rho \in \mathrm{In}(V,\sigma)$ and $(\sigma,\rho) \notin \mathrm{In}(V)$. There cannot be $(\sigma,\rho) \in \mathrm{In}(U)$, as $\rho \in \mathrm{Cycl}(V)$, and so $\sigma \in \mathrm{Cycl}(U)$ by rule (N2).

(3) Finally assume $\rho \in \mathrm{Cycl}(U)\cap\mathrm{Cycl}(V)$, and $(\rho,\sigma) \notin \mathrm{In}(V)$. There cannot be $(\rho,\sigma) \in \mathrm{In}(U)$, as $\sigma \in \mathrm{Cycl}(V)$, and so $\rho \in \mathrm{Cycl}(U)$, by rule (N2). □

Corollary 5.4 now yields a variant of the preceding proposition.

Proposition 5.8. *Let U and V be separating multigons. If U abuts V, then the domains of U and V have no common face.*

6. Decomposition of multigons

Let U, V and W be multigons. Put $\Pi = (\mathrm{Prof}(V)\cup\mathrm{Prof}(W))\setminus\mathrm{Prof}(U)$. Say that U *decomposes into* V and W if

(1) U divides into V and W on every $\sigma \in \mathrm{Prof}(U)$,
(2) V complements W on every $\rho \in \Pi$, and
(3) $\mathrm{Prof}(U)\cap\mathrm{Prof}(V) \neq \emptyset$ and $\mathrm{Prof}(U)\cap\mathrm{Prof}(W) \neq \emptyset$.

Lemma 6.1. *Suppose that U decomposes into V and W. Then U wraps both V and W, and V abuts W. If $\{P_1,P_2,P_3\} = \{\mathrm{Prof}(U),\mathrm{Prof}(V),\mathrm{Prof}(W)\}$ and $\{i,j,k\} = \{1,2,3\}$, then $P_i\cap P_j \neq \emptyset$, $P_i\setminus(P_j\cup P_k) = \emptyset$ and $P_i\cup P_j\cup P_k = P_i\cup(P_j\cap P_k)$.*

Proof. We shall first show $\mathrm{Prof}(V)\cap\mathrm{Prof}(W) \neq \emptyset$. Assume contrariwise, ande let $(\mu_1,\dots,\mu_n)$ be the profile of U. We have $\mu_1 \in \mathrm{Prof}(V)\cup\mathrm{Prof}(W)$, let us assume $\mu_1 \in \mathrm{Prof}(V)$. If all μ_i belong to $\mathrm{Prof}(V)$, then $U = V$, which cannot be. We can hence find $\mu_j \in \mathrm{Prof}(V)$ with $\mu_{j+1} \notin \mathrm{Prof}(V)$. Then $\mu_{j+1} \in \mathrm{Prof}(W)\setminus\mathrm{Prof}(V)$, and $\mu_j \in \mathrm{Prof}(V)\setminus\mathrm{Prof}(W)$, by our assumptions. Let μ_j and μ_{j+1} intersect in the vertex v, and let ρ be the third cycle that moves v. Then clearly $v \in \mathrm{Vert}(V)\cap\mathrm{Vert}(W)$, and $\rho \in \mathrm{Prof}(V)\cap\mathrm{Prof}(W)$.

The rest is easy. The multigon U wraps both V and W, by (1) and (3). We have $\Pi = (\mathrm{Prof}(V) \cap \mathrm{Prof}(W)) \setminus \mathrm{Prof}(U)$, by (2), and $\mathrm{Prof}(U) \subseteq \mathrm{Prof}(V) \cup \mathrm{Prof}(W)$, by (1). From that and from (3) one can deduce the assertions about P_1, P_2 and P_3. Finally, to see that V abuts W, use (1), (2) and the fact that $\mathrm{Prof}(V) \cap \mathrm{Prof}(W) \neq \emptyset$. □

Lemma 6.2. *Suppose that U decomposes into V and W. Then V^{op} decomposes into U^{op} and W, and W^{op} decomposes into U^{op} and V.*

Proof. We shall prove that V^{op} decomposes into U^{op} and W, and that means to verify conditions (1–3) with respect to these multigons. From Lemma 6.1 we get immediately the validity of condition (3). From the same lemma one also gets that V^{op} wraps U^{op} and that V^{op} wraps W, by Corollary 5.4. By the same argument, U^{op} abuts W. Note also that $\Pi' = (\mathrm{Prof}(U) \cup \mathrm{Prof}(W)) \setminus \mathrm{Prof}(V)$ is equal to $(\mathrm{Prof}(U) \cap \mathrm{Prof}(W)) \setminus \mathrm{Prof}(V)$, by Lemma 6.1. If $\sigma \in \Pi'$, then U^{op} complements W on σ, by Lemma 5.5, and so (2) holds.

For $\sigma \in \mathrm{Prof}(V)$ consider first the case $\sigma \notin \mathrm{Prof}(U)$. Then V^{op} divides into U and W on σ, again by Lemma 5.1. Finally, let us have $\sigma \in \mathrm{Prof}(V) \cap \mathrm{Prof}(U)$. We need to show that an edge e of V^{op} that is induced by σ, is an edge of U^{op} or an edge of W. If this were not true, then the edge e would be an edge of U and not an edge of W. But then it would be an edge of V, since U divides into V and W on σ, a contradiction. □

Lemma 6.3. *Suppose that U decomposes into V and W. If x is a vertex in one of the multigons, then it is a vertex in at least two of them. Set*

$$s(x) = \frac{1}{3}(\mathrm{sgn}_V(x) + \mathrm{sgn}_W(x) - \mathrm{sgn}_U(x)).$$

Then $s(x) = 1$ if $x \in \mathrm{Vert}(U) \cap \mathrm{Vert}(V) \cap \mathrm{Vert}(W)$, and $s(x) = 0$ in all other cases.

Proof. Note first that $3s(x) = \mathrm{sgn}_{U^{op}}(x) + \mathrm{sgn}_W(x) - \mathrm{sgn}_{V^{op}}(x)$. Therefore one can assume $x \in \mathrm{Vert}(U)$, by Lemma 6.2. It is clear that $x \in \mathrm{Vert}(V) \cup \mathrm{Vert}(W)$, since U divides into V and W on each $\sigma \in \mathrm{Prof}(U)$. Let σ and σ' be the consecutive cycles in the profile of U that meet at x. If x appears as a vertex in only one of polygons V and W, say in V, then σ and σ' clearly have to be consecutive cycles in the profile of V as well, and hence $\mathrm{sgn}_V(x) = \mathrm{sgn}_U(x)$. Suppose now that x is a vertex of both V and W, and let ρ be the third cycle that moves x. Neither V nor W has σ and σ' as consecutive cycles in their profiles since an edge of V or W that is induced by σ or σ' has to be an edge of U. The pairs of consecutive cycles in the profiles hence have to be (σ, ρ) and (ρ, σ). Since U wraps both V and W, we must have $\mathrm{sgn}(\sigma, \rho) = 1 = \sigma(\rho, \sigma')$. Therefore $3s(x) = 1 + 1 - (-1) = 3$. □

Proposition 6.4. *Let U, V and W be multigons such that U decomposes into V and W. If any two of the multigons are separating, then the third multigon is separating as well.*

Proof. By Lemma 6.2 and Proposition 2.5 it suffices to consider the case when V and W are separating multigons. We shall show that $\mathcal{N} = \Pi \cup \mathrm{Cycl}(V) \cup \mathrm{Cycl}(W)$ is closed under rules (N1) and (N2) when $C = U$, and that no element of $\mathrm{Prof}(U)$ belons to $\mathcal{N}$. This will suffice, by Proposition 3.6.

Suppose that $(\rho,\sigma) \in \mathrm{In}(U)$ and $(\sigma,\rho) \notin \mathrm{In}(U)$. We wish to show $\rho \in \mathcal{N}$. Let us first consider the case $(\rho,\sigma) \in \mathrm{In}(V)$. If $(\sigma,\rho) \notin \mathrm{In}(V)$, then $\rho \in \mathrm{Cycl}(V)$, by rule (N1). Assume $(\sigma,\rho) \in \mathrm{In}(V)$. Then $\rho \in \mathrm{Prof}(V)$, and only the case $\rho \notin \Pi$ needs to be considered. In that case $\rho \in \mathrm{Prof}(U)$, and $(\sigma,\rho) \in \mathrm{In}(V)$ since $\mathrm{In}(V,\rho) \subseteq \mathrm{In}(U,\rho)$. However, this contradicts our assumptions. Secondly, let us have $(\rho,\sigma) \notin \mathrm{In}(V) \cup \mathrm{In}(W)$. Then Lemma 5.6 applies, and we get $\rho \in \Pi$ in cases (1) and (2). Let us consider case (3). Denote by x the intersecting point of σ and ρ. Since $\mathrm{sgn}_V(x) = \mathrm{sgn}_W(x) = 1$, by Lemma 5.6, there must be $\mathrm{sgn}_U(x) = -1$, by Lemma 6.3. The point x is hence an intersecting point of consecutive cycles from the profile of U, say of σ' and σ''. The edges of σ that move x are edges of C, and hence $\sigma \notin \{\sigma',\sigma''\}$. Cycle ρ thus equals σ' or σ'', and x is on the inner arc of ρ in C. Hence $(\sigma,\rho) \in \mathrm{In}(U)$, contrary to our assumption. We have verified that $\mathcal{N}$ is preserved by rule (N1).

When dealing with rule (N2) we shall consider apart cases $\rho \in \mathrm{Cycl}(V)$ and $\rho \in \Pi$. We assume that ρ meets a cycle σ such that $(\rho,\sigma) \notin \mathrm{In}(U)$.

If $\rho \in \mathrm{Cycl}(V)$ and $(\rho,\sigma) \notin \mathrm{In}(V)$, then $\sigma \in \mathrm{Cycl}(V)$, by (N2). If $(\rho,\sigma) \in \mathrm{In}(V)$, then $\sigma \notin \mathrm{Prof}(U)$, since $\mathrm{In}(V,\sigma) \subseteq \mathrm{In}(U,\sigma)$ for every $\sigma \in \mathrm{Prof}(U)$ and since we assume $(\rho,\sigma) \notin \mathrm{In}(U)$. But that means $\sigma \in \Pi \subseteq \mathcal{N}$.

