ON LEFT CONJUGACY CLOSED LOOPS IN WHICH THE LEFT MULTIPLICATION GROUP IS NORMAL

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Abstract. A loop Q is said to be left conjugacy closed (LCC) if the left translations form a set of permutations that is closed under conjugation. This papers investigates those LCC loops where the group generated by left translations is normal in the group generated by both left and right translations.

Left conjugacy closed (LCC) loops are those loops that have left translations closed under conjugation. Thus, if $L_x : x \mapsto ax$ is a left translation, and $L_b : x \mapsto bx$ is another left translation of the loop, then $L_a L_b L_a^{-1}$ is equal to some L_c. By applying $L_c = L_a L_b L_a^{-1}$ to a, one get $c = (ab)/a$. LCC loops are thus a variety of loops satisfying the identity

$$x(y(x\backslash z)) = ((xy)/x)z.$$

There are other equivalent identities [11, 6], and by one of them $L_x R_y R_z^{-1} L_x^{-1}$ equals $R_{xy} R_{xz}^{-1}$ for all x, y, z. Here R_y and R_z stand, of course, for the right translations $R_y : u \mapsto uy$ and $R_z : u \mapsto uz$, respectively.

In particular, $L_x R_y L_x^{-1} = R_{xy} R_x^{-1}$ for all $x, y \in Q$, whenever Q is an LCC loop. Thus the right multiplication group $\mathcal{R} = \mathcal{R}(Q) = \langle R_x;\, x \in Q\rangle$ is a normal subgroup of the (full) multiplication group $\operatorname{Mlt} Q = \langle L_x, R_x;\, x \in Q\rangle$, in every LCC loop Q.

In this paper we are concerned with LCC loops Q such that $\mathcal{L} = \mathcal{L}(Q) = \langle L_x;\, x \in Q\rangle \trianglelefteq \operatorname{Mlt} Q$. It turns out that these loops have many features common with (both sided) conjugacy closed loops (CC loops). This started to be apparent in [6], which lists a number of parallels in its Section 4. That list is extended in this paper substantially.

The properties of CC loops that served as a comparative material can be found in [4], [10] and [5] (the latter paper is in fact much earlier than [6], but it got stuck in the refereeing process).

An LCC loop Q is always an A_ℓ-loop (i.e., $\mathcal{L}_1 \le \operatorname{Aut} Q$) and satisfies $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$. If $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, then one can also prove that $N_\rho \le N_\lambda$, and that Q/N_λ is a group. Section 1 describes in Propositions 1.7, 1.8 and 1.10 certain facts about general A_ℓ-loops that fulfil such properties. Proposition 1.7 is based on a well known construction of Sabinin [12].

Other statements of Section 1 are well known facts, for which one needs to know only the very basics of loop thoery. In most cases proofs (which are short) are included for reader's comfort.

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In Section 2 there are recapitulated and extended some of the basic properties of LCC loops with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. In Section 3 we prove that in these loops $[\mathcal{L}, \mathcal{R}]$ equals $\langle \mathcal{R}_u;\, u \in Q \rangle$, which is a subgroup of $\mathcal{B} = \langle \mathcal{L}_u;\, u \in Q \rangle$. Group $\mathcal{B}$ is abelian and can be embedded into A^r, where $r = |Q : N|$. Here A means the associator subloop of Q, and N the nucleus of Q. The subloop A is an abelian group and Q/N can be embedded into $\operatorname{Aut} \mathcal{B}$.

Section 4 is devoted to the proof that nilpotency of $\mathcal{L}$ implies the nilpotency of $\operatorname{Mlt} Q$, in any LCC loop Q with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. In Section 5 we prove that LCC p-loops satisfying $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$ are centrally nilpotent.

Section 6 is concerned with examples of LCC loops Q that are not right conjugacy closed and satisfy $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. We show that such loops do not exist for orders p^2 and 8, but do exist for orders p^3 and 16, p an odd prime. The constructed loops are of nilpotency class two and their behaviour depends on trilinear mappings that are induced by the associator.

Recall that [2] and [1] are standard sources for loops and quasigroups.

1. Nuclei and inner automorphisms

Let Q be a loop. Its *left nucleus* $N_\lambda = N_\lambda(Q)$ is defined as a set of all $a \in Q$ that associate on the left; i.e. $N_\lambda = \{a \in Q;\, a(xy) = (ax)y \text{ for all } x, y \in Q\}$. The *middle nucleus* N_μ and the *right nucleus* N_ρ are defined by shifting the position of a to the right. The intersection $N_\rho \cap N_\mu \cap N_\lambda$ is known as the *nucleus* of Q. It is denoted by $N = N(Q)$. The *centre* $Z(Q)$ consists of all $a \in N$ with $ax = xa$ for all $x \in Q$.

There are various ways how to connect the nuclei to the multiplication groups $\mathcal{L} = \mathcal{L}(Q)$ and $\mathcal{R} = \mathcal{R}(Q)$. Some of them are mentioned below.

Recall that $\mathcal{L}_1$ is generated by all mappings $L(x,y) = L_{xy}^{-1} L_x L_y$, while $\mathcal{R}_1$ by all mappings $R(x,y) = R_{yx}^{-1} R_x R_y$. Recall also that $\operatorname{Inn} Q = (\operatorname{Mlt} Q)_1$, the *inner mapping group* of Q, is generated by mappings $L(x,y)$, $R(x,y)$ and $T_x = R_x^{-1} L_x$, where $x, y \in Q$. A subloop $S \le Q$ is said to be *normal* if it induces a congruence of Q. It is well known (and easy to prove) that a subloop S is normal if and only if it is retained by all $\varphi \in \operatorname{Inn} Q$ (alternatively, by all mappings $R(x,y)$, $L(x,y)$ and T_x).

Like in [6] or [5] we shall use the convention that $L_{(M)}$ means $\{L_a;\, a \in M\}$, for any $M \subseteq Q$, and, similarly, $R_{(M)} = \{R_a;\, a \in M\}$.

Lemma 1.1. *Let Q be a loop. Then $N_\lambda = \{a \in Q;\, \varphi(a) = a$ for all $\varphi \in \mathcal{R}_1\}$ and $N_\rho = \{a \in Q;\, \varphi(a) = a$ for all $\varphi \in \mathcal{L}_1\}$.*

Proof. Note that $a \in N_\rho$, by definition, if and only if $R(y,x)(a) = a$ for all $x, y \in Q$. □

Corollary 1.2. *If $\mathcal{R}_1 \le \mathcal{L}_1$, then $N_\rho \le N_\lambda$. If $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, then N_ρ is a normal subloop of Q.*

Proof. The first claim is an immediate consequence of Lemma 1.1. Assume $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, $a \in N_\rho$ and $\psi \in \operatorname{Inn} Q$. To show $\psi(a) \in N_\rho$ means, by Lemma 1.1, to prove $\varphi(\psi(a)) = \psi(a)$ for all $\varphi \in \mathcal{L}_1$. However, $\varphi(\psi(a)) = \psi(\psi^{-1}\varphi\psi)(a) = \psi(a)$, as $\psi^{-1}\varphi\psi \in \mathcal{L}_1 = \mathcal{L} \cap \operatorname{Inn} Q$. □

Lemma 1.3. *If $a \in N_\lambda$, then $L(ax, y) = L(x, y)$ for all $x, y \in Q$.*

Proof. We have $L(x, y)(u) = v$ exactly when $x(yu) = (xy)v$. But then $(ax)(yu) = a(x(yu)) = a((xy)v) = ((ax)y)v$, for every $a \in N_\lambda$. □

Proposition 1.4. *If N_λ is a normal subloop of Q, with $N_\lambda \le N_\mu$, then $L(x, y) = L(x', y')$ whenever $xN_\lambda = x'N_\lambda$ and $yN_\lambda = y'N_\lambda$. Similarly, $R(x, y)$ depends only on the classes of N_ρ, if N_ρ is normal and contained in N_μ.*

Proof. Assume $N_\lambda \le N_\mu$, and let $x(yu) = (xy)v$. By Lemma 1.3 it suffices to show $x(ay \cdot u) = (x \cdot ay)v$, for every $a \in N_\lambda$. There exists $b \in N_\lambda$ with $xa = bx$, since N_λ is normal. Now, $x(ay \cdot u) = x(a \cdot yu) = xa \cdot yu = bx \cdot yu = b(x \cdot yu) = b(xy \cdot v) = (b \cdot xy)v = (bx \cdot y)v = (xa \cdot y)v = (x \cdot ay)v$, as required. □

Lemma 1.5. *Let Q be a loop with an automorphism φ. Then $\varphi L_x \varphi^{-1} = L_{\varphi(x)}$, and $\varphi R_x \varphi^{-1} = R_{\varphi(x)}$, for all $x \in Q$.*

Proof. Indeed, $\varphi L_x(y) = \varphi(x)\varphi(y) = L_{\varphi(x)}\varphi(y)$, and $\varphi R_x(y) = \varphi(y)\varphi(x) = R_{\varphi(x)}\varphi(y)$, for all $x, y \in Q$. □

