NON-SINGULAR COVERS OVER MONOID RINGS

Ladislav Bican, Praha

Abstract. We shall introduce the class of strongly cancellative multiplicative monoids which contains the class of all totally ordered cancellative monoids and it is contained in the class of all cancellative monoids. If G is a strongly cancellative monoid such that $hG \subseteq Gh$ for each $h \in G$ and if R is a ring such that $aR \subseteq Ra$ for each $a \in R$, then the class of all non-singular left R-modules is a cover class if and only if the class of all non-singular left RG-modules is a cover class. These two conditions are also equivalent whenever we replace the strongly cancellative monoid G by the totally ordered cancellative monoid or by the totally ordered group.

In what follows, R stands for an associative ring with the identity element and R-mod denotes the category of all unitary left R-modules. If G is a multiplicative monoid with the unit element e, then RG will denote the monoid ring over R consisting of all elements of the form $\sum_{i=1}^{n} r_i g_i$ with $r_i \in R$, $g_i \in G$, $i = 1, \dots, n$, where the addition is given naturally and the multiplication is given by $(\sum_{i=1}^{n} r_i g_i)(\sum_{j=1}^{m} s_j h_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} r_i s_j g_i h_j$. Recall, that a monoid G is called *left cancellative* if for any three elements $h, g_1, g_2 \in G$ the equality $hg_1 = hg_2$ implies that $g_1 = g_2$. The *right cancellative* monoid is defined similarly and G is called *cancellative* if it is both left and right cancellative. The basic properties of rings and modules can be found in [1].

A class $\mathcal{G}$ of modules is called *abstract*, if it is closed under isomorphic copies. Recall that a *hereditary torsion theory* $\tau_R = (\mathcal{T}_\tau, \mathcal{F}_\tau)$, or simply $\tau = (\mathcal{T}, \mathcal{F})$, for the category R-mod consists of two abstract classes $\mathcal{T}$ and $\mathcal{F}$, the *τ-torsion class* and the *τ-torsionfree class*, respectively, such that $\operatorname{Hom}(T, F) = 0$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$, the class $\mathcal{T}$ is closed under submodules, factor-modules, extensions and arbitrary direct sums, the class $\mathcal{F}$ is closed under submodules, extensions and arbitrary direct products and for each module M there exists a short exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$. It is easy to see that every module M contains the unique largest τ-torsion submodule (isomorphic to T), which is called the *τ-torsion part of the module M* and it is usually denoted by $\tau(M)$. Associated to each hereditary torsion theory τ is the *Gabriel filter* $\mathcal{L}_\tau$ (or simply $\mathcal{L}$) of left ideals of R consisting of all the left ideals $I \le R$ such that $R/I \in \mathcal{T}$. Recall that τ is said to be of *finite type*, if $\mathcal{L}$ contains a cofinal subset of finitely generated left ideals, i.e. if every element of $\mathcal{L}$ contains a finitely generated left ideal of R lying in $\mathcal{L}$.

1991 *Mathematics Subject Classification.* 16S90, 18E40, 16D80.

Key words and phrases. Hereditary torsion theory, torsion theory of finite type, Goldie's torsion theory, non-singular module, non-singular ring, monoid ring, precover class, cover class.

The research has been partially supported by the Grant Agency of the Czech Republic, grant #GAČR 201/06/0510 and also by the institutional grant MSM 0021620839.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

For a module M, a submodule K is called *essential* in M if $K \cap L \neq 0$ for each non-zero submodule L of M and the *singular submodule* $Z(M)$ consists of all elements $a \in M$, the *annihilator left ideal* $(0 : a)_R = \{r \in R \mid ra = 0\}$, or simply $(0 : a)$, of which is essential in R. *Goldie's torsion theory* for the category R-mod is the hereditary torsion theory $\sigma = (\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = \{M \in R\text{-mod} \mid Z(M/Z(M)) = M/Z(M)\}$ and $\mathcal{F} = \{M \in R\text{-mod} \mid Z(M) = 0\}$. Note, that throughout this paper the letter σ will always denote Goldie's torsion theory and that the modules from the class $\mathcal{F}_\sigma$ are usually called *non-singular modules.* A ring R is said to be (*left*) *non-singular* if it is non-singular as a left R-module. For more details on torsion theories we refer to [11] or [10].