Assume $\rho \in \Pi$, and denote by x the intersection of ρ and σ. Start by considering the case when x is a point of the inner arc of ρ in V. This means $(\sigma,\rho) \in \mathrm{In}(V)$. Suppose that $(\rho,\sigma) \in \mathrm{In}(V)$ as well. If $\sigma \in \Pi$, then $\sigma \in \mathcal{N}$, and so $\sigma \in \mathrm{Prof}(U)$ can be assumed. But that contradicts $\mathrm{In}(V,\sigma) \subseteq \mathrm{In}(U,\sigma)$ because $(\rho,\sigma) \notin \mathrm{In}(U)$. The case $(\rho,\sigma) \in \mathrm{In}(V)$ is thus clear, and the case $(\rho,\sigma) \notin \mathrm{In}(V)$ gives $\sigma \in \mathrm{Cycl}(U)$, by rule (N1). Therefore we can turn to the case when x is a point on the inner arc of ρ neither in V nor in W. That means that x is an acute vertex of both V and W, since V and W complement each other on ρ. Let σ' and σ'' be the two cycles that move x and are different from ρ. The point x is an obtuse vertex of U, by Lemma 6.3. The cycles σ' and σ'' are neighbours in the profile of U and $\sigma \in \{\sigma',\sigma''\}$. Hence $(\rho,\sigma) \in \mathrm{In}(U)$, which contradicst our assumption. We have verified that $\mathcal{N}$ is preserved by rule (N2).

Finally we have to refute the possibility that $\mathcal{N}$ consists of all cycles in the connectivity component. Note that only when this is done, one can conclude that Proposition 3.6 really guarantees that the multigon U is separating.

The natural way seems to be to show that no $\sigma \in \mathrm{Prof}(U)$ belongs to $\mathcal{N}$. Assume contrariwise. We cannot have $\sigma \in \Pi$, by the definition of Π, and so we can assume $\sigma \in \mathrm{Cycl}(V)$. This means $\sigma \in \mathrm{Prof}(W) \setminus \mathrm{Prof}(V)$ since $\mathrm{Prof}(V) \cap \mathrm{Cycl}(V) = \emptyset$. Consider now a triangular face with an edge of W induced by σ. This is a border face of W that is also a face in the domain of V, a contradiction to Proposition 5.8.

Proposition 6.5. *Suppose that the multigon U decomposes to multigons V and W, and that all these multigons are separating. Then*

$$\mathrm{Cycl}(U) = \mathrm{Cycl}(V) \cup \mathrm{Cycl}(W) \cup \Pi, \ \textit{where } \Pi = (\mathrm{Prof}(V) \cap \mathrm{Prof}(W)) \setminus \mathrm{Prof}(U).$$

The domain of U contains domains of both V and W. Each face in the domain of U is either in the domain of V, or in the domain of W, or it is a cyclic face induced by some $\sigma \in \Pi$.

Proof. We know from the proof of the previous proposition that $\mathrm{Cycl}(U)$ is contained in $\mathrm{Cycl}(V) \cup \mathrm{Cycl}(W) \cup \Pi$. On the other hand $\mathrm{Cycl}(U)$ contains both $\mathrm{Cycl}(V)$ and $\mathrm{Cycl}(W)$, by Proposition 5.7. We shall prove $\Pi \subseteq \mathrm{Cycl}(U)$.

Let $(\nu_1, \dots, \nu_m)$ be the profile of V. We shall show that either $\nu_j \in \operatorname{Prof}(U)$, or $\nu_j \in \Pi \cap \operatorname{Cycl}(U)$. This can proceed by induction since $\operatorname{Prof}(U) \cap \operatorname{Prof}(V)$ is a nonempty set. If $\nu_{j+1} \in \operatorname{Prof}(U)$, then there is nothing to prove. Assume $\nu_{j+1} \in \Pi$, and let x be the intersection point of ν_j and ν_{j+1}. Let us first have $\nu_j \in \operatorname{Prof}(U)$. There cannot be $x \in \operatorname{Vert}(U) \setminus \operatorname{Vert}(W)$, as in such a case $\operatorname{sgn}_V(x) = \operatorname{sgn}_U(x)$, by Lemma 6.3, and that means $\nu_{j+1} \in \operatorname{Prof}(U)$. Assume $x \in \operatorname{Vert}(U) \cap \operatorname{Vert}(V)$. Then x is an obtuse vertex of U, but an acute vertex of V, by Lemma 6.3. Hence $(\nu_{j+1}, \nu_j) \in \operatorname{In}(U)$, and $(\nu_j, \nu_{j+1}) \notin \operatorname{In}(U)$, since we can assume $\nu_{j+1} \notin \operatorname{Prof}(U)$. Therefore $\nu_{j+1} \in \operatorname{Cycl}(U)$, by rule (N1). We have solved the case $x \in \operatorname{Vert}(U)$. If $x \notin \operatorname{Vert}(U)$, then x is a nonvertex point of U, and so $(\nu_{j+1}, \nu_j) \in \operatorname{In}(U)$, and rule (N1) can be applied again.

Suppose now that $\nu_j \in \Pi \cap \operatorname{Cycl}(U)$ and $\nu_{j+1} \in \Pi$. Then $(\nu_j, \nu_{j+1}) \notin \operatorname{In}(U)$, as $\nu_{j+1} \notin \operatorname{Prof}(U)$, and $\nu_{j+1} \in \operatorname{Cycl}(U)$ by rule (N2).

We have verified that $\operatorname{Cycl}(U)$ consists of $\operatorname{Cycl}(V)$, $\operatorname{Cycl}(W)$ and Π. We know that the domains of U contains the domains of V and W, by Proposition 5.7. These domains have no common face, by Proposition 5.8. The cyclic faces induced by elements of Π belong to the domain of U, but not to the domain of V or W, since $\Pi \subseteq \operatorname{Prof}(V) \cap \operatorname{Prof}(W)$. Hence it remains to show that the faces in the domains of V and W, plus the faces induced by elements of Π really constitute all faces within the domain of U. For that we shall use Proposition 3.6. We already know that for the cyclic faces the assertion holds. Each triangular face adjacent to a cyclic face induced by $\sigma \in \Pi$ is a border face of V or W, since V complements W on σ. Hence our set fulfils condition (F3). Each border face of U is a border face of either V or W, since U divides into V and W on each $\sigma \in \operatorname{Prof}(U)$. The conditon (F1) is hence fulfilled as well. □

Proposition 6.6. *Suppose that the multigon U decomposes to polygons V and W, and that the multigons V and W are planar. Put $\Pi = (\operatorname{Prof}(V) \cap \operatorname{Prof}(W)) \setminus \operatorname{Prof}(U)$. If $|\Pi| + 1 = |\operatorname{Vert}(V) \cap \operatorname{Vert}(W)|$, then U is a planar polygon as well.*

Proof. Put $\Delta_C = |\operatorname{Cycl}(V)| + |\operatorname{Cycl}(W)| - |\operatorname{Cycl}(U)|$, $\Delta_s = \operatorname{sgn}(V) + \operatorname{sgn}(W) - \operatorname{sgn}(U)$ and $\Delta_P = |\operatorname{Pnt}(V)| + |\operatorname{Pnt}(W)| - |\operatorname{Pnt}(U)|$. We need to show that

$$\Delta_C + \Delta_s = \Delta_P + 1.$$

Now, $\Delta_C = -|\Pi|$, by Proposition 6.5, and $\Delta_s = |\operatorname{Vert}(U) \cap \operatorname{Vert}(V) \cap \operatorname{Vert}(W)|$, by Lemma 6.3. Denote by $P(U)$, $P(V)$ and $P(W)$ the sets $\operatorname{Pnt}(U) \cup \operatorname{Vert}(U)$, $\operatorname{Pnt}(V) \cup \operatorname{Vert}(V)$ and $\operatorname{Pnt}(W) \cup \operatorname{Vert}(W)$, respectively. We have $P(U) = P(V) \cup P(W)$, by Proposition 6.5. Furthermore, $P(V) \cap P(W) = \operatorname{Vert}(V) \cap \operatorname{Vert}(W)$, as V and W complement each other. Hence $|P(U)| + |\operatorname{Vert}(V) \cap \operatorname{Vert}(W)| = |P(V)| + |P(W)|$, and so $\Delta_P = |\operatorname{Vert}(V) \cap \operatorname{Vert}(W)| - |\operatorname{Vert}(V)| - |\operatorname{Vert}(W)| + |\operatorname{Vert}(U)|$. However,

$$|\operatorname{Vert}(V)| = |\operatorname{Vert}(V) \cap \operatorname{Vert}(W)| + |\operatorname{Vert}(V) \cap \operatorname{Vert}(U)| - |\operatorname{Vert}(U) \cap \operatorname{Vert}(V) \cap (W)|,$$

and for both $\operatorname{Vert}(U)$ and $\operatorname{Vert}(W)$ one can get corresponding expressions as well, by Lemma 6.3. Hence $\Delta_P = |\operatorname{Vert}(U) \cap \operatorname{Vert}(V) \cap \operatorname{Vert}(W)| - |\operatorname{Vert}(V) \cap \operatorname{Vert}(W)|$. The equality $\Delta_C + \Delta_s = \Delta_P + 1$ thus holds if and only if $|\operatorname{Vert}(V) \cap \operatorname{Vert}(W)| = |\Pi| + 1$. However, that is exactly what we assume. □

7. The existence of nonseparating multigons

Let C be a multigon in the surface S of (τ_1, τ_2, τ_3), and let $(\sigma_1, \dots, \sigma_r)$ be the profile of C. Recall that the multigon is simple if and only if σ_i meets σ_j exactly when $j = i+1$, where $0 \le i < j < r$. (We use here the convention that $\sigma_0 = \sigma_r$.)

Simple multigons are simple oriented polygonal curves, or, as one would usually say, they are oriented polygons. However, an oriented polygon need not be a multigon, since a polygon can return to the same cyclic face twice. Note also that an oriented polygon that is a multigon need not be a simple multigon, since σ_i and σ_j can intersect in their outer arcs even when they do not intersect in a point of C.

In this section we shall show that the existence of a nonseparating multigon implies the existence of a nonseparating simple triangle free multigon.

Let C be a multigon and v a point of S. Denote by $\mathcal{E}_v(C)$ the set of all pairs (e_1, e_2), where $e_1 = (x_1, v)$ and $e_2 = (v, x_2)$ are oriented edges of S such that x_1, v and x_2 are consecutive points of C. (This can be expressed also by saying that e_1 and e_2 are consecutive edges of C.)

Note that $0 \le |\mathcal{E}_v(C)| \le 3$, and that at most one pair $(e_1, e_2) \in \mathcal{E}_v(C)$ can have the property that the cycles inducing e_1 and e_2 are distinct.

Lemma 7.1. *Let U, C and V be multigons, and let v be a point of S such that*

$$\mathcal{E}_v(C) = \mathcal{E}_v(U) \cup \mathcal{E}_v(V) \quad \text{and} \quad \mathcal{E}_v(U) \cap \mathcal{E}_v(V) = \emptyset.$$

Suppose that U is separating and that $(e, f) \in \mathcal{E}_v(V)$. Denote by E and F the border faces of V that are induced by e and f, respectively.

If one of E or F belongs to the domain of U, then both of them are in the domain.