Lemma 1.6. *Let Q be an A_ℓ-loop. Then $N_\rho = \{a \in Q;\ L_a \in C_{\mathcal{L}}(\mathcal{L}_1)\}$.*

Proof. The left translation L_a, $a \in Q$, centralizes $\mathcal{L}_1$ if and only if $\varphi L_a \varphi^{-1} = L_a$ for all $\varphi \in \mathcal{L}_1$. By Lemma 1.5 this means $\varphi(a) = a$ for all $\varphi \in \mathcal{L}_1$, which brings us to Lemma 1.1. □

Let Q be an A_ℓ-loop. From Lemma 1.5 one easily computes

$$(L_x\varphi)(L_y\psi) = L_{x\varphi(y)}L(x, \varphi(y))\varphi\psi,$$

for all $x, y \in Q$ and $\varphi, \psi \in \mathcal{L}_1$. Thus one can define a group operation on $Q \times \mathcal{L}_1$ by

$$(x, \varphi) \cdot (y, \psi) = (x\varphi(y),\ L(x, \varphi(y))\varphi\psi).$$

This seems to have been observed first in [12] by Sabinin. The group is clearly isomorphic to $\mathcal{L}$ (each element of $\mathcal{L}$ can be expressed as $L_x\varphi$, $\varphi \in \mathcal{L}_1$, in a unique way).

Proposition 1.7. *Let Q be an A_ℓ-loop such that N_λ is its normal subloop, Q/N_λ is a group and $N_\lambda \le N_\mu$. Then*

$$(xN_\lambda, \varphi) \cdot (yN_\lambda, \psi) = (xyN_\lambda, L(x, y)\varphi\psi)$$

is a correctly defined group operation on $Q/N_\lambda \times \mathcal{L}_1$. The mapping $L_x\varphi \mapsto (xN_\lambda, \varphi)$ is a homomorphism of $\mathcal{L}$ onto this group. The kernel is equal to $L_{(N_\lambda)}$.

Proof. The fact that Q/N_λ is a group implies that each $\varphi \in \mathcal{L}_1$ moves within the cosets of N_λ. Hence $\varphi(y) \equiv y \bmod N_\lambda$ for all $y \in Q$. This together with Proposition 1.4 mean that Sabinin's formula can be factorized in the suggested way. □

Proposition 1.8. *Let Q be an A_ℓ-loop such that N_λ is its normal subloop, Q/N_λ is a group and $N_\lambda \le N_\mu$. Then $\mathcal{L}_1$ is abelian, and*

$$L(xy, z)L(x, z) = L(x, yz)L(y, z)$$

for all $x, y, z \in Q$.

Proof. We get the result from Proposition 1.7 by means of the associative law. Indeed,

$$(xN_\lambda,\varphi)(yN_\lambda,\psi)\cdot(zN_\lambda,\mathrm{id}_Q) = (xyzN_\lambda, L(xy,z)L(x,y)\varphi\psi), \quad \text{and}$$
$$(xN_\lambda,\varphi)\cdot(yN_\lambda,\psi)(zN_\lambda,\mathrm{id}_Q) = (xyzN_\lambda, L(x,yz)\varphi L(y,z)\psi).$$

Setting $\varphi = \mathrm{id}_Q$ yields $L(x,yz)L(y,z) = L(xy,z)L(x,y)$ for all $x,y,z\in Q$. Hence $L(x,yz)L(y,z)\varphi\psi = L(xy,z)L(x,y)\varphi\psi = L(x,yz)\varphi L(y,z)\psi$ for all $\varphi,\psi\in\mathcal{L}_1$ and all $x,y,z\in Q$. Thus $L(y,z)$ centralizes every $\varphi\in\mathcal{L}_1$. However, $\mathcal{L}_1$ is generated by the mappings $L(y,z)$, and so $\mathcal{L}_1$ has to be abelian. □

For a loop Q put $M_\lambda = \{a\in Q;\ L_a\in Z(\mathcal{R})\}$ and $M_\rho = \{a\in Q;\ R_a\in Z(\mathcal{L})\}$.

Lemma 1.9. *Let Q be a loop, and put $G = \operatorname{Mlt} Q$. Then*

(i) *$C_G(\mathcal{R}) = L_{(N_\lambda)}$ and $C_G(\mathcal{L}) = R_{(N_\rho)}$;*
(ii) *$M_\lambda \le Z(N_\lambda)$ and $M_\rho \le Z(N_\rho)$ are subloops of Q;*
(iii) *$Z(\mathcal{R}) = L_{(M_\lambda)} = L_{(N_\lambda)}\cap\mathcal{R}$ and $Z(\mathcal{L}) = R_{(M_\rho)} = R_{(N_\rho)}\cap\mathcal{L}$; and*
(iv) *$a\mapsto R_a$ and $a\mapsto L_a$ are isomorphisms $M_\rho\cong Z(\mathcal{L})$ and $M_\lambda\cong Z(\mathcal{R})$, respectively.*

Proof. These are standard facts. See, e.g., [6, Lemma 1.4 and Proposition 1.6]. □

Proposition 1.10. *Let Q be an A_ℓ-loop such that N_λ is its normal subloop, Q/N_λ is a group and $N_\lambda\le N_\mu$. Suppose $\mathcal{R}\trianglelefteq\operatorname{Mlt}Q$, and consider the action of $\mathcal{L}$ on $\mathcal{R}$ by conjugation. The image of this action is isomorphic to the group on $Q/N_\lambda\times\mathcal{L}_1$ with $(xN_\lambda,\varphi)\cdot(yN_\lambda,\psi) = (xyN_\lambda, L(x,y)\varphi\psi)$ for all $x,y\in Q$ and $\varphi,\psi\in\mathcal{L}_1$.*

Proof. The latter group is a homomorphic image of $\mathcal{L}$, with kernel $L_{(N_\lambda)}$, by Proposition 1.7. The kernel of the action of $\mathcal{L}$ on $\mathcal{R}$ is equal to $L_{(N_\lambda)}$ as well, by point (i) of Lemma 1.9. □

Lemma 1.11. *Let U be a normal subloop of a loop Q, $U\le N_\lambda\cap N_\mu$. Then $T_x(uv) = T_x(u)T_x(y)$ for all $u\in u$ and $x,y\in Q$.*

Proof. We have $T_x(u)T_x(y)\cdot x = T_x(u)\cdot T_x(y)x = T_x(u)\cdot xv = T_x(u)x\cdot y = xu\cdot y = x\cdot uy = T_x(uy)\cdot x$. □

Proposition 1.12. *Let Q be a loop such that $\mathcal{L}\trianglelefteq\operatorname{Mlt}Q$ (or $\mathcal{R}\trianglelefteq\operatorname{Mlt}Q$). Then M_ρ (or M_λ) is a normal subloop of Q, respectively.*

Proof. Assume $\mathcal{L}\trianglelefteq\operatorname{Mlt}Q$. Then $Z(\mathcal{L})\trianglelefteq\operatorname{Mlt}Q$, and hence the orbits of $Z(\mathcal{L})$ yield blocks of a congruence of Q. Since M_ρ is such a block, by point (iii) of Lemma 1.9, it has to be a normal subloop. □

2. Homomorphisms, subloops and actions

In an LCC loop Q one has $N_\lambda = N_\mu \trianglelefteq \mathcal{R}$, and $T_x = R_x^{-1}L_x$ is an automorphism if and only if $x \in N_\rho$. The defining identity $L_xL_yL_x^{-1} = L_{T_x(y)}$ has an alternative expression $L_xR_yR_z^{-1}L_x^{-1} = R_{xy}R_{xz}^{-1}$, which yields $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$. There exists a (unique) group homomorphism $\Lambda : \mathcal{L} \to \operatorname{Inn} Q$ with $\Lambda(L_x) = T_x$ for all $x \in Q$. Its kernel is equal to $Z(\mathcal{L}) = R_{(M_\rho)}$ (c.f. Lemma 1.9).

The above facts are regarded as basic and will be applied without an explicit reference. All of them are proved in [6], and some also in [11] (an earlier paper).

Proposition 2.1. *Let Q be an LCC loop. Then*

$$(\operatorname{Mlt} Q)/\mathcal{R} \cong \mathcal{L}/(\mathcal{R} \cap \mathcal{L}) \cong (\operatorname{Inn} Q)/\mathcal{R}_1.$$

If, in addition, $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, then also

$$(\operatorname{Mlt} Q)/\mathcal{L} \cong \mathcal{R}/(\mathcal{R} \cap \mathcal{L}) \cong (\operatorname{Inn} Q)/\mathcal{L}_1.$$

Proof. The statement clearly holds for all loops with $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$, as then $\operatorname{Mlt} Q = \mathcal{L}\mathcal{R}$ and $\operatorname{Mlt} Q = \mathcal{R}(\operatorname{Inn} Q)$. □

Let Q be an LCC loop. An equivalent condition to $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$ is $\mathcal{R}_1 \leq \mathcal{L}_1$ [6, Proposition 4.3]. This fact will be used further on freely as well, without a reference.