If $\mathcal{G}$ is an abstract class of modules, then a homomorphism $\varphi : G \to M$ with $G \in \mathcal{G}$ is called a $\mathcal{G}$*-precover* of the module M, if for each homomorphism $f : F \to M$ with $F \in \mathcal{G}$ there exists a homomorphism $g : F \to G$ such that $\varphi g = f$. A $\mathcal{G}$-precover φ of M is said to be a $\mathcal{G}$*-cover*, if every endomorphism f of G with $\varphi f = \varphi$ is an automorphism of the module G. An abstract class $\mathcal{G}$ of modules is called a *precover* (*cover*) *class*, if every module has a $\mathcal{G}$-precover ($\mathcal{G}$-cover). A more detailed study of precovers and covers can be found in [15].

Recently, in [4; Corollary 3], it has been proved that for each hereditary torsion theory τ with $\tau \geq \sigma$ in the usual sense that $\mathcal{T}_\sigma \subseteq \mathcal{T}_\tau$ the class of all τ-torsionfree modules is a precover class if and only if it is a cover class and these conditions are satisfied exactly when the torsion theory τ is of finite type. Moreover, one of the main results in [5] states that these conditions are equivalent for Goldie's torsion theory for all members of the countable set $\mathfrak{M} = \{R, R/\sigma(R), R[x_1, \dots, x_n], R[x_1, \dots, x_n]/\sigma(R[x_1, \dots, x_n]), n < \omega\}$ of rings whenever they are equivalent for an arbitrary member of this set. Moreover, in [6] it has been shown that if RG is a non-singular ring such that the class of all non-singular RG-modules is a cover class, then so is that of R-modules. Furthermore, if G is a well-ordered cancellative monoid such that for all elements $g, h \in G$ with $g < h$ there is $l \in G$ such that $lg = h$, then the class of all non-singular R-modules is a cover class if and only if the class of all non-singular RG-modules is a cover class.

The purpose of this note is to introduce the class of strongly cancellative monoids which lies between the classes of all totally ordered cancellative monoids and the class of all cancellative monoids. The main results of this paper can be formulated as follows.

Theorem. *Let G be a monoid and let R be a ring such that $aR \subseteq Ra$ for each $a \in R$. Then the class of all non-singular R-modules is a cover class if and only if the class of all non-singular RG-modules is a cover class provided one of the following conditions holds:*

(i) G is a strongly cancellative monoid such that $hG \subseteq Gh$ for all $h \in G$;

(ii) G is a totally ordered cancellative monoid such that $hG \subseteq Gh$ for all $h \in G$;

(iii) G is a totally ordered group.

Proof. With respect to [4; Corollary 3] it suffices to use Theorems 10, 12 and 13 presented below. ■

Definition 1. We shall say that a monoid G is *strongly cancellative*, if to each finite subset $F = \{g_1, \dots, g_n\}$ of G it is associated an index $i \in \{1, \dots, n\}$ such that for any two finite subsets $F = \{g_1, \dots, g_n\}$ and $H = \{h_1, \dots, h_m\}$ of G with associated indices $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ the product $h_j g_i$ is different from all other products $h_k g_l$ with relevant indices k and l. We shall also say that g_i is the element corresponding to the set $F = \{g_1, \dots, g_n\}$ under the strong cancellation law.

Remark 2. Note, that the index in the above definition need not be determined uniquely. So, if we shall speak about this index in what follows, then we shall mean one of the indices having the property described in the definition. Further, every strongly cancellative monoid is cancellative, i.e. it satisfies the left and right cancellation laws. Clearly, for the subsets $\{g\}$ and $\{h_1, h_2\}$ of G we necessarily have $gh_1 \neq gh_2$ and $h_1 g \neq h_2 g$.

Lemma 3. *Let R be a ring and let G be a strongly cancellative monoid and let $u = \sum_{k=1}^n r_k g_k$ be a non-zero element of the ring RG such that $r_k \neq 0$ for each $k = 1, \dots, n$. If K is a left ideal of the ring R such that the left ideal $J = (RGK : u)$ is essential in RG, then the left ideal $I = (K : r_i)$ is essential in R, i being the index corresponding to the set $\{g_1, \dots, g_n\}$ under the strong cancellation law.*