Proof. Let ρ and ρ' be the cycles that induce e and f, respectively. If $\operatorname{sgn}(\rho, \rho') = 1$, then $E = F$, and nothing needs to be proved. If E or F is in the domain of U, then $\rho \in \operatorname{Cycl}(U)$ or $\rho' \in \operatorname{Cycl}(U)$, respectively, since neither e nor f is an edge of U. The case $\rho = \rho'$ is hence clear as well, and we can assume $\operatorname{sgn}(\rho, \rho') = -1$. Denote by σ the third cycle that moves v, and note that σ induces and edge of both E and F. Denote these edges by g and h, respectively. If $\sigma \in \operatorname{Cycl}(U)$, then both E and F are in the domain of U, and if $\sigma \in \operatorname{Cycl}(C^{op})$, then none of them is in the domain. Assume $\sigma \in \operatorname{Prof}(U)$. We have $(e, f) \in \mathcal{E}_v(C)$, and hence either $(h, g) \in \mathcal{E}_v(C)$, or none of g and h is an edge of C. By our assumptions this means that either $(h, g) \in \mathcal{E}_v(U)$, or $(h, g) \in \mathcal{E}_v(U^{op})$. Both E and F are in the domain of U in the former case, and none of them is in the domain in the latter case. □

In the next paragraphs we shall deal with the situation when C is not a simple multigon, i.e. when C contains a point that is repeated at least twice.

Let x be the point. There must be two consecutive edges of C, both incident to x, that are induced by the same cycle, say $\sigma = \sigma_j$. (Otherwise x would appear twice as a vertex, contrary to the definition of a multigon.) Since the point x gets repeated, there must be another pair of consecutive edges of C that are incident to x, say $(\rho^{-1}(x), x)$ and $(x, \rho'(x))$. We have $\rho = \sigma_i$ for some $i \ne j \pm 1$. The elements of the profile can be cyclically shifted, and hence we can assume $1 \le i < j-1 < r$, without loss of generality.

We shall define an oriented polygonal curve V by its profile: the profile will be equal to $(\sigma_{i+1}, \dots, \sigma_j)$ if $\rho' = \sigma_{i+1}$, and to $(\sigma_i, \sigma_{i+1}, \dots, \sigma_j)$ if $\rho' = \sigma_i = \rho$. All edges of V are also edges of C, and the edges $(\sigma^{-1}(x), x)$ and $(x, \rho'(x))$ are among them. The point x is a vertex of V, and all other vertices of V are vertices of C. The point x is a vertex of C if and only if $\rho \neq \rho'$. We see that V is a multigon.

Similar arguments can be used to show that there exists a multigon with profile $(\sigma_1, \dots, \sigma_i, \sigma_j, \dots, \sigma_r)$. This multigon will be denoted by U.

Suppose first that $\mathrm{sgn}(\rho, \rho') = 1$ (i.e., that x is an acute vertex of C) and that U is separating. Let $v_1 \dots v_k$ be the pointwise description of V, with $x = v_1$ and $v_k = \sigma^{-1}(x)$, and let E be the triangular face with edges induced by ρ and ρ'. Since $(\rho^{-1}(x), x)$ is an edge of U and $(x, \rho'(x)) = (v_1, v_2)$ is an edge of V, we see that E is a border face of both U and V. In particular, E is in the domain of U. Denote by E_i, $1 \le i \le k$, the border face of V determined by (v_i, v_{i+1}).

If $v = v_{i+1}$, $1 \le i < k$, then the assumptions of Lemma 7.1 are satisfied, and we can prove, by induction, that all E_i, $1 \le i \le k$, belong to the domain of U. The edge $(\sigma^{-1}(x), x)$ is an edge of both E_k and V. Since it is an edge of V, it is not an edge of U. Since E_k is the domain of U, there must be $\sigma \in \mathrm{Cycl}(U)$, a contradiction to $\sigma \in \mathrm{Prof}(U)$. We see that $\mathrm{sgn}(\rho, \rho') \le 0$ whenever U is separating.

Assume now $\mathrm{sgn}(\rho, \rho') \le 0$. Put $\mathit{\Pi} = (\mathrm{Prof}(U) \cup \mathrm{Prof}(V)) \setminus \mathrm{Prof}(C)$. This set is clearly empty, and hence $|\mathit{\Pi}| + 1 = |\mathrm{Vert}(U) \cap \mathrm{Vert}(V)|$. We easily see that C decomposes into V and W. (The only condition which needs to be taken care of is the fact that C divides into U and V on σ, and that follows from $\mathrm{sgn}(\rho, \rho') \le 0$.)

Propositions 6.5 and 6.6 therefore imply

Lemma 7.2. *If both multigons U and V are separating, then the multigon C is separating as well. If both multigons U and V are planar, then the multigon C is planar as well.*

We shall now turn to the case when neither C nor C^{op} possesses a repeated point. In other words, we assume that both C and C^{op} are oriented polygons. Suppose however that C is not a simple multigon.

That means that some $\sigma = \sigma_j \in \mathrm{Prof}(C)$ meets with some $\rho = \sigma_i \in \mathrm{Prof}(C)$ in a point y of C in such a way that $(\sigma^{-1}(y), y)$ and $(y, \sigma(y))$ are not edges of C, while $(\rho^{-1}(y), y)$ and $(y, \rho'(y))$ are consecutive edges of C. In particular, $i \neq j \pm 1$. Denote by z and x the points where σ meets σ_{j-1} and σ_{j+1}, respectively. Clearly, $x, z \in \mathrm{Vert}(C)$, and $y \notin \{x, z, \sigma_{j+1}(x), \sigma_{j-1}^{-1}(z)\}$. We can assume $1 \le i < j-1 < r$, with no loss of generality.

Let V be the closed oriented polygonal curve with profile $(\sigma_{i+1}, \dots, \sigma_j)$ when $\sigma_{i+1} = \rho' \neq \rho$, and profile $(\sigma_i, \dots, \sigma_j)$ when $\sigma_i = \rho' = \rho$. The edges of V which are induced by σ go from z to y over x, and hence form a single segment. All other edges of V are edges of C. The point y is a vertex of V, and other vertices of V are vertices of C (note however that x is not a vertex of V). If $\rho = \rho'$, then y is not a vertex of C, and we see that V is a multigon.

Similar arguments can be used to show that there exists a multigon with profile $(\sigma_1, \dots, \sigma_i, \sigma_j, \dots, \sigma_r)$. This multigon will be denoted by U.

Suppose now that the multigon U is separating and that $\mathrm{sgn}(\rho, \sigma) = 1$. Since $\sigma = \sigma_j$ moves points on the way from $y \in \mathrm{Vert}(U)$ to $x \in \mathrm{Vert}(U)$ through the point z, we see that triangular faces with edges $(\sigma^{-1}(z), z)$ and $(z, \sigma(z))$ are border faces of U. One of these faces has an edge moved by $\sigma_{j-1} \notin \mathrm{Prof}(U)$ and hence $\sigma_{j-1} \in \mathrm{Cycl}(U)$. Denote by E the triangular face that contains the edge $(\sigma_{j-1}^{-1}(z), z)$. This is a border face of V, which is in the domain of U. By working against the arrows of V we derive from Lemma 7.1 that the domain of U contains also the border face of V with the edge $(y, \rho'(y))$. That edge is not an edge of U. Denote the border face by E'. If $\rho = \rho'$, then we obtain $\rho' = \sigma_i \in \mathrm{Cycl}(U)$, while $\rho' = \sigma_{i+1}$ implies $\sigma_j = \sigma \in \mathrm{Cycl}(U)$, since $(\sigma^{-1}(y), y)$ is in this case another edge of E', and this edge is not an edge of U. We have obtained a contradiction in both cases since $\mathrm{Cycl}(U) \cap \mathrm{Prof}(U) = \emptyset$. Hence $\mathrm{sgn}(\rho, \sigma) \neq 1$ when U is a separating multigon.

Assume now that V is separating and $\mathrm{sgn}(\sigma, \rho') = 1$. Starting from the border face E of U with edge $(x, \sigma_{j+1}(x))$ we work along the arrows of U up to the border face E' with the edge $(\rho^{-1}(y), y)$. The multigon V yields edges $(\sigma^{-1}(x), x)$ and $(x, \sigma(x))$, and so $\sigma_{j+1} \notin \mathrm{Prof}(V)$. Hence $\sigma_{j+1} \in \mathrm{Cycl}(U)$, E belongs to the domain of V, and E' belongs to the domain of V by repeated applications of Lemma 7.1. By considering the triangular face E' we deduce that $\mathrm{Prof}(V) \cap \mathrm{Cycl}(V)$ contains σ or ρ, which is a contradiction to the assumption that V is separating. Thus $\mathrm{sgn}(\rho, \sigma') \leq 0$ when V is a separating multigon.

Lemma 7.3. *Suppose that both U and V are separating. Then U decomposes into C and V^{op}, and C is separating as well. If U and V^{op} are planar, then C is also planar.*

Proof. First note that if $\rho = \rho'$, then either $\mathrm{sgn}(\sigma, \rho') = 1$ or $\mathrm{sgn}(\rho, \sigma) = 1$. We have excluded both these alternatives, and hence there must be $\rho \neq \rho'$ and $1 = \mathrm{sgn}(\rho', \sigma) = \mathrm{sgn}(\sigma, \rho) = \mathrm{sgn}(\rho, \rho')$. Polygons U and V^{op} have profiles $(\sigma_1, \dots, \sigma_i, \sigma_j, \dots, \sigma_r)$ and $(\sigma_j, \sigma_{j-1}, \dots, \sigma_{i+1})$, respectively. Thus $\Pi = (\mathrm{Prof}(V) \cup \mathrm{Prof}(C)) \setminus \mathrm{Prof}(U) = \{\sigma_{i+1}, \dots, \sigma_{j-1}\}$, where $\sigma_{i+1} = \rho'$, $\sigma_i = \rho$ and $\sigma_j = \sigma$.

We have $\mathrm{In}(U, \sigma_k) = \mathrm{In}(C, \sigma_k)$ if $k = 1, \dots, i-1$ and $k = j+1, \dots, r$. Thus U divides into C and V^{op} on such σ_k.

The edges of V^{op} from y to z are those of U as well. The outcoming edge of V^{op} from z is the incoming edge of C. The outcoming edges of U and C agree at x, and $\rho' \in \mathrm{In}(U, \sigma)$, as $\mathrm{sgn}(\rho, \rho') = 1$. These facts together imply that U divides into C and V^{op} on $\sigma = \sigma_j$.

Since U leaves y by σ, C leaves it by means of ρ' and $\mathrm{sgn}(\rho', \sigma) = 1$, U divides into C and V^{op} on $\sigma_i = \rho$ as well, and hence on every element of $\mathrm{Prof}(U)$.