Corollary 2.2. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then $|\mathcal{L}| = |\mathcal{R}| \cdot |\mathcal{L}_1 : \mathcal{R}_1|$.*

Proof. We have $|\operatorname{Mlt} Q| = |\mathcal{L}| \cdot |\operatorname{Mlt} Q : \mathcal{L}| = |\mathcal{L}| \cdot |\operatorname{Inn} Q : \mathcal{L}_1|$ and $|\operatorname{Mlt} Q| = |\mathcal{R}| \cdot |\operatorname{Mlt} Q : \mathcal{R}| = |\mathcal{R}| \cdot |\operatorname{Inn} Q : \mathcal{R}_1| = |\mathcal{R}| \cdot |\mathcal{L}_1 : \mathcal{R}_1| \cdot |\operatorname{Inn} Q : \mathcal{L}_1|$. □

In every loop Q, $R_a \in \mathcal{L} \Leftrightarrow T_a \in \mathcal{L}_1$. If $a \in N_\rho$, then also $R_a \in \mathcal{L} \Leftrightarrow a \in M_\rho$, by point (iii) of Lemma 1.9. If Q is an LCC loop, then $T_a \in \mathcal{L}_1$ implies $a \in N_\rho$, since $\mathcal{L}_1 \leq \operatorname{Aut} Q$.

Proposition 2.3. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then $N_\rho = N$ is the nucleus of Q, and it is a normal subloop of Q. Furthermore, $Z(Q) = \{a \in Q;\ xa = ax \text{ for all } x \in Q\}$, $\operatorname{Inn} Q = \langle T_x;\ x \in Q\rangle$, $M_\rho \leq Z(N)$ is a normal subloop of Q, and $\mathcal{L} \cap \mathcal{R} = Z(\mathcal{L})\mathcal{R}_1 = R_{(M_\rho)}\mathcal{R}_1$. For all $a \in Q$*

$$T_a \in \mathcal{L}_1 \Leftrightarrow a \in M_\rho \Leftrightarrow R_a \in \mathcal{L}.$$

Proof. Use the paragraph preceding the proposition to obtain its last assertion. In every LCC loop $N_\lambda = N_\mu$ and $\mathcal{L}_1 \leq \operatorname{Im} \Lambda$. We assume $\mathcal{R}_1 \leq \mathcal{L}_1$, and hence $\operatorname{Im} \Lambda = \operatorname{Inn} Q$, and $N = N_\rho$ is a normal subloop of Q, by Corollary 1.2. The normality of M_ρ follows from Proposition 1.12. The formula for $\mathcal{R} \cap \mathcal{L}$ is clearly true if $R_a \in \mathcal{L}$ just for $a \in M_\rho$. But that we already know. The identity mapping is also an automorphism, and hence $T_a = \mathrm{id}_Q$ implies $a \in N_\rho = N$. Thus $Z(Q)$ consists of all $a \in Q$ with $T_a = \mathrm{id}_Q$. □

Proposition 2.4. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then Q/M_ρ is a group and $\mathcal{L}_1$ is abelian.*

Proof. For the associativiy of Q/M_ρ it suffices to prove that the left inner mapping group of M_ρ is normal in the left multiplication group of Q/M_ρ. Assume $x \in Q$ and $\varphi \in \mathcal{L}_1$. Then $L_x\varphi(M_\rho) = L_x(M_\rho)$ because M_ρ is a normal subloop, by Proposition 2.3. We also obtain $L_x(M_\rho) = M_\rho \Leftrightarrow x \in M_\rho \Leftrightarrow T_x \in \mathcal{L}_1$. Thus $\psi(M_\rho) = M_\rho$

if and only if $\Lambda(\psi) \in \mathcal{L}_1 \trianglelefteq \operatorname{Inn} Q$, and $\{\psi \in \mathcal{L};\ \psi(M_\rho) = M_\rho\} = \Lambda^{-1}(\mathcal{L}_1)$ is a normal subgroup of $\mathcal{L}$. The commutativity of $\mathcal{L}_1$ follows from Proposition 1.8. □

Proposition 2.4 appeared already in [6], as Proposition 4.6.

Proposition 2.5. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then $xM_\rho \mapsto T_x\mathcal{L}_1$ defines an isomorphism $Q/M_\rho \cong \operatorname{Inn} Q/\mathcal{L}_1$, and $xM_\rho \mapsto R_x^{-1}(\mathcal{R} \cap \mathcal{L})$ defines an isomorphism $Q/M_\rho \cong \mathcal{R}/\mathcal{R} \cap \mathcal{L}$.*

Proof. The mapping $L_x\varphi \mapsto xM_\rho$ is clearly a surjective homomorphism $\mathcal{L} \to Q/M_\rho$, since Q/M_ρ is a group. The kernel of this homomorphism equals $\{L_a\varphi;\ a \in M_\rho \text{ and } \varphi \in \mathcal{L}_1\}$, which contains $\operatorname{Ker} \Lambda = R_{(M_\rho)}$. Hence $T_x\varphi \mapsto xM_\rho$ is a surjective homomorphism $\operatorname{Inn} Q \to Q/M_\rho$. Its kernel is $\mathcal{L}_1$, by Proposition 2.3.

To get the second isomorphism consider the surjective homomorphism $\mathcal{R} \to Q/M_\rho$, $R_x^{-1}\varphi \mapsto xM_\rho$. The existence follows, again, from the fact that Q/M_ρ is a group. The kernel is equal to $\mathcal{R}_1 R_{(M_\rho)}$, which equals $\mathcal{L} \cap \mathcal{R}$, by Proposition 2.3. □

Proposition 2.6. *Let U be a subloop of an LCC loop Q. If U is normal, then $\langle L_u;\ u \in U\rangle \trianglelefteq \mathcal{L}$. Assume, in addition, $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. If $\langle L_u;\ u \in U\rangle \trianglelefteq \mathcal{L}$, then U is normal.*

Proof. First note that $L_x^{\pm 1} L_u L_x^{\mp 1}$ equals $L_{T_x^{\pm 1}(u)}$ for all $u \in U$ and $x \in Q$. The normality of U implies $T_x^{\pm 1}(u) \in U$, and $\langle L_u;\ u \in Q\rangle \trianglelefteq \mathcal{L}$ follows. Assume $\langle L_u;\ u \in Q\rangle \trianglelefteq \mathcal{L}$ and $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then $\operatorname{Inn} Q = \langle T_x;\ x \in Q\rangle$, by Proposition 2.3, and so it suffices to show $v = T_x^{\pm 1}(u) \in U$, for all $u \in U$ and $x \in Q$. We have $L_v \in \langle L_u;\ u \in U\rangle \trianglelefteq \mathcal{L}$, and hence v must be in that orbit of $\langle L_u;\ u \in U\rangle$ which contains 1. However, this orbit is equal to U. □

Corollary 2.7. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, and let U be a subgroup of N_λ. Then U is a normal subloop of $Q \Leftrightarrow L_{(U)} \trianglelefteq \mathcal{L} \Leftrightarrow L_{(U)} \trianglelefteq \operatorname{Mlt} Q$.*

Proof. The first equivalence follows from Proposition 2.6 immediately. It remains to prove $L_{(U)} \trianglelefteq \mathcal{L} \Rightarrow L_(U) \trianglelefteq \operatorname{Mlt} Q$. However, each L_u, $u \in U$, is centralized by every R_x, $x \in Q$, by point (i) of Lemma 1.9. □

Proposition 2.8. *If Q is an LCC loop, then $M_\lambda \le Z(N_\lambda)$ is a normal subloop of Q. If in addition $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, then $M_\lambda \le M_\rho$, and for all $a \in Q$*

$$T_a \in \mathcal{R}_1 \Leftrightarrow a \in M_\lambda \Leftrightarrow L_a \in \mathcal{R}.$$

Proof. The normality of M_λ follows from Proposition 1.12. Assume $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Clearly, $L_a \in \mathcal{R} \Leftrightarrow T_a \in \mathcal{R}_1$. Our assumption means $\mathcal{R}_1 \le \mathcal{L}_1$, and so $T_a \in \mathcal{R}_1$ implies that a belongs to $N_\rho = N$. This yields the implication $L_a \in \mathcal{R} \Rightarrow a \in M_\lambda$ (cf. Lemma 1.9). □

It seems to be an open question whether Q/M_λ has to be a group.

We shall finish this section by pointing out the relevance of Propositions 1.7, 1.8 and 1.10 to LCC loops. The formula

$$(xN_\lambda, \varphi) \cdot (yN_\lambda, \psi) = (xyN_\lambda, L(x, y)\varphi\psi)$$

defines a group operation on $Q/N_\lambda \times \mathcal{L}_1$ whenever Q is an LCC loop with Q/N_λ a group, by Proposition 1.7. The latter condition is satisfied when $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, by Proposition 2.4, since $M_\rho \le Z(N) \le N \le N_\lambda$.