Proof. Proving indirectly let us suppose that there exists a non-zero left ideal L of R such that $L \cap I = 0$. Now RGL is a non-zero left ideal of RG and we are going to show that $RGL \cap J = 0$. Assume, on the contrary, that $v = \sum_{l=1}^m s_l h_l$ is a non-zero element of $RGL \cap J$ with all the coefficients non-zero. By the hypothesis there is an index $j \in \{1, \dots, m\}$ such that the product $h_j g_i$ is different from all other products $h_l g_k$ with relevant indices l and k. Then $v \in J$ yields $vu = \sum_{k=1}^n \sum_{l=1}^m s_l r_k h_l g_k \in RGK$ and consequently the coefficient $s_j r_i$ at $h_j g_i$ belongs to K. On the other hand, $0 \neq s_j \in L$ means that $s_j \notin I$, hence $s_j r_i \notin K$, which is a contradiction finishing the proof. ∎

Theorem 4. *If G is a strongly cancellative monoid, then the equalities $Z(RG) = Z(R)G$ and $\sigma(RG) = \sigma(R)G$ hold. Especially, a ring R is non-singular if and only if the ring RG is so.*

Proof. We start with the equality $Z(RG) = Z(R)G$. The inclusion $Z(R)G \subseteq Z(RG)$ holds by [6; Proposition 6]. In order to prove the converse let $u = \sum_{k=1}^n r_k g_k \in Z(RG)$ be an arbitrary non-zero element such that $r_k \neq 0$ for each $k = 1, \dots, n$. Then $(0 : u) \leq' RG$ and so $(0 : r_i) \leq' R$ by Lemma 3 for the index $i \in \{1, \dots, n\}$ corresponding to the subset $\{g_1, \dots, g_n\}$ of G by the strong cancellation property. Hence $r_i \in Z(R)$ yields that $r_i g_i \in Z(R)G \subseteq Z(RG)$. Thus $u - r_i g_i \in Z(RG)$ and continuing by the induction we finally obtain that $u = \sum_{k=1}^n r_k g_k \in Z(R)G$, as we wished to show.

Now we are going to finish the proof in the similar way. By [6; Proposition 6] we know that $\sigma(R)G \subseteq \sigma(RG)$ and thus we shall prove the equality. So, let $0 \neq u = \sum_{k=1}^n r_k g_k$ be an arbitrary element of $\sigma(RG)$ such that $r_k \neq 0$ for each $k = 1, \dots, n$. Then $(Z(RG) : u)$ is essential in RG and so the left annihilator ideal $(Z(R) : r_i)$ is essential in R by Lemma 3 in view of the equality $Z(RG) = Z(R)G$ proved in the first part of the proof, $i \in \{1, \dots, n\}$ being the index corresponding to the subset $\{g_1, \dots, g_n\}$ of G under the strong cancellation property of G. Thus $r_i \in \sigma(R)$ gives that $r_i g_i \in \sigma(R)G \subseteq \sigma(RG)$. From this we infer that $u - r_i g_i \in \sigma(RG)$ and we can proceed by the induction. Finally we obtain that $u = \sum_{k=1}^n r_k g_k \in \sigma(R)G$, as we wished to show. The rest is now obvious. ∎

Corollary 5. *Let G be a strongly cancellative monoid and let $u = \sum_{k=1}^n r_k g_k$ be a non-zero element of the ring RG. If the left annihilator ideal $(0 : u)$ is essential in RG, then the intersection $\cap_{k=1}^n (0 : r_k)$ is essential in R.*

Proof. In the proof of Theorem 4 we have shown that $(0 : r_i) \leq' R$ for the index $i \in \{1, \dots, n\}$ corresponding to the set $\{g_1, \dots, g_n\}$ under the strong cancellation property and that $u - r_i g_i \in Z(RG)$. Continuing by the induction we shall obtain that $(0 : r_k) \leq' R$ for each $k = 1, \dots, n$, from which the assertion follows immediately. ∎

Notation. Let G be a strongly cancellative monoid and let $J \leq RG$ be an arbitrary left ideal. We shall define the subset $J[g]$ of the ring R for an arbitrary element $g \in G$ in such a way that $a \in J[g]$ if and only if there is an element $u = ag + \sum_{j=1}^{m} a_j h_j$ in J such that g is the element corresponding to the subset $\{g, h_1, \dots, h_m\}$ under the strong cancellation law. Note, that the set $J[g]$ is non-empty for each $g \in G$. Clearly, $J[g]$ contains the zero element 0 of R since $0g \in J$ is the zero element of the ring RG.

