THE ORDERING OF COMMUTATIVE TERMS

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ABSTRACT. By a commutative term we mean an element of the free commutative groupoid F of infinite rank. For two commutative terms a, b write $a \leq b$ if b contains a subterm that is a substitution instance of a. With respect to this relation, F is a quasiordered set which becomes an ordered set after the appropriate factorization. We study definability in this ordered set. Among other thing, we prove that every commutative term (or its block in the factor) is a definable element. Consequently, the ordered set has no automorphisms except the identity.

0. INTRODUCTION

The investigation of definability in the quasiordered set of terms, or in the ordered set of term patterns, is motivated by an effort to solve the questions of definability in the lattice of equational theories. Let us say that a variety V has positive definability if the lattice L_V of equational theories of V-algebras (or the lattice of all subvarieties of V, which is antiisomorphic to L_V) has the following properties:

(1) the lattice L_V has no automorphisms except the obvious ones,
(2) every finitely based element of L_V is definable up to the obvious automorphisms,
(3) the set of finitely based elements of L_V is definable,
(4) the set of one-based elements of L_V is definable,
(5) the equational theory of every finite algebra from V is definable in L_V up to the obvious automorphisms, and
(6) the set of equational theories of finite algebras of V is a definable subset of L_V.

It has been proved in a series of papers [1], [2], [3], [4] that for an arbitrary fixed signature, the variety of all algebras of that signature has positive definability. The series can serve as a prototype for the investigation of definability for some other interesting varieties. However, the technique used there can be applicable only to the balanced varieties, i.e., varieties based on balanced equations (equations where every variable and every operation

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symbol has the same number of occurrences at the left as at the right). Examples are the variety of semigroups, the variety of commutative semigroups, the variety of commutative groupoids, the variety of medial groupoids, etc.

An attempt [5] to prove that the variety of semigroups has positive definability was not completely successful. There are many partial results in support of the conjecture, and at least the items (5) and (6) have been answered in the positive.

The least balanced variety of groupoids is the variety of commutative semigroups. Surprisingly, in the recent paper [6] it was discovered that in this case the lattice of equational theories has uncountably many automorphisms, so that the variety has negative definability.

It seems that no other balanced variety has been considered in this context. The most natural candidate is the variety of commutative groupoids. In the present paper we are going to make a first step in this direction.

When trying to imitate the process outlined in [1]–[4], one crucial step is to investigate definability in the ordered set of term patterns; it was the part [2] in which this was done for universal algebras. For semigroups, this part was quite short, as elements of the free semigroup have more simple structure than elements of the free groupoid. For commutative groupoids, the structure might seem to be of about the same complexity as in the case of (general) groupoids. There is an advantage, making the matter even less technically complicated, consisting in the absence of obvious nonidentical automorphisms, so that we will not need to introduce a special parameter in formulas for the purpose of handling those automorphisms. On the other hand, it turns out that not much from [2] can be taken over. An essential drawback is that while elements of the free groupoid can be imagined as static binary trees, where each branch has a fixed position, in the commutative case we should imagine the same trees but with all branches rotating at different speeds.

Although it is not consistent with the generally accepted terminology, by a term we mean in this paper an element of the free commutative groupoid (rather than an element of the free groupoid). If we wished to set it right, we should replace every occurrence of the word 'term' by 'commutative term' in the following text.

1. Preliminaries

Let X be a (fixed) infinite countable set. Its elements will be called variables. We denote by F the free commutative groupoid over X. Its characteristic properties are that it is a commutative groupoid generated by X, $ab \notin X$ for all $a, b \in F$, and whenever $ab = cd$ in F then either $(a, b) = (c, d)$ or $(a, b) = (d, c)$. The elements of F will be called terms.

The unique homomorphism of F into the additive semigroup of natural numbers sending all variables to 1 will be denoted by λ. The number $\lambda(a)$ is called the length of a term a.

We will write $a_1a_2\dots a_n$ instead of $((a_1a_2)\dots)a_n$. Similarly, $ab\cdot cd\cdot efg$ stands for $((ab)(cd))((ef)g)$, etc.

A term b is said to be a subterm of a term a if a can be written as $a=bc_1\dots c_n$ for some $n\ge 0$ and some terms $c_1,\dots,c_n$. We write $b\subseteq a$ if b is a subterm of a; we write $b\subset a$ if b is a proper subterm of a, i.e., $b\subseteq a$ and $b\ne a$. The set of subterms of a is a finite subset of F. It could be also defined by induction on the length of a as follows: if a is a variable, then a is the only subterm of a; if $a=bc$, then a term is a subterm of a if and only if it either equals a or is a subterm of either b or c. We denote by $\mathbf{S}(a)$ the set of the variables that are subterms of a; its elements are called variables occurring in a.