If Q/N_λ is a group, then the group on $Q/N_\lambda \times \mathcal{L}_1$ is isomorphic to the image of $\mathcal{L}$ acting on $\mathcal{R}$ by conjugation, by Proposition 1.10. This means that Q/N_λ embeds into $\operatorname{Aut} Q$. We also have $L(x, yz)L(x, z) = L(x, yz)L(y, z)$ for all $x, y, z \in Q$, by Proposition 1.8.

If Q is LCC and satisfies $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, then further results can be obtained by using left-right symmetric versions of Propositions 1.7, 1.8 and 1.11. Thus

$$(xN, \varphi) \cdot (yN, \psi) = (yxN, R(x, y)\varphi\psi)$$

defines a group on $Q/N \times \mathcal{R}_1$ isomorphic to the image of $\mathcal{R}$ acting on $\mathcal{L}$ by conjugation, Q/N embeds into $\operatorname{Aut} \mathcal{L}$, and $R(xy, z)R(x, z) = R(x, yz)R(y, z)$ for all $x, y, z \in Q$.

3. Embeddings and the associator subloop

The associator subloop $A(Q)$ of a loop Q is the least normal subloop A with Q/A a group.

Let Q be an LCC loop. Then

$$[\mathcal{L}, \mathcal{R}] = \langle R_u;\ u \in Q\rangle \trianglelefteq \operatorname{Mlt} Q,$$

by [6, Theorem 3.5]. The orbits of $\langle R_u;\ u \in Q\rangle$ coincide with the cosets modulo the associator subloop $A(Q)$. The latter fact is true in any loop (the proof is easy, c.f. [5]).

Of course, the cosets modulo $A(Q)$ also coincide with the orbits of $\langle L_u;\ u \in Q\rangle$. We shall denote the latter group by $\mathcal{B}$, and will work with it throughout this section. Let Q be LCC with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then $\mathcal{R}_1 \le \mathcal{L}_1$ implies $\mathcal{R}_u \le \mathcal{L}_u = R_u \mathcal{L}_1 R_u^{-1}$ for all $u \in Q$, and $[\mathcal{L}, \mathcal{R}] \le \mathcal{B} \trianglelefteq \operatorname{Mlt} Q$ follows. Since $[\mathcal{L}, \mathcal{R}]$ and $\mathcal{B}$ share orbits, one immediately gets $|\mathcal{B} : [\mathcal{L}, \mathcal{R}]| = |\mathcal{L}_1 : \mathcal{R}_1|$. (Note that $\mathcal{L}_1$ is the stabilizer of $\mathcal{B} \le \mathcal{L}$ and $\mathcal{R}_1$ is the stabilizer of $[\mathcal{L}, \mathcal{R}] \le \mathcal{L} \cap \mathcal{R}$.) We shall record these facts in an extended form as a lemma:

Lemma 3.1. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then $[\mathcal{L}, \mathcal{R}] \le \mathcal{L} \cap \mathcal{R}$ and $[\mathcal{L}, \mathcal{R}] \le \mathcal{B} = \langle L_u;\ u \in Q\rangle$. Furthermore, $|(\mathcal{L} \cap \mathcal{R}) : [\mathcal{L}, \mathcal{R}]| = |M_\rho : A(Q)|$ and $|\mathcal{B} : [\mathcal{L}, \mathcal{R}]| = |\mathcal{L}_1 : \mathcal{R}_1|$.*

Proof. By Proposition 2.4, $\mathcal{L} \cap \mathcal{R} = R_{(M_\rho)}\mathcal{R}_1$. Since $[\mathcal{L}, \mathcal{R}] = R_{(A(Q)}\mathcal{R}_1$, we see that $|(\mathcal{L} \cap \mathcal{R}) : [\mathcal{L}, \mathcal{R}]| = |M_\rho : A(Q)|$. For the rest see above. □

Lemma 3.2. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then $C_{\mathcal{L}}(\mathcal{L}_1) = \mathcal{L}_1 L_{(N)}$ and $L_{(N)}$ centralizes $\mathcal{B} = \langle L_u;\ u \in Q\rangle$.*

Proof. We know that L_a centralizes $\mathcal{L}_1$ if and only if $a \in N_\rho$, by Lemma 1.6. Since $N_\rho = N$, by Proposition 2.3, and since $\mathcal{L}_1$ is abelian, by Proposition 2.4, we see that $L_a\varphi$, $\varphi \in \mathcal{L}_1$, centralizes $\mathcal{L}_1$ if and only if $a \in N$.

We shall now show that each L_a, $a \in N$, centralizes every $\psi \in \mathcal{L}_u$, $u \in Q$. Express ψ as $L_v^{-1}\varphi L_v$, where $v = 1/u$ and $\varphi \in \mathcal{L}_1$. Then $L_a \psi L_a^{-1} = L_v^{-1}(L_v L_a L_v^{-1})\varphi(L_v L_a^{-1} L_v^{-1})L_v = L_v^{-1} L_{T_v(a)} \varphi L_{T_v(a)}^{-1} L_v = L_v^{-1}\varphi L_v = \psi$, since $T_v(a) \in N$. □

Lemma 3.3. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Let Γ be a coset modulo N. Then N acts regularly on Γ by left translations. Denote by J the image of this action, and denote by I the image of the action of $\mathcal{B} = \langle \mathcal{L}_u;\ u \in Q\rangle$ on Γ. Then $I \cong A(Q)$, $I \le Z(J)$ and the orbits of I coincide with those cosets of $A(Q)$ that are contained in Γ.*

Proof. We have $\Gamma = L_u(N)$ for some $u \in Q$. Every $\psi \in J$ has the form $L_u\psi' L_u^{-1}$, where $\psi' \in L_{(N)}$, since the latter group is normal in $\mathcal{L}$, by Corollary 2.7. We also have $A(Q) \le Z(N) \le N$, by Proposition 2.4, and $\mathcal{B} \trianglelefteq \mathcal{L}$. Hence, similarly, every $\psi \in I$ has the form $L_u\psi' L_u^{-1}$, where ψ' is a restriction to N of a permutation from $\mathcal{B}$. Thus $\Gamma = N$ can be assumed for the rest of the proof.

The group $L_{(N)}$ acts on N regularly, and so J is a regular normal permutation group on N, $J \cong N$. By Lemma 3.2 this group is centralized by I. Hence I is semiregular and completely determined by any of its orbits, say by $A(Q)$. Since $L_{(A(Q))} \le \mathcal{B}$, we see that I is equal to the action of $L_{(A(Q))}$ on N. The rest follows from $A(Q) \le Z(N)$. □

Theorem 3.4. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Let $N = N(Q)$ be its nucleus and $A = A(Q)$ its associator subloop. Then $\mathcal{B} = \langle L_u;\ u \in Q\rangle$ embeds into A^r, $r = |Q : N|$.*

Proof. Map each $\psi \in \mathcal{B}$ to the list of restrictions of ψ to Γ, where Γ runs through the cosets modulo N. Each coordinate of the list can be identified with an element of A, by Lemma 3.3. Hence the mapping yields an embedding to A^r, since there are r coordinates. □

Corollary 3.5. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then the groups $[\mathcal{L}, \mathcal{R}]$ and $\mathcal{B} = \langle \mathcal{L}_u;\ u \in Q\rangle$ are abelian.*

Each $\psi \in \mathcal{B}$ uniquely decomposes as $L_a\varphi$, where $\varphi \in \mathcal{L}_1$ and $a \in A(Q)$. Now, $(L_a\varphi)(L_b\psi) = L_{ab}(\varphi\psi)$ for all $a, b \in A(Q)$, and $\varphi, \psi \in \mathcal{L}_1$, as $A(Q) \le N$ and $L_{(N)}$ centralizes $\mathcal{L}_1$ (see Lemma 3.2). We can thus state

Proposition 3.6. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Put $\mathcal{B} = \langle \mathcal{L}_u;\ u \in Q\rangle$. Then $\mathcal{B} = \{L_a\varphi;\ a \in A(Q)$ and $\varphi \in \mathcal{L}_1\} \cong A(Q) \times \mathcal{L}_1$.*

Corollary 3.7. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. If the nucleus N is of a prime index, then $\mathcal{L}_1$ embeds into $A(Q)$ and $\mathcal{B}$ into $A(Q) \times A(Q)$.*

Proof. Each $\varphi \in \mathcal{L}_1$ is an automorphism of Q that fixes every element of $N = N_\rho$ pointwise. Since the points fixed by $\varphi \in \operatorname{Aut} Q$ form a subloop of Q, and since, by assumption, there is no subloop between N and Q, $\mathcal{L}_1$ must act semiregularly on $Q \setminus N$. Hence each of its orbits is faithful. The rest follows from Lemma 3.3. □