Lemma 6. *Let G be a strongly cancellative monoid and let J be a left ideal of the ring RG. Then*

(i) $J[g]$ is a left ideal of the ring R for each $g \in G$;

(ii) $J[g] \subseteq J[hg]$ for any two elements $g, h \in G$;

(iii) if $hG \subseteq Gh$ for each $h \in G$, then the set $\{J[g] \mid g \in G\}$ is directed upward.

Proof. (i) For arbitrary elements $a, b \in J[g]$ and $r \in R$ there are elements $u = ag + \sum_{j=1}^{m} a_j h_j$ and $v = bg + \sum_{r=1}^{t} b_r k_r$ from J such that the element g corresponds to the subsets $\{g, h_1, \dots, h_m\}$ and $\{g, k_1, \dots, k_t\}$ under the strong cancellation law. So, $u - v = (a - b)g + \sum_{j=1}^{m} a_j h_j - \sum_{r=1}^{t} b_r k_r \in J$, $ru = rag + \sum_{j=1}^{m} ra_j h_j \in J$ and consequently $a - b, ra \in J[g]$ once we verify that the element g corresponds to the set $\{g, h_1, \dots, h_m, k_1, \dots, k_t\}$ under the strong cancellation law. However, if $F = \{g_1, \dots, g_n\} \subseteq G$ is arbitrary with the index $i = 1$, then $gg_1 \neq h_j g_i$ and $gg_1 \neq k_r g_i$ for all relevant indices i, j, r, as we wished to show.

(ii) Using the above notations we have $hu = ahg + \sum_{j=1}^{m} a_j hh_j \in J$ and $hgg_1 \neq hh_j g_i$ for all relevant indices by the left cancellation law and so $a \in J[hg]$.

(iii) For $g, h \in G$ we have $hg = \tilde{g}h$ for a suitable element $\tilde{g} \in G$ by the hypothesis and so (ii) yields $J[g] \subseteq J[hg]$ and $J[h] \subseteq J[\tilde{g}h] = J[hg]$, as desired. ∎

Lemma 7. *Let R be a non-singular ring such that Goldie's torsion theory σ for the category R-*mod *is of finite type. Further, let G be a monoid and let J be an essential left ideal of the ring RG such that the set $\{J[g] \mid g \in G\}$ is directed upward with respect to the inclusion. Then there is an element $g \in G$ such that $J[g]$ is essential in R.*

Proof. First of all we are going to verify that the union $\tilde{J} = \cup_{g \in G} J[g]$ is an essential left ideal of the ring R. Clearly, if $r \in R \setminus \tilde{J}$ is an arbitrary element, then $r \notin J[g]$ for each $g \in G$. Especially, $r = re$ is not in $J[e] = J$, and so there is an element $u \in RG$ such that $0 \neq ur \in J$. Now if $u = \sum_{j=1}^{m} b_j h_j$, where all the coefficients $b_1, \dots, b_m$ are non-zero, then there is an index $i \in \{1, \dots, m\}$ such that $0 \neq b_i r \in J[h_i] \subseteq \tilde{J}$, as we wished to show. Assume first now, that there exists an element $g \in G$ such that $J[g]$ is essential in $J[h]$ for each $h \in G$ with $J[g] \subseteq J[h]$. Now if $0 \neq r \in R$ is an arbitrary element, then there is an element $s \in R$ with $0 \neq sr \in \tilde{J}$, hence $0 \neq sr \in J[h]$ for some $h \in G$ and consequently $0 \neq sr \in J[k]$ for some $k \in G$ with $J[g], J[h] \subseteq J[k]$. Thus $0 \neq tsr \in J[g]$ for a suitable element $t \in R$, which means that $J[g]$ is essential in R. So, to finish the proof let $g_0 \in G$ be arbitrary. Continuing by the induction let us suppose that the elements $g_0, g_1, \dots, g_i$ of G

have been found for some $i < \omega$ in such a way that $J[g_j]$ is not essential in $J[g_{j+1}]$ for each $j < i$. By the preceding part of the proof there is an element $g_{i+1} \in G$ such that $J[g_i] \subseteq J[g_{i+1}]$ and $J[g_i]$ is not essential in $J[g_{i+1}]$. Then there is a left ideal $L_i \leq R$ such that $0 \neq L_i \leq J[g_{i+1}]$ and $J[g_i] \cap L_i = 0$ for each $i < \omega$. Obviously, the ideals L_i are σ-torsionfree left ideals of the ring R and they form the infinite direct sum $\oplus_{i<\omega} L_i$ in R, which contradicts [14; Theorem 2.1] stating that σ is of finite type if and only if the ring R contains no infinite direct sum of σ-torsionfree left ideals. ■