For the proof that V^{op} complements C on each element of Π we thus only need to show that V wraps C, by Corollary 5.4, and this clearly can be reduced to teh proof that V wraps C on $\sigma = \sigma_j$ and on $\rho' = \sigma_{i+1}$. The case of σ is obvious since V and C coincide on the star of z, and $\sigma_{j+1} \in \mathrm{In}(V, \sigma)$, since x is on the inner arc of V. The case of ρ' follows from the fact that y is an obtuse vertex of V, and so $\rho \in \mathrm{In}(V, \rho')$.

We have verified that U indeed decomposes into C and V^{op}. Every common vertex of V and C appears on the common way of V and C from y to z. This way uses $|\Pi|$ different cycles, and so $|\mathrm{Vert}(V^{op}) \cap \mathrm{Vert}(C)| = |\Pi| + 1$. Propositions 6.5 and 6.6 can be thus applied. □

Theorem 7.4. *Let S be a connected combinatorial surface determined by a triple of permutations (τ_1, τ_2, τ_3) of a set X that satisfy conditions (P1) and (P2). Then S is a combinatorial sphere if and only if every multigon of S is separating. If S is a combinatorial sphere, then every multigon of S is planar. If S is not a combinatorial sphere, then there exists a simple triangle free nonseparating multigon.*

Proof. Suppose first that there exists a nonseparating multigon C of profile $(\sigma_1, \dots, \sigma_r)$. Choose C in such a way that r is minimal possible. If C is not simple, consider U and V as defined in the earlier parts of this section. Both U

and V have shorter profiles, and in Lemmas 7.2 and 7.3 we observed that C has to be separating if both U and V are separating. The existence of U and V is thus excluded when r is supposed to be the minimum among possible values of $|\operatorname{Prof}(C)|$.

Recall that Lemma 7.2 covered the case when C is not a simple polygon. By transition to C^{op} it covers also the situation when C^{op} is not a simple polygon. The remaining possible violations of multigon simplicity of C are treated by Lemma 7.3. Hence C has to be a simple nonseparating multigon. From Proposition 4.5 we see that C has to be triangle free.

In a combinatorial sphere there cannot exist a nonseparating polygon, and so all multigons have to be separating if S is a combinatorial sphere. If we find a multigon C such that both C and C^{op} are planar, then S is a combinatorial sphere, by Proposition 2.6. To finish we thus need to prove that every multigon is planar whenever S has the property that every multigon C is separating.

We shall prove this by induction on the number of faces in the domain of C. Since triangles are planar, we can proceed by induction. However, conceptually it seems to be easier to argue by contradiction. Let thus C be a counterexample with the least domain. From Lemma 7.2 we see that C cannot have repeated points, and so it is a simple polygon. Knowing that, one could argue by means of Eulerian characteristic. However, we shall take a more elementary approach, using some of our earlier results.

Let $(\sigma_1, \dots, \sigma_r)$ be the profile of C, and let $(v_1, \dots, v_r)$ be its vertices. Note that we have $\operatorname{In}(C, \sigma_j) \neq \emptyset$, $1 \leq j \leq r$, by Lemma 4.3. (In other words $(\sigma_{j-1}, \sigma_j, \sigma_{j+1})$ is never a profile of a triangle.) Suppose first that C contains an acute vertex, say v_j. Let ρ be the cycle with $\sigma_{j-1}\rho\sigma_j(v_j) = v_j$. The edge $(\sigma_j(v_j), \sigma_{j-1}^{-1}(v_j))$ is not an edge of C, since C is triangle free and without repeated points. Denote by V the triangle $\sigma_{j-1}^{-1}(v_j)\ v_j\ \sigma_j(v_j)$, and by W the polygon with profile

$$(\sigma_1, \dots, \sigma_{j'}, \rho, \sigma_{j''}, \dots, \sigma_r),$$

where $j' \in \{j-1, j-2\}$ and $j'' \in \{j, j+1\}$ are chosen so that $j' = j-2$ when $\sigma_{j-1}^{-1}(v_j) = v_{j-1}$ and $j'' = j+1$ when $\sigma_j(v_j) = v_{j+1}$. We have obtained W by "chopping" V from C, and it is easy to see that C decomposes into V and W. We also see that the assumptions of Proposition 6.6 are fulfilled, and so C is planar, a contradiction.

We can thus assume that all vertices of C are obtuse. Instead of "chopping" a triangle of C, we shall "drag" it out of the domain. We shall describe the triangle V and the multigon W, the rest then follows easily from Proposition 6.6, like in the previous case.

Choose j, $1 \leq j \leq r$, and consider the cycles ρ and ρ' with $\rho\rho'\sigma_j(v_j) = v_j$. Let V be the triangle with profile (σ_j, ρ', ρ). We have $\rho' \neq \sigma_{j+1}$ and $\rho \neq \sigma_{j-1}$ since $\operatorname{sgn}(\sigma_j, \rho') = 1 = \operatorname{sgn}(\rho, \sigma_j)$. From the three edges of V only the edge $(v_j, \sigma_j(v_j))$ is an edge of C. Hence $\rho, \rho' \in \operatorname{Cycl}(C)$, and one can define W by profile

$$(\sigma_1, \dots, \sigma_{j-1}, \rho, \rho', \sigma_{j'}, \dots, \sigma_r),$$

where $j' \in \{j, j+1\}$, with $j' = j+1$ if and only if $\sigma_j(v_j) = v_{j+1}$. □

8. Flip-flops

Let S be the combinatorial surface determined by (τ_1, τ_2, τ_3), where (τ_1, τ_2, τ_3) satisfies properties (P1) and (P2). Note first that the triple $(\tau_2^{-1}, \tau_1^{-1}, \tau_3^{-1})$ satisfies the properties as well. Denote by $\overline{A_i}$ the set of cycles of τ_i^{-1}, $1 \le i \le 3$. Clearly $\rho \in \overline{A_i} \Leftrightarrow \rho^{-1} \in A_i$.

The arrows in $\overline{S}$, the combinatorial surface of $(\tau_2^{-1}, \tau_1^{-1}, \tau_3^{-1})$, are exactly opposite to the arrows in S. Hence $x_1 \ \ldots \ x_k$ is an oriented polygonal curve in S whenever $x_k \ \ldots \ x_1$ is an oriented polygonal curve in $\overline{S}$. Thus for every multigon C in S of profile $(\sigma_1, \ldots, \sigma_r)$ we get a multigon $\overline{C}$ in $\overline{S}$ of profile $(\sigma_k^{-1}, \ldots, \sigma_1^{-1})$. Since $\tau_2^{-1}\tau_1^{-1}\tau_3^{-1}$ is the identity mapping, we see that $\overline{\mathrm{sgn}}(\rho^{-1}, \sigma^{-1}) = -\operatorname{sgn}(\rho, \sigma)$, for all cycles $\rho, \sigma \in \bigcup A_i$, $1 \le i \le 3$. Therefore $\overline{\mathrm{sgn}}(\sigma_{j+1}^{-1}, \sigma_j^{-1}) = \operatorname{sgn}(\sigma_j, \sigma_{j+1})$, and we see that $\overline{\mathrm{sgn}}_{\overline{C}}(v) = \operatorname{sgn}_C(v)$ for all $v \in \mathrm{Vert}(C) = \mathrm{Vert}(\overline{C})$. The inner arc of σ_j in C hence consists of exactly the same points as the inner arc of σ_j^{-1} in $\overline{C}$.

There is nothing surprising in the facts above: when (τ_1, τ_2, τ_3) is seen as a latin bi-trade, then the transition to $(\tau_2^{-1}, \tau_2^{-1}, \tau_3^{-1})$ corresponds to the exchange of rows and columns. It is sometime worth to consider the additional symmetries offered by this transition, and we shall do so in this section.

We shall be describing a surgery on the surface S that results in a combinatorial surface that can be interpreted as a latin bi-trade as well. There can be changes of the genus, and for that purposes we need to allow S to have more components of connectivity (that is formally easier than considering a surgery that would interconnect different surfaces).

Let C and D be multigons in S. A mapping $f : \mathrm{Prof}(C) \to \mathrm{Prof}(D)$ will be called an *isogony of C onto D* if it is bijective, for each $\rho \in \mathrm{Prof}(C)$ with consecutive cycle ρ' there holds $\operatorname{sgn}(\rho, \rho') = \operatorname{sgn}(f(\rho), f(\rho'))$, $f(\rho')$ is a consecutive cycle to $f(\rho)$ in the profile of D, and both ρ and $f(\rho)$ belong to the same A_i, $1 \le i \le 3$. The multigons C and D are said to be *isogonic* if there exists at least one isogony $f : \mathrm{Prof}(C) \to \mathrm{Prof}(D)$.

We shall assume that C and D have profiles $(\sigma_1, \ldots, \sigma_r)$ and $(\rho_1, \ldots, \rho_r)$, respectively, and that $f : \sigma_j \mapsto \rho_j$ is an isogony. The vertices will be denoted by $(v_1, \ldots, v_r)$ and $(w_1, \ldots, w_r)$. The isogony implies that $\operatorname{sgn}_C(v_j) = \operatorname{sgn}_D(w_j)$ for all j, $1 \le j \le r$.

To define the surgery we shall have to impose several assumptions on C and D. At the outset we shall assume that both C and D are triangle free and that $\sigma_j \ne \rho_j$ for every j, $1 \le j \le r$.

Since C and D are triangle free, all inner and outer arcs are nonempty. For every j define a_j, b_j, c_j and d_j in such a way that

$$\sigma_j(c_j), \sigma_j^2(c_j), \ldots, a_j \quad \text{and} \quad \rho_j(d_j), \rho_j^2(d_j), \ldots, b_j$$

are inner arcs of C and D, respectively, and the outer arcs are

$$\sigma_j(a_j), \sigma_j^2(a_j), \ldots, c_j \quad \text{and} \quad \rho_j(b_j), \rho_j^2(b_j), \ldots, d_j.$$

Note that an inner arc or an outer arc can have only one point, so one has to consider cases $\sigma_j(c_j) = a_j$ etc.

Suppose that σ_j and ρ_j are cycles of τ_i. Then τ_i sends the terminal points of the arcs as follows:

$$a_j \mapsto \sigma_j(a_j) \quad c_j \mapsto \sigma_j(c_j) \quad b_j \mapsto \rho_j(b_j) \quad d_j \mapsto \rho_j(d_j).$$

Define a new mapping τ_i' in such a way that it differs from τ_i exactly on a_j, c_j, b_j and d_j, for those j, $1 \le j \le r$, such that σ_j (and hence also ρ_j) is a cycle of τ_i. The mapping τ_i' maps the points a_j, b_j, c_j and d_j as follows:

$$a_j \mapsto \rho_j(b_j) \quad c_j \mapsto \rho_j(d_j) \quad b_j \mapsto \sigma_j(a_j) \quad d_j \mapsto \sigma_j(c_j).$$

The mapping τ_i' is clearly a permutation with cycles

$$(\sigma_j(c_j)\ \sigma_j^2(c_j)\ \ldots\ a_j\ \rho_j(b_j)\ \rho_j^2(b_j)\ \ldots\ d_j),$$

and

$$(\rho_j(d_j)\ \rho_j^2(d_j)\ \ldots\ b_j\ \sigma_j(a_j)\ \sigma_j^2(a_j)\ \ldots\ c_j).$$

(The cycles of τ_i which do not appear in $\mathrm{Prof}(C) \cup \mathrm{Prof}(D)$ are also cycles of τ_i', of course.)