Lemma 3.8. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then $\mathcal{L}$ acts on $\mathcal{B} = \langle \mathcal{L}_u;\ u \in Q\rangle$ by conjugation, and the kernel of this action is equal to $C_{\mathcal{L}}(\mathcal{L}_1) = L_{(N)}\mathcal{L}_1$.*

Proof. By Lemma 3.2 and Corollary 3.5, $\mathcal{B}$ is centralized by $L_{(N)}\mathcal{L}_1$, indeed. On the other hand, for each $x \in Q \setminus N$ one finds $\varphi \in \mathcal{L}_1$ with $L_x\varphi L_x^{-1} \ne \varphi$, by Lemma 1.6. □

By Lemma 3.8, the action of $\mathcal{L}$ on $\mathcal{B}$ has the same kernel as the homomorphism $\mathcal{L} \to Q/N$, $L_x\varphi \mapsto xN$. Hence we can state

Corollary 3.9. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then Q/N embeds into $\operatorname{Aut} \mathcal{B}$, where $\mathcal{B} = \langle L_u;\ u \in Q \rangle$.*

4. Nilpotency

The goal of this section is to prove

Theorem 4.1. *Let Q be an LCC loop with $\mathcal{L}(Q) \trianglelefteq \operatorname{Mlt} Q$. If any of the groups $\operatorname{Mlt} Q$, $\mathcal{L}(Q)$ and $\operatorname{Inn} Q$ is nilpotent, then all of them are nilpotent.*

Since $\operatorname{Inn} Q$ is isomorphic to $\mathcal{L}/Z(\mathcal{L})$, one only needs to prove the implication

$$\mathcal{L} \text{ is nilpotent} \quad \Rightarrow \quad \operatorname{Mlt} Q \text{ is nilpotent.}$$

Some preparatory statements will be needed. The first one is a standard fact of group theory. As such it is stated without a proof.

Lemma 4.2. *Let G be a group with normal subgroups $A = \langle X \rangle$ and $B = \langle Y \rangle$. Then $[A, B] = \langle [a, b];\ a \in A \text{ and } b \in B \rangle$ is a normal subgroup of G. It is the least normal subgroup of G that contains all $xyx^{-1}y^{-1}$, where $x \in X$ and $y \in Y$.*

Lemma 4.3. *Let Q be a loop. Then both $a = \varphi(x)/x$ and $b = x\backslash\varphi(x)$ belong to $A(Q)$, for all $x \in X$ and all $\varphi \in \langle \mathcal{L}_1 \cup \mathcal{R}_1 \rangle$. If $A(Q) \le N_\lambda$, then $L_{\varphi(x)} L_x^{-1} = L_a$. If $A(Q) \le N_\rho$, then $R_{\varphi(x)} R_x^{-1} = R_b$.*

Proof. We have $\varphi(x) \equiv x \bmod A(Q)$ for all $\varphi \in \mathcal{L}_1 \cup \mathcal{R}_1$, which implies $a, b \in A(Q)$. If $a \in N_\lambda$, then $L_{ax} = L_a L_x$. If $b \in N_\rho$, then $R_b R_x = R_{xb}$. □

Lemma 4.4. *Let Q be an LCC loop, and let U be a normal subloop of Q, $U \le N = N_\lambda \cap N_\rho$. Then $R_x^{-1} R_u R_x = R_{T_x(u)}$ for all $u \in U$ and $x \in Q$.*

Proof. From $u \in N_\lambda$ one gets $R_x R_u R_x^{-1} = R_{ux} R_x^{-1}$, which equals $L_x R_v L_x^{-1}$, $v = T_x^{-1}(u)$. The element v belongs to N_ρ since $U \le N_\rho$ is normal. Hence $L_x R_v = R_v L_x$, by point (i) of Lemma 1.9. □

Corollary 4.5. *Let Q be an LCC loop, and let U be a normal subloop of Q, $U \le N = N_\lambda \cap N_\rho$. Consider $x \in X$ and $u \in U$. Then $L_x R_u = R_u L_x$ and $R_x L_u = L_u R_x$. Furthermore,*

$$L_x L_u L_x^{-1} = L_{T_x(u)} \quad \text{and} \quad R_x^{-1} R_u R_x = R_{T_x(u)}.$$

Lemma 4.6. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. If $U \le N$ is a subloop, then $L_{(U)} R_{(U)} = R_{(U)} L_{(U)} = \{L_u R_v;\ u, v \in Q\}$ is a subgroup of $\operatorname{Mlt} Q$. Furthermore, if $U \le N$ is a normal subloop of Q, then $L_{(U)} R_{(U)}$ is a normal subgroup of $\operatorname{Mlt} Q$.*

Proof. We have $L_u R_v = R_v L_u$, $L_{uv} = L_u L_v$ and $R_{vu} = R_u R_v$ for all $u, v \in U \le N$ (c.f. Lemma 1.9). Hence $L_{(U)} R_{(U)} = R_{(U)} L_{(U)}$ is really a subgroup of $\operatorname{Mlt} Q$. By Corollary 2.7, U is normal in Q if and only if $L_{(U)} \trianglelefteq \operatorname{Mlt} Q$. A similar approach based on Lemma 4.4 yields that U is normal in Q if and only if $R_{(U)} \trianglelefteq \operatorname{Mlt} Q$. □

Lemma 4.7. *Let Q be an LCC loop, and let U be a normal subgroup of Q such that $A(Q) \le U \le N = N_\lambda \cap N_\rho$. If $R_{(U)} \mathcal{R}_1$ is a normal subgroup of $\operatorname{Mlt} Q$ and $\mathcal{R}_1 \le \operatorname{Aut} Q$, then $[\operatorname{Mlt} Q, R_{(U)} \mathcal{R}_1] \le L_{(U)} R_{(U)}$.*

Proof. We have $\operatorname{Mlt} Q = \mathcal{L}\mathcal{R}$, and $L_{(U)}R_{(U)} \trianglelefteq \mathcal{L}\mathcal{R}$ follows from Lemma 4.6. By Lemma 4.2 it hence suffices to show that $L_{(U)}R_{(U)}$ contains each of the elements $L_xR_uL_x^{-1}R_u^{-1} = \mathrm{id}_Q$, $R_x^{-1}R_uR_xR_u^{-1} = R_{T_x(u)/u}$, $L_x\varphi L_x^{-1}\varphi^{-1} = L_xL_{\varphi(x)}^{-1} = (L_{\varphi(x)}L_x^{-1})^{-1}$ and $R_x\varphi R_x^{-1}\varphi^{-1} = (R_{\varphi(x)}R_x)^{-1}$, for every $u \in U$, $\varphi \in \mathcal{R}_1$, and $x \in Q$. The identities follow from Lemma 1.9, Lemma 4.4 and Lemma 1.5. The inclusion to $L_{(U)}R_{(U)}$ follows from the normality of U and from Lemma 4.3 (we assume $A(Q) \le U$). □

Lemma 4.8. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Let $U \le N(Q)$ be a normal commutative subloop of Q. Then $V = \{T_x(u)u^{-1};\ u \in U \text{ and } x \in Q\}$ is a normal subloop of Q and $[\mathcal{L}, L_{(U)}] = [\operatorname{Mlt} Q, L_{(U)}] = L_{(V)}$.*

Proof. We have $T_x(u) \in U$ for every $u \in U$ and $x \in Q$, since U is normal. Furthermore, $T_x(u)u^{-1}T_x(v)v^{-1} = T_x(u)T_x(v)u^{-1}v^{-1} = T_x(uv)(uv)^{-1}$ because of the commutativity and Lemma 1.11. Thus V is a subloop of Q. One has $T_y(T_x(u)u^{-1}) = T_yT_x(u) \cdot (T_y(u))^{-1} = T_yT_xT_y^{-1}(T_y(u)) \cdot (T_y(u))^{-1} = T_{(yx)/y}(T_y(u)) \cdot (T_y(u))^{-1}$ for all $x, y \in Q$, by Lemma 1.11. This means that V is a normal subloop. Now, $R_xL_uR_x^{-1}L_u^{-1} = \mathrm{id}_Q$, by Lemma 1.9, and $L_xL_uL_x^{-1}L_u^{-1} = L_{T_x(u)}L_u^{-1} = L_{T_x(u)u^{-1}}$. The rest follows from Corollary 2.7 and Lemma 4.2. □

Lemma 4.9. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, and let $U \le N$ be a normal commutative subloop of Q. Suppose $[\mathcal{L}, L_{(U)}] = L_{(V)}$ for some $V \subseteq Q$. Then $[\operatorname{Mlt} Q, L_{(U)}R_{(U)}] = L_{(V)}R_{(V)}$.*

Proof. We know that $V = \{T_x(u)u^{-1};\ u \in U \text{ and } x \in Q\}$ is a normal subloop of Q, by Lemma 4.8. From Lemma 4.6 we thus see that $L_{(V)}R_{(V)}$ is a normal subloop of $\operatorname{Mlt} Q$. From Corollary 4.5 we obtain $R_x^{-1}R_uR_xR_u^{-1} = R_{T_x(u)}R_u^{-1} = R_{u^{-1}T_x(u))}$, and the rest follows from Lemma 4.2 and from the commutativity of U. □

Let G be a group and let A and B be its normal subgroups. Define

$$\gamma_1[A, B] = [A, B] \quad \text{and} \quad \gamma_{k+1}[A, B] = [A, \gamma_k[A, B]].$$

Then $\gamma_k[A, B]$ are normal subgroups of G and the groups $\gamma_k[G, G]$ form the upper central series of G.