For a variable x we denote by ν_x the homomorphism of F into the additive semigroup of natural numbers sending x to 1 and all other variables to 0. For $a\in F$, $\nu_x(a)$ is called the number of occurrences of x in a.

Let t,a,b be three terms. If t can be written as $t=ac_1\dots c_n$ for some $c_1,\dots,c_n$ then $bc_1\dots c_n$ is said to be a term obtained from t by replacing (one occurrence of) a with b. Observe that it is not uniquely determined by the triple t,a,b.

By a linear term we mean a term a such that $\nu_x(a)\le 1$ for all variables x.

By a slim term we mean a term that can be written as $x_1x_2\dots x_n$ for some $n\ge 1$ and some (not necessarily distinct) variables $x_1,\dots,x_n$. A slim term $x_1x_2\dots x_n$ is said to be rooted at x_1. (If $n\ge 2$, then it is also rooted at x_2.)

By a unary term we mean a term a such that $\mathbf{S}(a)=\{x\}$ for a variable x.

By the depth of a term a we mean the largest positive integer n such that a can be written as $a=b_1b_2\dots b_n$ for some terms $b_1,\dots,b_n$. The depth of a will be denoted by $\delta(a)$.

By a substitution we mean an endomorphism of the groupoid F. By a substitution instance of a term a we mean any term that can be expressed as $f(a)$ for a substitution f. Given a variable x and a term a, we denote by σ^x_a the substitution f such that $f(x)=a$ and $f(y)=y$ for every variable $y\ne x$.

If a term a is written as $a=a(x_1,\dots,x_n)$ then we assume that $x_1,\dots,x_n$ are pairwise distinct variables and $\mathbf{S}(a)\subseteq\{x_1,\dots,x_n\}$. In that case, for any n-tuple $b_1,\dots,b_n$ of terms we denote by $a(b_1,\dots,b_n)$ the term $f(a)$ where f is (any) substitution such that $f(x_i)=b_i$ for $i=1,\dots,n$.

For $a,b\in F$ we write $a\le b$ if there exists a substitution f such that $f(a)$ is a subterm of b. This relation is a quasiordering of F satisfying the minimal condition. We write $a<b$ if $a\le b$ and $b\not\le a$. We write $a\parallel b$ (and say that the two terms are incomparable) if neither $a\le b$ nor $b\le a$.

If $a\le b$ and $b\le a$, we write $a\sim b$ and say that the terms a,b are similar (or also, that b can be obtained from a by renaming variables). Clearly, $a\sim b$ if and only if $b=\alpha(a)$ for an automorphism α of the groupoid F. The relation $\sim$ is an equivalence on F (it is not a congruence).

The quasiordering $\leq$ of F induces an ordering on the set $F/{\sim}$, which will be also denoted by $\leq$. The elements of $F/{\sim}$ are called patterns; $a/{\sim}$ is the pattern of a term a.

For a term a we denote by $O(a)$ the ordered set of the patterns that are less or equal to $a/{\sim}$. For example, if x, y, z are three distinct variables then $O(x)$ is the one-element ordered set, $O(xy)$ is the two-element chain and $O(xx)$ and $O(xyz)$ are three-element chains.

Since two similar terms have the same length, it makes sense to speak about the length of a pattern. Similarly, we can speak about the depth of a pattern and about linear, unary and slim patterns. On the other hand, there is nothing like a product of two patterns.

It is easy to see that for every term a there exists a linear term b, unique up to similarity, such that $a = f(b)$ for a substitution f sending variables to variables. This linear term will be called the linear hull of a and denoted by $\mathbf{lh}(a)$. Since it is determined only up to similarity, it is better to write $b \sim \mathbf{lh}(a)$. For example, $\mathbf{lh}(xyx \cdot zy) \sim x_1x_2x_3 \cdot x_4x_5$.

By the unary hull of a term a we mean the term $f(a)$ where f is a substitution sending all variables to one fixed variable. It is again determined by a uniquely up to similarity. If b is the unary hull of a, we write $b \sim \mathbf{uh}(a)$.

Let P be an ordered set. An n-ary relation R on P is called definable if there exists a first-order formula $\varphi(x_1, \dots, x_n)$ with free variables $x_1, \dots, x_n$ in the language of ordered sets, such that for any elements $a_1, \dots, a_n$ of P, $\varphi(a_1, \dots, a_n)$ is satisfied in P if and only if $(a_1, \dots, a_n) \in R$. A subset of P is called definable if it is definable as a unary relation. An element a of P is called definable if the set $\{a\}$ is definable.