Let us label the above two cycles σ_j' and ρ_j', respectively.

We see that there exists a bijection $\hat{f} : \bigcup A_i \to \bigcup A_i'$, where A_i and A_i' are the cycle sets of τ_i and τ_i', $1 \le i \le 3$, that can be defined by $\hat{f}(\sigma_j) = \sigma_j'$, $\hat{f}(\rho_j) = \rho_j'$, $1 \le j \le r$, and by $\hat{f}(\gamma) = \gamma$ when $\gamma \notin \mathrm{Prof}(C) \cup \mathrm{Prof}(D)$. Note that the definition of $\hat{f}$ does not depend, in fact, on the labelling of cycles that we use, but only on the isogony f.

The triple $(\tau_1', \tau_2', \tau_3')$ is said to be the *flip-flop* of (τ_1, τ_2, τ_3) *by* f. We shall also be saying that $(\tau_1', \tau_2', \tau_3')$ is the result of the *flip-flop surgery along* f.

Note that the mapping $\sigma_j^{-1} \mapsto \rho_j^{-1}$ yields an isogony $\overline{f}$ in $\overline{S}$, the surface determined by $(\tau_2^{-1}, \tau_1^{-1}, \tau_3^{-1})$. The inner arc of σ_j^{-1} in $\overline{S}$ is given by $a_j, \ldots, \sigma_j^2(c_j), \sigma_j(c_j)$, and the outer arc by $c_j, \ldots, \sigma_j^2(a_j), \sigma_j(a_j)$. We thus have $\overline{a_j} = \sigma_j(c_j)$ and $\overline{c_j} = \sigma_j(a_j)$. Similarly, $\overline{b_j} = \rho_j(d_j)$ and $\overline{d_j} = \rho_j(b_j)$. Assume $\sigma_j, \rho_j \in A_i$, $1 \le i \le 3$. The equalities $\tau_i'(a_j) = \rho_j(b_j)$, $\tau_i'(c_j) = \rho_j(d_j)$, $\tau_i'(b_j) = \sigma_j(a_j)$ and $\tau_i'(d_j) = \sigma(c_j)$ can be thus expressed also in the form

$$(\tau_i')^{-1}(\overline{d_j}) = a_j = \sigma_j^{-1}(\overline{c_j}), \quad (\tau_i')^{-1}(\overline{b_j}) = c_j = \sigma_j^{-1}(\overline{a_j}),$$
$$(\tau_i')^{-1}(\overline{c_j}) = b_j = \rho_j^{-1}(\overline{d_j}) \quad \text{and} \quad (\tau_i')^{-1}(\overline{a_j}) = d_j = \rho_j^{-1}(\overline{b_j}).$$

We see that the flip-flop surgery along $\bar{f}$ yields $((\tau_2')^{-1}, (\tau_1')^{-1}, (\tau_3')^{-1})$.

We have already observed that τ_i', $1 \le i \le 3$, are fixed point free permutations. Our goal now is to prove that $\tau_1'\tau_2'\tau_3'$ is the identity mapping.

Suppose that $\tau_1'\tau_2'\tau_3'(x) \ne x$ and $\tau_3'(x) = \tau_3(x)$. Setting $x' = \tau_3(x)$ yields $\tau_3'\tau_2'\tau_1'(x') \ne x'$. This can be repeated until we find i, $1 \le i \le 3$, and a point y such that $\tau_{i+1}'\tau_{i-1}'\tau_i'(y) \ne y$, and $\tau_i'(y) \ne \tau_i(y)$. We thus need to prove $\tau_{i+1}'\tau_{i-1}'\tau_i'(x) = x$ only for $x \in \{a_j, b_j, c_j, d_j\}$, $1 \le j \le r$.

When multigons C and D are exchanged, then the definition of τ_i', $1 \le i \le 3$, does not change. Transition to C^{op} and D^{op} does not influence it as well. The inner and outer arcs are swapped in the latter transition, and so only the case $x = a_j$ needs to be considered. Let σ_j be a cycle of τ_i, $1 \le i \le 3$.

Suppose first $\mathrm{sgn}(\sigma_j, \sigma_{j+1}) = 1$. Then $v_{j+1} = \sigma_j(a_j) = c_{j+1}$ is an acute vertex of C. Hence $\tau_i'(a_j) = \rho_j(b_j) = \sigma_j'(a_j) = w_{j+1}$ is equal to d_{j+1}, and we obtain $\tau_{i-1}'\tau_i'(a_j) = \tau_{i-1}'(d_{j+1}) = \sigma_{j+1}'(d_{j+1}) = \sigma_{j+1}(c_{j+1}) = \tau_{i-1}(c_{j+1}) = \tau_{i-1}(v_{j+1}) = \tau_{i-1}\tau_i(a_j)$. Therefore we need to show that τ_{i+1} agrees with τ_{i+1}' on $\sigma_{j+1}(c_{j+1})$.

Let γ be the cycle of τ_{i+1} that moves the point $x = \sigma_{j+1}(c_{j+1})$. If $\gamma \notin \mathrm{Prof}(C) \cup \mathrm{Prof}(D)$, then τ_{i+1} and τ_{i+1}' agree on x. We can hence assume that either (1)

$\gamma = \sigma_k$ and $x \in \{a_k, c_k\}$, or (2) $\gamma = \rho_k$ and $x \in \{b_k, d_k\}$. In the former case $\{x, \gamma(x)\} \cap \{v_k, v_{k+1}\} \neq \emptyset$, and in the latter case $\{x, \gamma(x)\} \cap \{w_k, w_{k+1}\} \neq \emptyset$.

Assume $\gamma(x) = v_{k+1}$ or $\gamma(x) = w_{k+1}$. Then $x = a_k$ or $x = b_k$, and $\gamma(x)$ is an acute vertex of C or D, respectively. Since $\operatorname{sgn}(\gamma, \sigma_j) = 1$, we get $\gamma = \sigma_{j-1}$ or $\sigma_j = \rho_{k+1}$. However $\gamma = \sigma_{j-1}$ contradicts the fact that C is triangle free, and $\sigma_j = \rho_{k+1}$ contradicts $\operatorname{Prof}(C) \cap \operatorname{Prof}(D) = \emptyset$.

Assume $\gamma(x) = v_k$ or $\gamma(x) = w_k$. Then $x = c_k$ or $x = d_k$, and $\gamma(x)$ is an obtuse vertex of C or D, respectively. Since $\operatorname{sgn}(\sigma_j, \gamma) = -1$, we get $\sigma_j = \sigma_{k-1}$ or $\sigma_j = \rho_{k-1}$. None of these equalities can hold.

Assume $x = v_{k+1}$ or $x = w_{k+1}$. Then $x = a_k$ or $x = b_k$, and x is an obtuse vertex of C or D, respectively. If $x = v_{k+1}$, then $\sigma_{j+1} = \sigma_{k+1}$, as $\operatorname{sgn}(\gamma, \sigma_{j+1}) = -1$, a contradiction to $\sigma_j \neq \gamma = \sigma_k$. If $x = w_{k+1}$, then $\sigma_{j+1} = \rho_{k+1}$, a contradiction to $\operatorname{Prof}(C) \cap \operatorname{Prof}(D) = \emptyset$.

Assume $x = v_k$ or $x = w_k$. Then $x = c_k$ or $x = d_k$, and x is an acute vertex of C or D, respectively. If $x = v_k$, then $\sigma_{k-1} = \sigma_{j+1}$ and $\gamma = \sigma_{j+2}$, which cannot be since C is a triangle free multigon. If $x = w_k$, then $\gamma_{k-1} = \sigma_{j+1}$, a contradiction again.

It remains to consider the case $\operatorname{sgn}(\sigma_j, \sigma_{j+1}) = -1$. The cycles ρ_{j+1} and σ_{j+1} belong to τ_{i+1}, $a_j = v_{j+1} = \sigma_{j+1}(c_{j+1})$ and $b_j = w_{j+1} = \rho_{j+1}(d_{j+1})$. Recall that $\rho_j(b_j)$ can be expressed as $\overline{d_j}$. We have $\rho_j^{-1}(\overline{d_j}) = b_j$ and $\overline{\operatorname{sgn}}_{\overline{D}}(b_j) = -1$, as both v_{j+1} and w_{j+1} are obtuse vertices of C and D, respectively. By turning from C and D to $\overline{C}$ and $\overline{D}$, one gets $\rho_j(b_j)$ in the role of d_j. Hence by turning from C and D to $\overline{D}^{op}$ and $\overline{C}^{op}$, one gets $\rho_j(b_j)$ in the role of a_j. Since $\rho_j^{-1}(\rho_j(b_j)) = b_j$ and $\overline{\operatorname{sgn}}_{\overline{D}^{op}}(b_j) = 1$ we can apply the preceding part of the proof to $\rho_j(b_j)$. We obtain

$$(\tau_{i-1}^{-1})'\,(\tau_{i+1}^{-1})'\,(\tau_i^{-1})'\,(\rho_j(b_j)) = \rho_j(b_j),$$

which is the same as

$$(\tau'_{i-1})^{-1}\,(\tau'_{i+1})^{-1}\,(\tau'_i)^{-1}\,(\rho_j(b_j)) = \rho_j(b_j),$$

and so $\rho_j(b_j) = \tau'_i\tau'_{i+1}\tau'_{i-1}\,(\rho_j(b_j))$. Now, $(\tau'_i)^{-1}(\rho_j(b_j)) = a_j$, and hence $a_j = \tau'_{i+1}\tau'_{i-1}\tau'_i(a_j)$, as required.

Theorem 8.1. *Let S be a combinatorial surface determined by permutations τ_i, $1 \le i \le 3$, such that (τ_1, τ_2, τ_3) fulfils conditions (P1) and (P2). Suppose that C and D are triangle free multigons in this surface.*

Let f be an isogony of C onto D, and let $(\tau'_1, \tau'_2, \tau'_3)$ be the result of the flip-flop surgery along f. Put $P = \operatorname{Prof}(C) \cup \operatorname{Prof}(D)$, and assume $\operatorname{Prof}(C) \cap \operatorname{Prof}(D) = \emptyset$. For $\gamma \in \operatorname{Prof}(C)$ put $g(\gamma) = f(\gamma)$, and denote by $I_1(\gamma)$ and $I_{-1}(\gamma)$ the inner and outer arcs of γ in C. For $\gamma \in \operatorname{Prof}(D)$ put $g(\gamma) = f^{-1}(\gamma)$, and denote by $I_1(\gamma)$ and $I_{-1}(\gamma)$ the inner and outer arcs of γ in D, respectively.