The following lemma follows from elementary group theory, and hence it is stated without a proof.

Lemma 4.10. *Let $\pi : G \to H$ be a surjective group homomorphism. and let A and B be normal subgroups of G. Then $\gamma_k[A, B] \le \pi^{-1}(\gamma_k[\pi(A), \pi(B)])$ for all $k \ge 1$.*

Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, Q nontrivial. Suppose that $\mathcal{L}$ is nilpotent of degree $k+1$ (thus $\operatorname{Inn} Q \cong \mathcal{L}/Z(\mathcal{L})$ is nilpotent of degree k).

We have $\operatorname{Mlt} Q = \mathcal{L}\mathcal{R}$. Consider first the natural homomorphism $\pi : \mathcal{L}\mathcal{R} \to \mathcal{L}/(\mathcal{L} \cap \mathcal{R})$, $\operatorname{Ker} \pi = \mathcal{R}$. Since $\mathcal{L} \cap \mathcal{R}$ contains $Z(\mathcal{L})$, by Proposition 2.3, we see that $\mathcal{L}/(\mathcal{L} \cap \mathcal{R})$ is nilpotent of degree at most k. Hence $\gamma_k[\mathcal{L}\mathcal{R}, \mathcal{L}\mathcal{R}] \le \mathcal{R}$, by Lemma 4.10.

We shall now investigate $\gamma_j[\mathcal{L}\mathcal{R}, \mathcal{R}]$, by means of a homomorphism $\mathcal{L}\mathcal{R} \to Q/M_\rho$ that will be now described. The homomorphism will be denoted again by π, anticipating thus another application of Lemma 4.10. To get π compose the natural homomorphism $\mathcal{L}\mathcal{R} \to \mathcal{R}/(\mathcal{L} \cap \mathcal{R})$, $L_xR_y \mapsto R_y(\mathcal{L} \cap \mathcal{R})$, with the isomorphism $\mathcal{R}/(\mathcal{R} \cap \mathcal{L}) \cong Q/M_\rho$, $R_y(\mathcal{R} \cap \mathcal{L}) \mapsto y^{-1}M_\rho$ of Proposition 2.5. Thus

$\pi(L_xR_y) = y^{-1}M_\rho$. Since $Q/M_\rho \cong \operatorname{Inn} Q/\mathcal{L}_1$, again by Proposition 2.5, we see that Q/M_ρ is nilpotent of class at most k. Now,

$$\gamma_j[\mathcal{L}\mathcal{R}, \mathcal{R}] \le \pi^{-1}(\gamma_j[Q/M_\rho, Q/M_\rho]),$$

by Lemma 4.10. With respect to Proposition 2.5 we hence obtain

$$\gamma_{2k}[\mathcal{L}\mathcal{R}, \mathcal{L}\mathcal{R}] \le \gamma_k[\mathcal{L}\mathcal{R}, \mathcal{R}] \le (\operatorname{Ker} \pi) \cap \mathcal{R} = \mathcal{L} \cap \mathcal{R}.$$

Put $U = M_\rho$. From $\mathcal{L} \cap \mathcal{R} = R_{(U)}\mathcal{R}_1$ it follows $[\mathcal{L}\mathcal{R}, \mathcal{L} \cap \mathcal{R}] \le L_{(U)}R_{(U)}$, by Lemma 4.7. Since U is abelian, by Proposition 2.3, we can employ Lemma 4.9 in an iterative way. We see that $\gamma_j[\mathcal{L}, L_{(U)}] = 1$ if and only if $\gamma_j[\mathcal{L}\mathcal{R}, L_{(U)}R_{(U)}] = 1$. The latter hence certainly holds for $j = k + 1$.

We have descended from $\mathcal{L}\mathcal{R}$ to $\mathcal{R}$, from $\mathcal{R}$ to $\mathcal{L} \cap \mathcal{R}$, from $\mathcal{L} \cap \mathcal{R}$ to $L_{(U)}R_{(U)}$, $U = M_\rho$, and from $L_{(U)}R_{(U)}$ to 1. The described process gives

Proposition 4.11. *Let Q be a nontrivial LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. If Q is nilpotent of class $k+1$, then $\operatorname{Mlt} Q$ is nilpotent of class at most $3k + 2$.*

Theorem 4.1 follows directly from Proposition 4.11. The estimate $3k + 2$ is probably too large, but at this point we do not see how it should be lowered.

5. Loops of prime power order

The aim of this short section is to prove that finite LCC p-loops are nilpotent if $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. The proof is practically the same as that for CC loops [10], which in turn mimicks the standard proof of group theory.

First we need to establish the action of $\operatorname{Inn} Q$ upon N.

Lemma 5.1. *Let Q be a loop with elements u, x and y. If $u \in N_\rho$, $T_y(u) \in N_\mu$ and $T_xT_y(u) \in N_\lambda$, then*

$$T_xT_y(u) = T_xT_y(u).$$

Proof. $T_xT_y(u) \cdot xy = (T_xT_y(u) \cdot x)y = (xT_y(u)) \cdot y = x \cdot (T_y(u) \cdot y) = x \cdot yu = xy \cdot u$. □

Proposition 5.2. *Let Q be an LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Then $Q/Z(N)$ acts upon the nucleus N by $xZ(N) \mapsto J_x$, where J_x is the restriction of T_x to N. The mapping J_x is an automorphism of N, and $T_x \mapsto J_x$ yields a homomorphism $\operatorname{Inn} Q \to \operatorname{Aut} N$ that contains $\mathcal{L}_1$ in its kernel.*

Proof. The nucleus N is a normal subloop of Q, by Proposition 2.3. If $a \in N$, then $L_xL_a = L_{xa}$ for all $x \in Q$, and so $T_{xa} = \Lambda(L_{xa}) = \Lambda(L_x)\Lambda(L_a) = T_xT_a$. Thus $J_{xa} = J_x$ if $a \in Z(N)$, and the mapping $xZ(N) \to J_x$ is a homomorphism $Q/Z(N) \to \operatorname{Aut} N$, by Lemmas 1.11 and 5.1. Now, $M_\rho \le Z(N)$, Q/M_ρ is a group (Proposition 2.4) and $Q/M_\rho \cong \operatorname{Inn} Q/\mathcal{L}_1$ (Proposition 2.5) gives the rest. □

Proposition 5.3. *Let Q be a finite LCC loop with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. If $|Q|$ is a prime power, then Q is nilpotent.*

Proof. Let p be the prime that divides $|Q|$. The nucleus N is nontrivial, by Proposition 2.4, and the orbits induced by the action of $Q/Z(N)$ upon N are of prime power sizes. The number of elements fixed by the action pointwise is hence divisible by the prime p. These elements are those nuclear elements that are fixed by all T_x, $x \in Q$. They form the centre of $Z(Q)$ since the mappings T_x, $x \in Q$, generate $\operatorname{Inn} Q$, by Proposition 2.3. □

6. Nilpotency class two

Up to now we did not give any examples of LCC loops with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$. Of course, CC loops and groups could be regarded as such examples. However, our effort would be in vain, if there were no other examples. To generate them, we shall consider a construction that gives many LCC loops of nilpotency class two. We shall see that the condition $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$ leads to a large class of examples that are in no sense pathological. Their nature seems to suggest that LCC A_r-loops might be worth of study. The first step could be to see which results of this paper can be generalized to such a wider class.

Let G be an abelian group, and let $b : G \times G \to G$ be a mapping. Say that b is *zero preserving* if $b(0, x) = b(x, 0) = 0$ for all $x \in G$. Say that b is *additive on the right*, if $b(x, y + z) = b(x, y) + b(x, z)$. Mapping b is *biadditive*, if it is addtive both on the left and the right (similarly one defines *triadditive* mappings etc.). The set of all $r \in G$ with

$$b(x + r, y) = b(x, y) = b(x, y + r) \quad \text{for all} \quad x, y \in G$$

is called the *radical* of b, and is denoted by $Rad(b)$. It is clear that $Rad(b)$ is a subgroup of G.

Theorem 6.1. *Let G be an abelian group and let $b : G \times G \to G$ be a zero preserving mapping with $Rad(b) \geq Im(b)$ that is additive on the right. Then*

$$x \cdot y = x + y + b(x, y)$$

defines on G a left conjugacy closed loop of nilpotency class two. Put $f(x, y, z) = b(x + y, z) - b(x, z) - b(y, z)$, for all $x, y, z \in Q$. Then

$$L(x, y)(z) = z - f(x, y, z) \quad \textit{and}$$
$$R(x, y)(z) = z + f(z, y, x).$$

Proof. All facts of the lemma can be verified easily in a direct way. One can also use [3, Theorem 5.3], where the construction seems to have appeared for the first time. (The formulas for $L(x, y)$ and $R(x, y)$ follow from [3, Lemma 5.2].) □

The loop of Theorem 6.1 will be denoted by $G[b]$.