Lemma 8. *Let G be a monoid such that $hG \subseteq Gh$ for each $h \in G$ and let R be a ring such that $aR \subseteq Ra$ for each $a \in R$. Then to any two elements $u, v \in RG$ there exists an element $\tilde{u} \in RG$ such that $vu = \tilde{u}v$.*

Proof. If $u = \sum_{i=1}^{n} r_i g_i$ and $v = \sum_{j=1}^{m} s_j h_j$, then $vu = \sum_{j=1}^{m} \sum_{i=1}^{n} s_j r_i h_j g_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{r}_i s_j \tilde{g}_i h_j = \tilde{u}v$, as desired. ■

Lemma 9. *Let G be a strongly cancellative monoid such that $hG \subseteq Gh$ for each $h \in G$. Further, let R be a non-singular ring such that $aR \subseteq Ra$ for each $a \in R$. If every essential left ideal of R essentially contains a finitely generated left ideal, then every essential left ideal of the ring RG essentially contains a finitely generated left ideal, too.*

Proof. Let J be an essential left ideal of the ring RG. The set $\{J[g] \mid g \in G\}$ is directed upward by Lemma 6(iii) and so Lemma 7 yields the existence of an element $g \in G$ such that $J[g]$ is essential in R. By the hypothesis there is a finitely generated left ideal $K = \sum_{i=1}^{s} Ra_i$ of R which is essential in $J[g]$. So, for each $i = 1, \dots, s$ there is an element $u_i \in J$ of the form $u_i = a_i g + \sum_{j=1}^{s_i} a_{ij} h_{ij}$ such that the element $g \in G$ corresponds to the set $\{g, h_{i1}, \dots, h_{is_i}\}$ under the strong cancellation law for each $i = 1, \dots, s$. Now we put $L = \sum_{i=1}^{s} RGu_i$ and we are going to show that L is essential in J. So, let $u = \sum_{r=1}^{n} b_r h_r$ be an arbitrary element of J such that $b_r \neq 0$ for each $r = 1, \dots, n$ and $1 \in \{1, \dots, n\}$ is the index corresponding to the set $\{h_1, \dots, h_n\}$ under the strong cancellation law. Assume first, that $u_i u \neq 0$ for some $i \in \{1, \dots, s\}$. Then $0 \neq u_i u = \tilde{u} u_i \in L$ for a suitable element $\tilde{u} \in RG$ by Lemma 8 and we are through. So, let $u_i u = 0$ for each $i = 1, \dots, s$. By the strong cancellation law we have $gh_1 \neq gh_i$ and $gh_1 \neq h_{ij} h_r$ for all relevant indices and consequently the coefficient $a_i b_1$ at the element gh_1 in the product $u_i u$ is equal to zero for each $i = 1, \dots, s$. This means that $K \subseteq (0 : b_1)$, which is a contradiction with the non-singularity of the ring R and the proof is complete. ■

Now we are ready to prove one of the main results.

Theorem 10. *Let G be a strongly cancellative monoid such that $hG \subseteq Gh$ for each element $h \in G$ and let R be a ring such that $aR \subseteq Ra$ for each element $a \in R$. Then Goldie's torsion theory for the category R-*mod *is of finite type if and only if Goldie's torsion theory for the category RG-*mod *is of finite type.*

Proof. By [5; Theorem 5] Goldie's torsion theory σ for the category R-mod is of finite type if and only if Goldie's torsion theory for the category $R/\sigma(R)$-mod is of finite type. By Theorem 4 we have $\sigma(RG) = \sigma(R)G$, from which easily follows the ring isomorphism $(R/\sigma(R))G \cong RG/\sigma(RG)$ and consequently we may restrict ourselves to the case of non-singular rings. So, if Goldie's torsion theory for the category RG-mod is of finite type, then so is that for the category R-mod by [6; Theorem 8], while the converse follows immediately from Lemma 9. ■

Now we are going to present some applications of this result to the case of totally ordered monoids and groups. These results are in some sense related to that in [6], namely when the well-ordering is replaced by the total order on G and the condition that for all elements $g, h \in G$ with $g < h$ there is $l \in G$ such that $lg = h$ is replaced by the conditions $hG \subseteq Gh$ and $aR \subseteq Ra$ for all elements $h \in G$ and $a \in R$, respectively.