The triple $(\tau'_1, \tau'_2, \tau'_3)$ fulfils conditions (P1) and (P2) if and only if the following holds:

(H1) *Let $\gamma, \gamma' \in P$ satisfy $\gamma' \neq \gamma$ and $\gamma' \neq g(\gamma)$. Then for any $\varepsilon, \eta \in \{-1, 1\}$ at least one of the sets*

$$I_\varepsilon(\gamma) \cap I_\eta(\gamma') \quad \textit{and} \quad I_\varepsilon(g(\gamma)) \cap I_\eta(g(\gamma'))$$

is empty.

(H2) *Let* $\gamma, \gamma' \in P$ *satisfy* $\gamma' \neq \gamma$ *and* $\gamma' \neq g(\gamma)$. *Then for both* $\varepsilon \in \{-1, 1\}$ *at least one of the sets*

$$I_1(\gamma) \cap I_\varepsilon(\gamma') \quad \textit{and} \quad I_{-1}(g(\gamma)) \cap I_\varepsilon(\gamma')$$

is empty.

(H3) *If* $\gamma \in P$ *and* μ *is a cycle,* $\mu \notin P$*, then* μ *never connects* $I_1(\gamma)$ *and* $I_{-1}(g(\gamma))$.

Proof. Let μ be a cycle of τ_i', and ν a cycle of $\tau_{i'}'$, where $i, i' \in \{1, 2, 3\}$ and $i \neq i'$. We have to show that μ and ν meet at most in one point.

If μ is a cycle of τ_i and ν of $\tau_{i'}$, then they certainly meet at most in one point. We can hence assume that ν is not a cycle of τ_i. By the construction of τ_i' there exists $\gamma \in P$ such that the set of points moved by ν equals $I_1(\gamma) \cup I_{-1}(g(\gamma))$. By condition (H3), ν meets μ in at most one point when μ is cycle of τ_i.

Let $I_1(\gamma') \cup I_{-1}(g(\gamma'))$ be the set of points moved by μ, where $\gamma' \in P$ and $\gamma' \neq \gamma$. Since both γ and γ' move the points of $J_1 = I_1(\gamma) \cap I_1(\gamma')$, this set contains at most one point. Similarly we see that each of the sets $J_2 = I_1(\gamma) \cap I_{-1}(g(\gamma'))$, $J_3 = I_{-1}(g(\gamma)) \cap I_1(\gamma')$ and $J_4 = I_{-1}(g(\gamma)) \cap I_{-1}(g(\gamma'))$ has at most one element. The set $J = J_1 \cup J_2 \cup J_3 \cup J_4$ coincides with the set of elements that are moved both by μ and ν.

Condition (H2) expresses the fact that none of the pairs (J_1, J_2), (J_2, J_4), (J_3, J_4) and (J_1, J_3) consists of two nonempty sets. We also see that the condition (H2) is necessary. The set J is hence equal to $J_1 \cup J_4$ or to $J_2 \cup J_3$. By condition (H1) none of the pairs (J_1, J_4) and (J_2, J_3) can consist of two nonempty sets, and hence $|J| \leq 1$. Note that condition (H1) does not change if one sets $\varepsilon = 1$, and so we see easily that a violation of the condition yields a two element intersection. □

We shall continue our discussion of the flip-flop surgery under the assumption that (τ_1, τ_2, τ_3) satisfies conditions (H1)–(H3). Hence $(\tau_1', \tau_2', \tau_3')$ yields a combinatorial surface and this surface will be denoted by S'.

Let γ and γ' be two distinct elements of $\mathrm{Prof}(C) \cup \mathrm{Prof}(D)$. The preceding proof shows that the sets $I_1(\gamma) \cup I_{-1}(g(\gamma))$ and $I_1(\gamma') \cup I_{-1}(g(\gamma'))$ have at most one common point. However, that also means that these sets are distinct, since each of τ_i', $1 \leq i \leq 3$, is a fixed point free permutations. Since these sets express the sets of points moved by cycles σ_j' and ρ_j', $1 \leq j \leq r$, we see that $\sigma_j' \neq \rho_k'$ for all $j, k \in \{1, \ldots, r\}$, and that $\sigma_j' \neq \sigma_k'$ and $\rho_j' \neq \rho_k'$ if $1 \leq j < k \leq r$.

Fix now j, $1 \leq j \leq r$. If v_{j+1} is an obtuse vertex of C, then it is a common point of σ_j' and σ_{j+1}', while the common point of ρ_j' and ρ_{j+1}' is w_{j+1}. If v_{j+1} is an acute vertex of C, then it is a common point of ρ_j' and ρ_{j+1}', while the common point of σ_j' and σ_{j+1}' is v_{j+1}.

From $\mathrm{Prof}(C) \cap \mathrm{Prof}(D) = \emptyset$ we get $\mathrm{Vert}(C) \cap \mathrm{Vert}(D) = \emptyset$, and so we see that each of the intersections of σ_j' and σ_{j+1}' is different from an intersection of σ_k' and σ_{k+1}', whenever $1 \leq j < k \leq r$. The same is true for ρ_j' and ρ_k', and so both $(\sigma_1', \ldots, \sigma_r')$ and $(\rho_1', \ldots, \rho_r')$ are profiles of multigons in S'. Clearly $\mathrm{sgn}(\sigma_j, \sigma_{j+1}) = \mathrm{sgn}(\sigma_j', \sigma_{j+1}')$ and $\mathrm{sgn}(\rho_j, \rho_{j+1}) = \mathrm{sgn}(\rho_j', \rho_{j+1}')$.

The notion of isogony was defined for two multigons in the same surface. The definition extends in a clear way to a pair of multigons in two different surfaces.

Proposition 8.2. *Let* S *be a surface determined by* (τ_1, τ_2, τ_3)*, and let* C *and* D *be triangle free multigons in* S *such that* $\mathrm{Prof}(C) \cap \mathrm{Prof}(D) = \emptyset$. *Let* f *be an isogony of* C *onto* D *that fulfils conditions (H1)–(H3). In the surface* S' *of* $(\tau_1', \tau_2', \tau_3')$ *there* *exist (uniquely determined) multigons C' and D' such that $\hat{f}$ yields isogonies of C onto C', and D onto D'. These multigons are triangle free,* $\mathrm{Prof}(C') \cap \mathrm{Prof}(D') = \emptyset$ *and $\hat{f}(\sigma) \mapsto \hat{f}(f(\sigma))$, $\sigma \in \mathrm{Prof}(C)$, defines an isogony f' of C' onto D'.*

For $\sigma \in \mathrm{Prof}(C)$ put $\rho = f(\sigma)$ and denote by $I(\sigma)$, $O(\sigma)$, $I(\rho)$ and $O(\rho)$ the inner and outer arcs of σ in C and ρ in D, respectively. Similarly, let $I'(\sigma)$, $O'(\sigma)$, $I'(\rho)$ and $O'(\rho)$ be the inner and outer arcs of $\hat{f}(\sigma)$ in C' and $\hat{f}(\rho)$ in D'. Then $I'(\sigma) = I(\sigma)$, $O'(\sigma) = O(\rho)$, $I'(\rho) = I(\rho)$ and $O'(\rho) = O(\sigma)$.

Proof. Nearly all statements of the proposition have been proved above. The part about arcs follows from the definition of τ_i', $1 \le i \le 3$, if we take into account that C, C', D and D' are isogonic. The arcs in C' and D' are thus nonempty, and hence both C' and D' are triangle free. □

Let C' and D', and the isogony f' of C' onto D' be the same as in Proposition 8.2.

Proposition 8.3. *The isogony f' fulfils conditions (H1)–(H3) with respect to C' and D'. The flip-flop surgery of $(\tau_1', \tau_2', \tau_3')$ along f' yields (τ_1, τ_2, τ_3).*

Proof. The relationships of arcs described in Proposition 8.2 imply that the flip-flop surgery along f' really yields (τ_1, τ_2, τ_3). The conditions (H1)–(H3) have to be fulfilled since in Theorem 8.1 we have proved that these conditions are necessary to obtain a triple that satisfies (P1)–(P2). □

Theorem 8.4. *Let S be a combinatorial surface determined by permutations τ_i, $1 \le i \le 3$, such that (τ_1, τ_2, τ_3) fulfils conditions (P1) and (P2). Suppose that C and D are triangle free simple multigons in this surface.*

Let f be an isogony of C onto D, and let $(\tau_1', \tau_2', \tau_3')$ be the result of the flip-flop surgery along f. Suppose that no cycle $\sigma \in \mathrm{Prof}(C)$ meets a cycle $\rho \in \mathrm{Prof}(D)$, and suppose that there is no cycle μ that connects the inner arc of σ in C with the outer arc of ρ in D. Then $(\tau_1', \tau_2', \tau_3')$ fulfils (P1)–(P2), the multigons C' and D' are simple, and no $\sigma' \in \mathrm{Prof}(C')$ meets any $\rho' \in \mathrm{Prof}(D')$.

Proof. Note that condition (H2) is true trivially since cycles of C and D never meet. Condition (H1) is clearly satisfied when $\gamma \in \mathrm{Prof}(C)$ and $\gamma' \in \mathrm{Prof}(D)$, by the same reason. We can hence assume $\gamma, \gamma' \in \mathrm{Prof}(C)$. This can be further simplified to $\gamma = \sigma_j$ and $\gamma' = \sigma_{j+1}$, as C and D are assumed to be simple. However, for this choice of γ and γ' the condition (H2) always holds, since f is an isogony. Condition (H3) is assumed explicitly in the text of theorem.

The triple $(\tau_1', \tau_2', \tau_3')$ thus satisfies (P1)–(P2), and we see that our assumptions imply that both C' and D' are simple, and that the cycles of $\mathrm{Prof}(C')$ do not meet the cycles of $\mathrm{Prof}(D')$. □

9. Flip-flops and separating multigons

Throughout this section we shall assume that (τ_1, τ_2, τ_3) is a triple of permutations that satisfies (P1)–(P2), that S is the combinatorial surface derived from (τ_1, τ_2, τ_3), that C and D are triangle free multigons in S with $\mathrm{Prof}(C) \cap \mathrm{Prof}(D) = \emptyset$, and that f is an isogony of C onto D that satisfies (H1)–(H3). Furthermore, by $(\tau_1', \tau_2', \tau_3')$ we shall denote the result of flip-flop surgery along f, by S' the combinatorial surface derived from $(\tau_1', \tau_2', \tau_3')$, and by C' and D' the multigons such that $\hat{f}$ induces an isogony of C onto C', and of D onto D'.

When convenient we shall also assume that $(\sigma_1, \ldots, \sigma_r)$ and $(\rho_1, \ldots, \rho_r)$ are the profiles of C and D, and that $(v_1, \ldots, v_r)$ and $(w_1, \ldots, w_r)$ are the vertices, respectively.