Theorem 6.2. *Let $Q = G[b]$, where G is an abelian group, and $b : G \times G \to G$ is a zero preserving mapping that is additive on the right and satisfies $Rad(b) \geq Im(b)$. Put $f(x, y, z) = b(x + y, z) - b(x, z) - b(y, z)$, for all $x, y, z \in Q$. Then*

(i) *Q is a group if and only if b is biadditive;*
(ii) *Q is a CC loop if and only if f is symmetric triadditive; and*
(iii) *Q is a A_r-loop if and only if f is triadditive.*

Proof. Point (i) is immediate from Theorem 6.1. Point (ii) is not difficult as well. The proof (see [7, Corollary 2.2]) is based on the fact that Q is RCC if and only if $R(x, y) = R(y, x)$ for all $x, y \in Q$ (see [3, Corollary 3.2]). From Theorem 6.1 we see that this takes place if and only if $f(u, v, w) = f(u, w, v)$ for all $u, v, w \in G$. Nothing more is needed, since we always have $f(u, v, w) = f(v, u, w)$ and $f(u, v, w_1 + w_2) = f(u, v, w_1) + f(u, v, w_2)$.

To prove point (iii) one needs to compose the equality which expresses the compatibility of $R(x,y)$. Now $R(x,y)(uv) = R(x,y)(u+v+b(u,v)) = u+v+b(u,v)+f(u+v,y,x)$ and $R(x,y)(u)\cdot R(x,y)(v) = (u+f(u,y,x))\cdot(v+f(v,y,x)) = u+v+b(u,v)+f(u,y,x)+f(v,y,x)$. Hence $f(u+v,y,x) = f(u,y,x)+f(v,y,x)$ is the sought equality, and the rest is clear. $\square$

The number of triadditive mappings that are symmetric in the first two coordinates is generally much larger than the number of (completely) symmetric triadditive mappings. However, some discussion is needed to see that every triadditive mapping f can be obtained from some $b : G\times G \to G$ that satisfies the assumptions of Theorem 6.2.

We shall restrict the discussion to the loops $V(b)$ that are defined on $V \times F$, where V is a finite vector space over a prime field F, $b : V \times V \to F$ is a zero preserving mapping that is additive on the right, and the loop operation is given by

$$(u,i)\cdot(v,j) = (u+v, i+j+b(u,v)).$$

Clearly $V(b) = G[\bar{b}]$, where $G = V \times F$ and $\bar{b}((u,i),(v,j)) = (0, b(u,v))$.

Put $f_b(u,v,w) = b(u+v,w) - b(u,w) - b(v,w)$. Mappings $L(x,y)$ get the form

$$(w,i) \mapsto (w, i - f_b(u,v,w)),$$

and mappings $R(y,x)$ are of the form

$$(w,i) \mapsto (w, i + f_b(w,v,u)).$$

Thus $V(b)$ is CC when f_b is symmetric trilinear, and $V(b)$ is an A_r-loop when f_b is trilinear, by Theorem 6.2.

Lemma 6.3. *Let $f : V\times V\times V \to F$ be a trilinear form with $f(u,v,w) = f(v,u,w)$ for all $u,v,w \in V$. If $|F| = 2$, also assume $f(u,u,v) = 0$ for all $u,v \in V$. Then there exists $b : V \times V \to F$ that is zero preserving, right linear, and satisfies*

$$f(u,v,w) = b(u+v,w) - b(u,w) - b(v,w)$$

for all $u,v,w \in V$.

Proof. For $|F| \geq 3$ set $b(u,v) = \frac{1}{2}f(u,u,v)$. For $|F| = 2$ choose a basis $e_1, \ldots, e_n \in V$ and an $n \times n$ matrix $\eta = (\eta_{ik})$ over F. Define $b = b_\eta$ by

$$b(\sum \alpha_i e_i, \sum \beta_k e_k) = \sum_{i<j} \alpha_i\alpha_j\beta_k f(e_i,e_j,e_k) + \sum \alpha_i\beta_k\eta_{ik},$$

and verify that b satisfies the required equality. $\square$

A proof identical with that of [7, Lemma 2.5] can be used to show that every b fulfilling the equality of Lemma 6.3 coincides with some b_η of the proof.

Proposition 6.4. *Let V be a finite vector space over a prime field F, and let $b : V\times V \to F$ be a zero preserving right linear form. Put $Q = V(b)$ and $f(u,v,w) = b(u+v,w) - b(u,w) - b(v,w)$. If f is not trilinear, then $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$ does not hold. Assume that f is trilinear. Define subspaces L^* and R^* of $V^* = \operatorname{Hom}(V,F)$ by*

$$L^* = \langle f(u,v,-);\ u,v \in V\rangle \quad \textit{and} \quad R^* = \langle f(-,u,v);\ u,v \in V\rangle.$$

Set also

$$Rad_L = \{x \in V;\ f(x,u,v) = 0 \quad \text{for all} \quad u,v \in V\}$$
$$\text{and} \quad Rad_R = \{x \in V;\ f(u,v,x) = 0 \quad \text{for all} \quad u,v \in V\}.$$

Then $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q \Leftrightarrow \mathcal{R}_1 \le \mathcal{L}_1 \Leftrightarrow R^* \le L^* \Leftrightarrow Rad_R \le Rad_L$*, and* $R^* = L^* \Leftrightarrow \mathcal{L}_1 = \mathcal{R}_1 \Leftrightarrow Rad_R = Rad_L$*. Furthermore, the loop* Q *is conjugacy closed if and only if* f *is symmetric.*

Proof. Facts that do not concern L^* and R^* have been already explained (see the discussion preceding Lemma 6.3).

The mapping $f(u,v,-) : w \mapsto f(u,v,w)$ is always a linear form, indeed. (If f is trilinear, then one gets linear forms by fixing any two coordinates.) The left inner mapping group is generated by mappings $(w,i) \mapsto (w, i - f(u,v,w))$. Now, if φ_1, φ_2 are linear forms, and ψ_j are defined by $\psi_j(w,i) = (w, i + \varphi_j(w))$, $j \in \{1,2\}$, then $\psi_1\psi_2(w,i) = (w, i + \varphi_1(i) + \varphi_2(i)) = (w, i + (\varphi_1 + \varphi_2)(w))$. The composition in $\mathcal{L}_1$ (and similarly in $\mathcal{R}_1$) thus corresponds to the addition in L^* (or R^*, respectively). This couples the properties of $\mathcal{L}$ and $\mathcal{R}$ with L^* and $R*$. The subspaces of V^* are characterized by their annihilators, and that gives the connection to Rad_R and Rad_L. □

We shall be now looking for a nonsymmetric trilinear f with $R^* < L^*$ (or $R^* = L^*$) for which V attains the least possible dimension. By Lemma 6.3 the only condition on f is that $f(u,v,w) = f(v,u,w)$ for all $u,v \in V$; for $|F| = 2$ there also must be $f(u,u,v) = 0$.

Put $p = |F|$, and let $e = (e_1, \dots, e_n)$ be a basis of V. A trilinear form f that satisfies our conditions is completely determined by all values $f(e_i, e_j, e_k)$, where $i \le j$ if $p > 2$, and $i < j$ if $p = 2$. Note that the dimension of Rad_R (or Rad_L) is n if and only if f is constant zero.

Lemma 6.5. *Assume* $R^* < L^*$*. If* $p \ge 3$*, then* $n \ge 3$*. If* $p = 2$*, then* $n \ge 5$*.*

Proof. Denote the dimension of Rad_L by k, and assume $Rad_R < Rad_L$. If $p \ge 3$, then L^* is generated by $g = \binom{n-k+1}{2}$ linear forms, and if $p = 2$, then by $g = \binom{n-k}{2}$ forms. The dimension of Rad_R is equal to the codimension of L^*, and hence it is not less than $n - g$. Thus $Rad_R < Rad_L$ implies $k > n - g$, and our assumptions also give $n > k \ge 1$.

If $k = n-1$, then $g \le 1$ and there cannot be $k + g > n$. Thus $n - 2 \ge k \ge 1$, and hence $n \ge 3$. Suppose now $p = 2$. If $k = n-2$, then $k + g = n - 1$, and if $k = n - 3$, then $k + g = n$. Therefore $n - 4 \ge k \ge 1$, and we get $n \ge 5$. □

It is not difficult to construct f such that $R^* < L^*$ and n attains the limit values of Lemma 6.5. In both cases we shall have $Rad_L = \langle e_n \rangle$ and $Rad_R = 0$. For $p \ge 3$ set

$$\begin{aligned}
(\, f(e_1,e_1,e_1),\ f(e_1,e_1,e_2),\ f(e_1,e_1,e_3)\,) &= (0,0,1),\\
(\, f(e_2,e_2,e_1),\ f(e_2,e_2,e_2),\ f(e_2,e_2,e_3)\,) &= (0,1,0), \quad \text{and}\\
(\, f(e_1,e_2,e_1),\ f(e_1,e_2,e_2),\ f(e_1,e_2,e_3)\,) &= (1,0,0).
\end{aligned}$$

This determines f completely, since we assume $e_3 \in Rad_L$.