Let Q be a quasiordered set. Then $Q/{\sim}$ is an ordered set, where $a \sim b$ means $a \leq b$ and $b \leq a$. An n-ary relation R on Q is called definable if it is invariant under $\sim$ and the relation $R/{\sim}$, defined by $(a_1/{\sim}, \dots, a_n/{\sim}) \in R/{\sim}$ if and only if $(a_1, \dots, a_n) \in R$, is definable in $R/{\sim}$. This is the same as to say that there exists a first-order formula $\varphi(x_1, \dots, a_n)$ with free variables $x_1, \dots, x_n$ without equality sign in the language of ordered sets, such that for any elements $a_1, \dots, a_n$ of Q, $\varphi(a_1, \dots, a_n)$ is satisfied in Q if and only if $(a_1, \dots, a_n) \in R$.

So, to investigate definability in the quasiordered set of terms is the same as to investigate definability in the ordered set of patterns. The differences are only technical. It is more safe to think in patterns.

Clearly, the binary relations $\leq, <, ||, \sim$ are definable.

1.1. **Lemma.** *Let a, b be two terms and f be a substitution. If $a \subseteq b$ then $f(a) \subseteq f(b)$. If $a \subset b$ then $f(a) \subset f(b)$.*

Proof. It is obvious. □

1.2. **Lemma.** *Let $a \leq b$. If b is linear, then a is also linear. If b is slim, then a is also slim. All slim linear terms are comparable with each other.*

Proof. This is obvious. □

1.3. **Lemma.** *Let a be a term and $x_1, \dots, x_n$ be pairwise distinct variables not occurring in a. Then every term b such that $a \le b \le ax_1 \dots x_n$ is similar to $ax_1 \dots x_i$ for some $i \in \{0, \dots, n\}$.*

Proof. By induction on the length of a. For $a \in X$ this follows from 1.2. Let $a \notin X$. Suppose that there is a term b such that $a < b < ax_1 \dots x_n$ and $b \not\sim ax_1 \dots x_i$ for all i, and take a minimal such term b. There are two substitutions f, g such that $f(a) \subseteq b$ and $g(b) \subseteq ax_1 \dots x_n$. Clearly, $g(b) = ax_1 \dots x_m$ for some $1 \le m \le n$. From this it follows that $b = cy$ for a term c and a variable $y \notin \mathbf{S}(c)$. If $f(a) \subseteq c$ then $a \le c < b \le ax_1 \dots x_n$; by the minimality of b, $c \sim ax_1 \dots x_i$ for some $i < n$; but then $b \sim ax_1 \dots x_{i+1}$. So, $f(a) \not\subseteq c$ and then $f(a) = b$. We have $a = a_1 a_2$ for two terms a_1, a_2 such that $f(a_1) = c$ and $f(a_2) = y$. Since $y \notin \mathbf{S}(c)$, a_2 is a variable not contained in $\mathbf{S}(a_1)$; denote this variable by x_0. Since $a_1 \le b \le a_1 x_0 x_1 \dots x_n$, by the induction assumption we get $b \sim a_1 x_0 x_1 \dots x_i = ax_1 \dots ax_i$ for some i. □

2. Covers

For two terms a, b we write $a \prec b$ if $a < b$ and there is no term c with $a < c < b$. If $a \prec b$, we say that a is covered by b or also that b is an (upper) cover of a or also that a is a lower cover of b.

We write $a \prec_1 b$ if $b \sim ax$ for a variable $x \notin \mathbf{S}(a)$.

We write $a \prec_2 b$ if $b \sim \sigma^x_{xy}(a)$ for a variable $x \in \mathbf{S}(a)$ and a variable $y \notin \mathbf{S}(a)$.

We write $a \prec_3 b$ if $b \sim \sigma^x_y(a)$ for two different variables $x, y \in \mathbf{S}(a)$.

2.1. **Theorem.** *Let a, b be two terms. Then $a \prec b$ if and only if either $a \prec_1 b$ or $a \prec_2 b$ or $a \prec_3 b$. We can never have $a \prec_3 b$ and $a \prec_i b$ for $i \in \{1, 2\}$ at the same time. If $a \prec_1 b$ and $a \prec_2 b$ then the terms a, b are both slim and linear.*

The proof of this theorem will be divided into several lemmas and will be finished at the end of this section.