Note that neither S nor S' needs to be a connected surface.

Proposition 9.1. *Suppose that the multigon C is separating and that $\mathrm{Prof}(D) \cap \mathrm{Cycl}(C) = \emptyset$. Then C' is separating as well, $\mathrm{Cycl}(C) = \mathrm{Cycl}(C')$ and $\mathrm{Pnt}(C) = \mathrm{Pnt}(C')$. If C is in addition planar, then C' is planar as well.*

Proof. We shall start by showing that

$$\mathrm{In}(C') = \{(\hat{f}(\rho), \hat{f}(\sigma));\ (\rho, \sigma) \in \mathrm{In}(C)\}.$$

Recall that $(\rho, \sigma) \in \mathrm{In}(C)$ if $\rho \neq \sigma$ meets $\sigma = \sigma_j$ in the inner arc of σ, $\rho \neq \sigma_{j\pm1}$. If $(\rho, \sigma) \in \mathrm{In}(C)$, then $\rho \in \mathrm{Cycl}(C) \cup \mathrm{Prof}(C)$, by rule (N1). No $\rho \in \mathrm{Prof}(D)$ can thus meet $\sigma \in \mathrm{Prof}(C)$ in the inner arc of σ since we assume $\mathrm{Prof}(D) \cap \mathrm{Cycl}(C) = \emptyset$. If $(\sigma_j, \sigma_k) \in \mathrm{In}(C)$, then σ_j and σ_k meet in the inner arcs, by Lemma 3.2. We see that $(\sigma_j, \sigma_k) \in \mathrm{In}(C) \Leftrightarrow (\sigma_j', \sigma_k') \in \mathrm{In}(C')$ and that (ρ_j', σ_k') is never in $\mathrm{In}(C')$, for all $j, k \in \{1, 2, \ldots, r\}$. A cycle $\mu \notin \mathrm{Prof}(C) \cup \mathrm{Prof}(D)$ meets σ_j in its inner arc if and only if it meets σ_j' in its inner arc, wince the arcs coincide, and that finishes the verification of the formula for $\mathrm{In}(C')$.

Put now $\mathcal{N} = \hat{f}(\mathrm{Cycl}(C))$ and note that our expression of $\mathrm{In}(C')$ implies that $\mathcal{N}$ is closed by rule (N1), with respect to C'. We shall prove that $\mathcal{N}$ is closed with respect to (N2) as well. If $\rho \in \mathrm{Cycl}(C)$ and $\sigma \in \bigcup A_i$ are such that $\hat{f}(\rho)$ meets $\hat{f}(\sigma)$, and $(\hat{f}(\rho), \hat{f}(\sigma)) \notin \mathrm{In}(C')$, then certainly $(\rho, \sigma) \notin \mathrm{In}(C)$. We do not have $\rho \in \mathrm{Prof}(C) \cup \mathrm{Prof}(D)$, and so $\hat{f}(\rho) = \rho$. If ρ meets σ, then $\sigma \in \mathrm{Cycl}(C)$, by rule (N2), and $\hat{f}(\sigma) \in \mathcal{N}$. It remains to show that ρ really meets σ. This is clear when $\hat{f}(\sigma) = \sigma$. There cannot be $\sigma \in \mathrm{Prof}(D)$, since ρ belongs to $\mathrm{Cycl}(C)$, and hence it neither meets some σ_k in its outer arc, nor it meets any ρ_k. If $\sigma = \sigma_j$, then ρ meets σ_j'. If the intersection is in the outer arc of σ_j', then ρ meets $\rho_j \notin \mathrm{Prof}(C)$, and so $\rho_j \in \mathrm{Cycl}(C)$, a contradiction. Hence ρ meets $\sigma = \sigma_j$ in every case. We have proved that $\mathcal{N}$ is closed under both (N1) and (N2). Thus $\hat{f}(\mathrm{Cycl}(C)) \supseteq \mathrm{Cycl}(C')$, by Proposition 3.6.

Since C' is separating we can use the reverse construction of Proposition 8.3 to show $\hat{f}^{-1}(\mathrm{Cycl}(C')) \supseteq \mathrm{Cycl}(C)$. Both inclusions together yield the equality $\mathrm{Cycl}(C') = \hat{f}(\mathrm{Cycl}(C))$. Since $\mathrm{Cycl}(C)$ contains no element of $\mathrm{Prof}(C)$ or $\mathrm{Prof}(D)$, this means $\mathrm{Cycl}(C') = \mathrm{Cycl}(C)$.

Let P and P' be sets of points that occur in domains of C and C', respectively, and let V and V' be the sets of acute vertices in C and C', respectively. Then $P \setminus V$ consists of points moved by $\rho \in \mathrm{Cycl}(C)$ and of points on the inner arcs of σ_j, $1 \leq j \leq r$. Hence $P \setminus V = P \setminus V'$, which implies $\mathrm{Pnt}(C) = \mathrm{Pnt}(C')$. Multigons C and C' are isogonic, and hence

$$|\,\mathrm{Cycl}(C)| - |\,\mathrm{Pnt}(C)| + \mathrm{sgn}(C) = |\,\mathrm{Cycl}(C')| - |\,\mathrm{Pnt}(C')| + \mathrm{sgn}(C').$$

□

According to Proposition 8.2 the multigon C' is determined by isogony obtained from $\hat{f}$ by restricting the mapping to $\mathrm{Prof}(C)$.

Put now $f_1 = f^{-1}$ and consider f_1 as an isogony of $C_1 = D^{op}$ onto $D_1 = C^{op}$. We know that the result of the flip-flop surgery along f_1 is $(\tau_1', \tau_2', \tau_3')$ again.

Furthermore, $\widehat{f_1}(\rho_j) = \sigma'_j$, as ρ_j and σ'_j agree in the inner arcs of C_1 and D_1 (since they agree in the outer arcs of C and D). By restricting $\widehat{f_1}$ to C_1 we see that C_1, which is of profile $(\rho_r, \dots, \rho_1)$, determines C'_1 as a polygon of profile $(\sigma'_r, \dots, \sigma'_1)$. Thus $C'_1 = (C')^{op}$ and similarly $D'_1 = (D')^{op}$. Proposition 9.1 can be thus alternatively expressed as

Proposition 9.2. *Suppose that the multigon D is separating and that* $\mathrm{Prof}(C) \cap \mathrm{Cycl}(D^{op}) = \emptyset$. *Then C' is separating as well,* $\mathrm{Cycl}((C')^{op}) = \mathrm{Cycl}(D^{op})$ *and* $\mathrm{Pnt}(D^{op}) = \mathrm{Pnt}((C')^{op})$. *If D^{op} is in addition planar, then $(C')^{op}$ is planar as well.*

We shall now bind together the propositions above.

Proposition 9.3. *Suppose that both multigons C and D are separating and that*

$$\mathrm{Prof}(D) \cap \mathrm{Cycl}(C) = \emptyset = \mathrm{Prof}(C) \cap \mathrm{Cycl}(D^{op}).$$

Then C' is separating, $\mathrm{Cycl}(C') = \mathrm{Cycl}(C)$, $\mathrm{Cycl}((C')^{op}) = \mathrm{Cycl}(D^{op})$, *and the connectivity component of C' does not contain the cyclic faces induced by the profile of D'. If both C and D^{op} are planar multigons, then the connectivity component of C' is spherical.*

Proof. The domains of C' and $(C')^{op}$ have their structure determined by Propositions 9.1 and 9.2. The cyclic faces of their connecting component are exactly those that are induced by an element of $\mathrm{Cycl}(C') \cup \mathrm{Cycl}((C')^{op}) \cup \mathrm{Prof}(C')$, as follows from Proposition 2.5. None of them is an element of $\mathrm{Prof}(D')$, as $\mathrm{Cycl}(C') = \mathrm{Cycl}(C)$ and $\mathrm{Cycl}((C')^{op}) = \mathrm{Cycl}(D^{op})$. If C and D^{op} are planar, then C' and $(C')^{op}$ are planar as well, by Propositions 9.1 and 9.2, and the component is spherical by Proposition 2.6. □

Proposition 9.4. *Suppose that the multigon C is separating, and that* $\mathrm{Prof}(D) \cap \mathrm{Cycl}(C^{op}) = \emptyset$. *If D is not separating, then C' and D' are in the same connectivity component of S'.*

Proof. Since D is not separating, there exists a way how to obtain an element of $\mathrm{Prof}(D)$ by means of rules (N1) and (N2) (see Proposition 3.6). By considering a sequence of the shortest length that produces such an element, we obtain a sequence $\gamma_0, \dots, \gamma_t$ of cycles, $t \geq 1$, such that $\gamma_0 \in \mathrm{Prof}(D)$, $\gamma_t \in \mathrm{Prof}(D)$, $\gamma_i \notin \mathrm{Prof}(D)$ if $1 \leq i < t$, and such that γ_1 is obtained from γ_0 by means of rule (N1), while γ_i, $t \geq i > 1$, are obtained from γ_{i-1} by means of rule (N2).

Let us have $\gamma_0 = \rho_j$ and $\gamma_t = \rho_k$. If $t = 1$, then the inner arc of ρ_j meets the outer arc of ρ_k, by Lemma 3.3, and so ρ'_j meets σ'_k. Hence $t > 1$ can be assumed.

Suppose first that none of elements $\gamma_1, \dots, \gamma_{t-1}$ belongs to the profile of C. Then each of $\gamma_1, \dots, \gamma_{t-1}$ is a cycle of τ'_1 or τ'_2 or τ'_3. Note that γ_1 meets $\gamma_0 = \rho_j$ in the inner arc of ρ_j, by rule (N1), while γ_{t-1} meets $\gamma_t = \rho_k$ in the outer arc of ρ_k, by rule (N2). Therefore ρ'_j meets γ_1 and γ_{t-1} meets σ'_k. We see that C' and D' are in the same component.

Let there exist t', $1 \leq t' < t$, with $\gamma_{t'} \in \mathrm{Prof}(C)$, say $\gamma_{t'} = \sigma_s$. Choose t' to be the least possible. If $\gamma_{t'-1}$ meets σ_s in the inner arc of σ_s, then we obtain a path from ρ'_j to σ'_s. If $\rho_{t'-1}$ meets σ_s in the outer arc of σ_s, then, by working backwards along rules (N1) and (N2), we get $\gamma_0 = \rho_j \in \mathrm{Cycl}(C^{op})$, a contradiction to our assumptions. □

We conclude this section by an easy lemma which does not seem to require a proof.