The case $p = 2$ can be treated similarly. Let $u(i,j)$ equal $(f(e_i,e_j,e_1),\dots,f(e_i,e_j,e_5))$. Set

$$u(1,2) = (0,0,0,0,1),\quad u(1,3) = (0,0,0,1,0),$$
$$u(1,4) = (0,0,1,0,0),\quad u(2,3) = (0,1,0,0,0),$$
$$u(2,4) = (1,0,0,0,0),\quad \text{and}\quad u(3,4) = 0.$$

Note that f is never (completely) symmetric when $Rad_L \neq Rad_R$.

We shall now consider the problem of the least orders, for which there exist proper LCC loops Q with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$ (i.e., LCC loops Q that are not right conjugacy closed and satisfy $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$.)

A loop Q is said to be *extraspecial* if $Z(Q)$ is of order p, p a prime, and $Q/Z(Q)$ is an elementary abelian p-group.

Lemma 6.6. *Let Q be a nilpotent LCC loop of order 8. Then either $|N_\rho| \geq 4$, or Q is an extraspecial loop.*

Proof. Start from the contrary. Then clearly $N_\rho = Z(Q) = \{1, z\}$ for some $z \in Q$, $z \neq 1$. Put $Z = Z(Q)$. The loop Q/Z must be a cyclic group of order 4. Hence there exists a subloop $U < Q$ of index 2 such that $Z < U$ and $x^2 \notin Z$ for all $x \in Q \setminus U$.

Suppose first $x \cdot x^2 = x^2 \cdot x$ for all $x \in Q \setminus V$. Then $L_x L_{x^2} L_x^{-1} = L_{T_x(x^2)} = L_{x^2}$ for each such x. Since $U = \{x^2, x^2 z, z, 1\}$, we see that in fact each L_u, $u \in U$, commutes with L_x. Subloop U is an abelian group, and so $L_u L_v = L_v L_u$ for all $u, v \in U$. Hence $L_u \in Z(\mathcal{L})$ for each $u \in U$, and thus $U \leq M_\rho \leq N_\rho$, a contradiction.

Fix now $x \in Q \setminus U$ with $xx^2 \neq x^2 x$. The quotient Q/Z is an abelian group, and hence $xx^2 = x^2 xz$, $L_x L_{x^2} L_x^{-1} = L_{x^2 z}$ and so $L_x L_{x^2} = L_{x^2} L_x L_z$.

Denote by φ the automorphism $L_{x^2}^{-1} L_x L_x$. From $xx^2 = x^2 xz$ one gets $\varphi(x) = xz$, and $\varphi(xx^2) = xz \cdot (xz \cdot xz) = xx^2 z = x^2 x$ follows. We also have $\varphi(x^2) = xz \cdot xz = x^2$. Thus φ sends each $y \in Q \setminus U$ to yz, and φ fixes each $u \in U$. We cannot have $T_{x^2} = \varphi$, since in an LCC loop $T_y \in \operatorname{Aut} Q$ if and only if $y \in N_\rho$. However, below we shall prove $T_{x^2} = \varphi$, and that will constitute the contradiction.

Clearly, $T_{x^2}(u) = u$ for all $u \in Q$, and $T_{x^2}(x) = xz$. It remains to show $x^2 \cdot x^2 x = xx^2 \cdot x^2$. We shall prove $L_{x^2} L_{x^2 x} = L_{xx^2} L_{x^2}$, from which the sought identity follows if both sides are applied to 1. To get $L_{x^2} L_{x^2 x} L_{x^2}^{-1} = L_{xx^2}$ express L_{x^2} as $L_x^2 \varphi^{-1} = L_x^2 \varphi$. Thus $L_{x^2} L_{x^2 x} L_{x^2}^{-1} = L_x^2 L_{\varphi(x^2 x)} L_x^{-2} = L_x^2 L_{xx^2} L_x^{-2}$. There exists $\epsilon \in \{0, 1\}$ such that $L_x L_{xx^2} L_x^{-1} = L_{xx^2} L_z^\epsilon$, as $T_x(xx^2) \equiv xx^2 \bmod Z$. Hence $L_x^2 L_{xx^2} L_x^{-2} = L_{xx^2} L_z^{2\epsilon} = L_{xx^2}$, and we are done. □

Theorem 6.7. *Let p be an odd prime. If Q is an LCC loop of order p^2 or 8 such that $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, then Q is conjugacy closed. There exist LCC loops Q with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$ that are not conjugacy closed and are of orders p^3 and 16.*

Proof. Note that the considered loops Q of orders p^2 or 8 are nilpotent, by Proposition 5.3. All nonassociative nilpotent LCC loops Q of order p^2 are of the form $G[b]$, by [3, Theorem 6.3]. Such loops have the associated mapping f triadditive, by point (iii) of Theorem 6.2, and the mapping is symmetric since $Q/Z(Q)$ is a one-dimensional vector space. Our assertion thus follows in this case from point (ii) of Theorem 6.2.

If $|Q| = 8$, $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$ and $|N_\rho| = 4$, then $N = N_\rho$, by Proposition 2.3. However, loops with nucleus of index two are conjugacy closed, by [8]. Thus we can assume that $N_\rho = Z(Q)$ consists of two elements. But that means that Q is extraspecial, by Lemma 6.6. Now, extraspecial LCC loops are described in [7], and Proposition 4.7 of that paper claims that in the even case all extraspecial LCC loops are of the form $V(b)$. If the order of such a loop is equal to 8, then $\dim V = 2$, and so $Rad_R \le Rad_L$ implies $k + g \ge 2$, where $k = \dim Rad_R$ and $g = \binom{2-k}{2}$ is the number of forms that generate L^*. However, $k + g < 2$ for both possible choices $k = 0$ and $k = 1$, and hence on eight elements there really exists no proper LCC loop Q with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$.

The constructive part of the theorem is easy: it suffices to construct an appropriate trilinear form $f : V \times V \times V \to F$, F a field of prime order, such that f is not (completely) symmetric and satisfies $R^* = L^* = V^*$, with $n = \dim V = 2$ in the odd case, and $n = \dim V = 3$ for $F = \{0, 1\}$.

Let $e_1, \dots, e_n$ be a basis of V. In the odd case consider f such that $f(e_i, e_j, e_k) \in \{0, 1\}$ for all $i, j, k \in \{1, 2\}$, and $f(e_i, e_j, e_k) = 1$ if and only if $i = j = k$ or $(i, j, k) = (1, 2, 2)$ or $(i, j, k) = (2, 1, 2)$. Then both $f(e_i, e_i, -)$ and $f(-, e_i, e_i)$, $i \in \{1, 2\}$, generate V^*.

In the even case set $f(e_i, e_{i+1}, e_{i+1}) = f(e_{i+1}, e_i, e_{i+1}) = 1$ for all $i \in \{1, 2, 3\}$ and assume that f gives zero on the other base triples (the index $i + 1$ is computed modulo 3, of course). Then V^* is generated by both $f(e_i, e_{i+1}, -)$ and $f(-, e_i, e_{i+1})$. $\square$

From the constructions following Lemma 6.5 we know that there exist LCC loops Q with $\mathcal{R}_1 < \mathcal{L}_1$ that are of order p^4, p an odd prime, and of order 64. At this point it does not seem to be clear whether such loops exist for orders p^3, 16 and 32. We guess that for orders p^3 and 16 they do not exist, and that should not be too difficult to prove. The order 32 might need a more thorough investigation.

Kiechle and Nagy described all involutorial Bol loops of order 16 (see [9]). Such loops are LCC and they form 34 isomorphism classe. Later Kiechle checked the list and found that it contains exactly two proper LCC loops Q with $\operatorname{Mlt} Q \trianglelefteq Q$. These were probably the first two explicit examples of the loops that are studied in this paper, and we thank Hubert Kiechle for the kindness with which he made the necessary computations and fulfilled thus a wish of one of the present authors.

It seems that the involutorial Bol loops Q with $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$ could be described, at least to some extent, by the methods of this paper. That is deferred to future research.

The condition $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, where Q is a left conjugacy closed loop, is equivalent to $\mathcal{R}_1 \le \mathcal{L}_1$. One could say that the results of this paper indicate that if $\mathcal{R}$ is small (relative to $\mathcal{L}$), then Q does not go far from groups. The question is, how much we could say about Q, when we $\mathcal{R}$ is big (say $\mathcal{R} \ge \mathcal{L}$). Some sort of answer to such a question could have impact on the quest for proper simple involutorial Bol loops.

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