Lemma 11. *Every totally ordered cancellative monoid is strongly cancellative.*

Proof. Let $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_m\}$ be finite subsets of G such that $g_1 < \dots < g_n$ and $h_1 < \dots < h_m$. Then $h_j g_1 < h_j g_i$ by the left cancellation law and $h_1 g_i < h_j g_i$ by the right cancellation law. Thus $h_1 g_1 < h_j g_i$ for all $j = 1, \dots, m$ and $i = 1, \dots, n$, where at least one of the indices i and j is different from 1 and the proof is complete, the elements g_1 and h_1 being the elements corresponding to the sets $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_m\}$ under the strong cancellation law, respectively. ∎

Theorem 12. *Let G be a totally ordered cancellative monoid such that $hG \subseteq Gh$ for each element $h \in G$ and let R be a ring such that $aR \subseteq Ra$ for each $a \in R$. Then Goldie's torsion theory for the category R-*mod *is of finite type if and only if Goldie's torsion theory for the category RG-*mod *is of finite type.*

Proof. It immediately follows from Lemma 11 and Theorem 10. ∎

Theorem 13. *Let G be a totally ordered group and let R be a ring such that $aR \subseteq Ra$ for each $a \in R$. Then Goldie's torsion theory for the category R-*mod *is of finite type if and only if Goldie's torsion theory for the category RG-*mod *is of finite type.*

Proof. For any two elements $g, h \in G$ we have $hg = hgh^{-1}h$ and it suffices to use Theorem 10, G being obviously cancellative on both sides. ∎

Corollary 14. *If G is a totally ordered group and if R is a commutative ring then Goldie's torsion theory for the category R-*mod *is of finite type if and only if Goldie's torsion theory for the category RG-*mod *is of finite type.*

Proof. Obvious. ∎

References

[1] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, vol.13, Springer-Verlag, 1974.

[2] L. Bican, *Torsionfree precovers*, Contributions to General Algebra 15, Proceedings of the Klagenfurt Conference 2003 (AAA 66), Verlag Johannes Heyn, Klagenfurt 2004 **15** (2004), 1–6.

[3] L. Bican, *Precovers and Goldie's torsion theory*, Math. Bohemica **128** (2003), 395–400; Zbl 1057.16027)].

[4] L. Bican, *On torsionfree classes which are not precover classes* (to appear).

[5] L. Bican, *Non-singular precovers over polynomial rings*, Comment.Math.Univ.Carolinae **47** (2006), 369–377.

[6] L. Bican, *Non-singular covers over ordered monoid rings*, Math. Bohemica **131** (2006), 95–104.

[7] L. Bican, R. El Bashir, E. Enochs, *All modules have flat covers*, Proc. London Math. Society **33** (2001), 385–390; Zbl 1029.16002).

[8] L. Bican, B. Torrecillas, *Precovers*, Czech. Math. J. **53 (128)** (2003), 191–203; Zbl 1016.16003).

[9] L. Bican, B. Torrecillas, *On covers*, J. Algebra **236** (2001), 645–650; Zbl 0973.16002).

[10] L. Bican, T. Kepka, P. Němec, *Rings, Modules, and Preradicals*, Marcel Dekker, New York, 1982; Zbl 0483.16026).
[11] J. Golan, *Torsion Theories*, Pitman Monographs and Surveys in Pure an Applied Matematics, 29, Longman Scientific and Technical, 1986.
[12] S. H. Rim, M. L. Teply, *On coverings of modules*, Tsukuba J. Math. **24** (2000), 15–20; Zbl 0985.16017).
[13] M. L. Teply, *Torsion-free covers II*, Israel J. Math. **23** (1976), 132-136; Zbl 0321.16014).
[14] M. L. Teply, *Some aspects of Goldie's torsion theory*, Pacif. J. Math. **29** (1969), 447-459; Zbl 0174.06803)].
[15] J. Xu, *Flat Covers of Modules*, Lecture Notes in Mathematics 1634, Springer Verlag Berlin-Heidelberg-New York, 1996; Zbl 0860.16002).

KA MFF UK,
Sokolovská 83,
186 00 Praha 8 – Karlín,
Czech Republik
E-mail address:
bican@karlin.mff.cuni.cz