Lemma 9.5. *Suppose that each connectivity component of S contains C or D. Then each connectivity component of S' contains C' or D'.*

10. Triangular construction

Let (τ_1, τ_2, τ_3) be a triple of permutations that satisfies (P1)–(P2), and suppose that $\mathbf{x} = x_3\, x_2\, x_1$ is a triangle in the combinatorial surface S determined by (τ_1, τ_2, τ_3). We can assume that τ_i induces the edge (x_{i-1}, x_{i+1}). Thus $\tau_1(x_3) = x_2$, $\tau_3(x_2) = x_1$ and $\tau_2(x_1) = x_3$. We shall define permutation $\tau_i^{\mathbf{x}}$, $1 \le i \le 3$, on the set $Y = X \cup \{y_1, y_2, y_3\}$, assuming that none of y_i belongs to X.

Set $\tau_i^{\mathbf{x}}(y_{i+1}) = y_{i-1}$, $\tau_i^{\mathbf{x}}(y_{i-1}) = y_{i+1}$, $\tau_i^{\mathbf{x}}(x_{i-1}) = y_i$ and $\tau_i^{\mathbf{x}}(y_i) = x_{i+1}$, and let $\tau_i^{\mathbf{x}}$ coincide with τ_i in all other cases, for all i, $1 \le i \le 3$.

The cyclic structure of $\tau_i^{\mathbf{x}}$, for a given i, $1 \le i \le 3$, differs from that of τ_i only slightly. There is a new cycle $(y_{i-1}\ y_{i+1})$ of length two, and the cycle $\gamma_i = (\dots\ x_{i-1}\ x_{i+1}\ \dots)$ is changed into $\gamma_i^{\mathbf{x}} = (\dots\ x_{i-1}\ y_i\ x_{i+1}\ \dots)$.

To see that $\tau_1^{\mathbf{x}}\tau_2^{\mathbf{x}}\tau_3^{\mathbf{x}}$ is the identity mapping it suffices to prove $\tau_{i+1}^{\mathbf{x}}\tau_{i-1}^{\mathbf{x}}\tau_i^{\mathbf{x}}(u) = u$ for all $u \in Y$ where $\tau_i^{\mathbf{x}}(u) \ne \tau_i(u)$ or where $\tau_i(u)$ is not defined. Now,

$$\tau_{i+1}^{\mathbf{x}}\tau_{i-1}^{\mathbf{x}}\tau_i^{\mathbf{x}}(x_{i-1}) = \tau_{i+1}^{\mathbf{x}}(y_{i+1}) = x_{i-1},\ \ \tau_{i+1}^{\mathbf{x}}\tau_{i-1}^{\mathbf{x}}\tau_i^{\mathbf{x}}(y_i) = \tau_{i+1}^{\mathbf{x}}(y_{i-1}) = y_i,$$
$$\tau_{i+1}^{\mathbf{x}}\tau_{i-1}^{\mathbf{x}}\tau_i^{\mathbf{x}}(y_{i-1}) = \tau_{i+1}^{\mathbf{x}}(y_i) = y_{i-1}, \quad \text{and} \quad \tau_{i+1}^{\mathbf{x}}\tau_{i-1}^{\mathbf{x}}\tau_i^{\mathbf{x}}(y_{i+1}) = \tau_{i+1}^{\mathbf{x}}(x_i) = y_{i+1}.$$

We have verified that $(\tau_1^{\mathbf{x}}, \tau_2^{\mathbf{x}}, \tau_3^{\mathbf{x}})$ fulfils conditions (P1)–(P2). Note that the number of both cycles and points increases by three. Hence the new surface, say $S^{\mathbf{x}}$, is connected of genus g if and only if S is connected of genus g. In particular, $S^{\mathbf{x}}$ is spherical when S is spherical.

Let $(v_1, \dots, v_r)$ be vertices of a multigon in S. The cyclic sequence

$$(\mathrm{sgn}_C(v_1), \dots, \mathrm{sgn}_C(v_k))$$

will be called the *signature* of C. We know that if $(\varepsilon_1, \dots, \varepsilon_r)$ is a signature of C, then $r \ge 3$ and $\sum \varepsilon_j$ is divisible by three, according to Lemma 2.1. The main purpose of this section is to show that for any such sequence $(\varepsilon_1, \dots, \varepsilon_r)$ there exist a spherical surface S and a triangle free multigon C that yields $(\varepsilon_1, \dots, \varepsilon_r)$ as its signature.

The next lemma shows that in our constructions one does not need to insist on triangle freeness.

Lemma 10.1. *Let C be a multigon of profile $(\sigma_1, \dots, \sigma_r)$. Consider j, $1 \le j \le r$, and assume that either $(\sigma_{j-1}, \sigma_j, \sigma_{j+1})$ or $(\sigma_{j+1}, \sigma_j, \sigma_{j-1})$ is a profile of a triangle, say $\mathbf{x} = x_3\, x_2\, x_1$. Then in $S^{\mathbf{x}}$ there exists a multigon $C^{\mathbf{x}}$ with profile that is obtained from the profile of C by replacing all occurences of σ_{j-1}, σ_j and σ_{j+1} by $\sigma_{j-1}^{\mathbf{x}}$, $\sigma_j^{\mathbf{x}}$ and $\sigma_{j+1}^{\mathbf{x}}$, respectively. The number of consecutive triples in the profile of $C^{\mathbf{x}}$ that induce a triangle in $S^{\mathbf{x}}$ is strictly smaller than the number of consecutive triples in the profile of C that induce a triangle in S.*

Proof. We can assume that σ_j is a cycle of τ_3 and that $\tau_i(x_{i-1}) = x_{i+1}$, $1 \le i \le 3$. If $(\sigma_{j-1}, \sigma_j, \sigma_{j+1})$ profiles a triangle, then x_3, x_2 and x_1 are consecutive vertices of C. Points y_1, y_3 and y_2 become points of the inner arcs of $\sigma_{j-1}^{\mathbf{x}}$, $\sigma_j^{\mathbf{x}}$ and $\sigma_{j+1}^{\mathbf{x}}$, respectively. The rest is easy. □

Proposition 10.2. *Let $(\varepsilon_1, \dots, \varepsilon_r)$ be a cyclic sequence of integers 1 and -1 such that 3 divides $\varepsilon_1 + \dots + \varepsilon_r$, and $r \geq 3$. Then there exists a triangle free multigon C in a spherical surface S such that $(\varepsilon_1, \dots, \varepsilon_r)$ coincides with the signature of C.*

Proof. Thanks to Lemma 10.1 we can prove the statement without taking into consideration the requirement of triangle freeness. Suppose first that the solution is already known for all alternating sequences $(1, -1, \dots, 1, -1)$.

To prove the rest we shall proceed by induction on r. Note that the soluton for $(\varepsilon_1, \dots, \varepsilon_r)$ yields a solution for $(-\varepsilon_r, \dots, -\varepsilon_1)$, by changing to C^{op}. The cases $(1, 1, 1)$ and $(-1, -1, -1)$ are clear, as one can take any triple in a spherical surface S. (Of course, for that one has to know that at least one spherical surface exists. The number of examples is abundant, the smallest one is obtained from two distinct latin squares of order two.)

Assume $r \geq 4$. We are assuming that $(\varepsilon_1, \dots, \varepsilon_r)$ is not alternating, and so we can assume that $\varepsilon_1 = \varepsilon_2 = -1$. Let C and S be a solution to $(1, \varepsilon_3, \dots, \varepsilon_r)$. Then there exist edges (x_3, x_2) and (x_2, x_1) of C such that $x_2 = v_1$ is the acute vertex. However that means that $\mathbf{x} = x_3\, x_2\, x_1$ is a triangle. Let us derive $C^{\mathbf{x}}$ from C in such a way that the consecutive edges (x_3, x_2) and (x_2, x_1) are replaced by (x_3, y_1), (y_1, y_3) and (y_3, x_1). The signature of $C^{\mathbf{x}}$ is clearly equal to $(-1, -1, \varepsilon_2, \dots, \varepsilon_r)$.

Note that $S^{\mathbf{x}}$ always contains a multigon with vertex sequence (x_3, y_1, y_3, y_2), which yields the shortest alternating signature $(1, -1, 1, -1)$. To show the existence of longer alternating sequences we need to expand one 1 to $1, -1, 1$. Proceeding like above we replace consecutive edges (x_3, x_2) and (x_2, x_1) by (x_3, y_1), (y_1, y_2), (y_2, y_3) and (x_3, x_1). □

11. Main Theorem

We are now ready to prove that any surface of positive genus that is derived from a separating latin bi-trade can be obtained by a flip-flop surgery from the surfaces of lower genus.

Lemma 11.1. *Let C and D be triangle free multigons in the surface S which is derived from (τ_1, τ_2, τ_3). Suppose that C and D are in different connectivity components of S. If C is simple and D separating, then every isogony of C onto D satisfies the conditions (H1)–(H3).*

Proof. Conditions (H2) and (H3) are fulfilled trivially, as C and D are in different components. Condition (H1) simplifies to the requirement that an outer arc and an inner arc never have a common point, both in C and D. For C this is true because C is simple, while for D this follows from Lemma 3.3. □

Call, like in the introduction, a combinatorial surface S a *trading surface* if it can be obtained from a triple of permutations that satisfy condition (P1) and (P2).

Theorem 11.2. *Let S_1 be a connected trading surface of genus $g \geq 1$. Then there exist a spherical trading surface S_2 and a connected trading surface S of genus $g-1$ such that the disjoint union of S_1 and S_2 can be obtained by a flip-flop surgery in S that maps a separating multigon onto a planar one.*

Proof. The surface S_1 is not spherical, and hence it contains a nonseparating simple triangle free multigon C, by Theorem 7.4. From Proposition 10.2 we know that there exists an isogonic multigon D in some spherical trading surface S_2. Let f be an isogony of C onto D. By Lemma 11.1, the flip-flop surgery is possible. We obtain a surface S which is connected, by Proposition 9.4 and Lemma 9.5. The reverse surgery thus yields the disjoint union of S_1 and S_2, by Proposition 8.3.

Let s, s_1 and s_2 be the sizes of S, S_1 and S_2, respectively. Furthermore, let r, r_1 and r_2 be their orders, and g, g_1 and g_2 their gene. We have $g_2 = 0$, $s = s_1 + s_2$, $r = r_1 + r_2$, $2 + s_2 = r_2$, $2 + s_1 = r_1 + 2g_1$ and $2 + s = r + 2g$, by Proposition 1.5. Hence $2g = 2 + s_1 + s_2 - r_1 - r_2 = 2(g_1 - 1)$.

The multigon D' is planar, by Proposition 9.1, and the multigon C' is separating, by Proposition 9.2. □

Corollary 11.3. *All trading surfaces can be obtained from the spherical ones by repeated applications of the flip-flop surgery.*

Note that in the process of constructing a trading surface of positive genus one has to "throw away a ball" at every stage when the genus is increased. In other words, one has to forget about the component S_2 of Theorem 11.2. The efficiency of the process in some sense depends on the size of S_2, and should be considered in detail in future. Let us just note here that the construction of Section 10 often will not be the most efficient one in the sense of finding the given signature on the trading spherical surface of the least possible size.

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