2.2. **Lemma.** *If $a \prec b$ then either $a \prec_1 b$ or $a \prec_2 b$ or $a \prec_3 b$.*

Proof. Let $a \prec b$. There exists a substitution f with $f(a) \subseteq b$. If $f(a) \subset b$ then $f(a) \sim a$ and we have $a < ax \le b$, so that $b \sim ax$ and $a \prec_1 b$. If $f(a) = b$, then f cannot map $\mathbf{S}(a)$ injectively into X; if $f(x) \notin X$ for some $x \in \mathbf{S}(a)$, then one can easily see that $a \prec_2 b$; if $f(x) \in X$ for all $x \in \mathbf{S}(a)$, then $a \prec_3 b$. □

2.3. **Lemma.** *Let a, b be two terms such that $a < b$. If $\lambda(a) = \lambda(b)$ then* $\operatorname{Card} \mathbf{S}(b) < \operatorname{Card} \mathbf{S}(a)$. *If $\lambda(b) = \lambda(a) + 1$ then* $\operatorname{Card} \mathbf{S}(b) \le \operatorname{Card} \mathbf{S}(a) + 1$.

Proof. We have $f(a) \subseteq b$ for a substitution f. If $\lambda(a) = \lambda(b)$ then f maps $\mathbf{S}(a)$ into X and this mapping cannot be injective, since $f(a) = b \not\sim a$. Let $\lambda(b) = \lambda(a) + 1$. If $f(a) \subset b$ then f maps $\mathbf{S}(a)$ into X and $b = f(a)x$ for a variable x. If $f(a) = b$ then f sends all variables from $\mathbf{S}(a)$ to variables except one, which is sent to the product of two variables. □

2.4. Lemma. *If either $a \prec_1 b$ or $a \prec_3 b$ then $a \prec b$. If $a \prec_2 b$ and $\lambda(b) = \lambda(a) + 1$ then $a \prec b$.*

Proof. This follows from 2.3. □

2.5. Lemma. *Let a be a term, b be a subterm of a, $x \in \mathbf{S}(b)$, $y \in X - \mathbf{S}(b)$ and let there exist a substitution f such that $f(a) = \sigma^x_{xy}(b)$. Then either $b = a$ or a is a slim linear term rooted at x.*

Proof. By induction on the length of a. Suppose that b is a proper subterm of a. If $b = x$, then clearly a is a product of two different variables, one of which must be x. Now let $b = b_1 b_2$ for two terms b_1 and b_2. We have $a = a_1 a_2$ for two terms a_1, a_2 such that $b \subseteq a_1$. Since $f(a_1)f(a_2) = \sigma^x_{xy}(b_1)\sigma^x_{xy}(b_2)$, we have $f(a_1) = \sigma^x_{xy}(b_i)$ for an $i \in \{1, 2\}$. But b_i is a proper subterm of a_1, so $x \in \mathbf{S}(b_i)$ and, by induction, a_1 is a slim linear term rooted at x. We have $a_1 = x_1 x_2 \dots x_n$ for some pairwise different variables $x_1, \dots, x_n$ where $x_1 = x$ and $b = x_1 x_2 \dots x_m$ for some m, $2 \le m \le n$. Now $f(a)$ is of depth at least $n+1$, while $\sigma^x_{xy}(b)$ is of depth $m+1$. Since $f(a) = \sigma^x_{xy}(b)$, it follows that $m = n$, $b = a_1$ and $b_i = x_1 \dots x_{n-1}$. Then $f(a_2)$ is a variable not occurring in $f(a_1)$. Consequently, a_2 is a variable not occurring in a_1 and a is a slim linear term rooted at x. □

2.6. Lemma. *Let $a \prec_3 b \prec_2 c$. Then either $a \prec_2 d \prec_3 c$ for some d or $a \prec_2 d_1 \prec_2 d_2 \prec_3 d_3 \prec_3 c$ for some d_1, d_2, d_3.*

Proof. Let $b = \sigma^x_y(a)$ and $c = \sigma^z_{zu}(b)$. If $z \ne y$ then $a \prec_2 \sigma^z_{zu}(a) \prec_3 c$. If $z = y$ then, for a new variable v, $a \prec_2 \sigma^x_{xv}(a) \prec_2 \sigma^y_{yu}\sigma^x_{xv}(a) \prec_3 \sigma^v_u\sigma^y_{yu}\sigma^x_{xv}(a) \prec_3 \sigma^x_y\sigma^v_u\sigma^y_{yu}\sigma^x_{xv}(a) = c$. □

For a term a and a variable $x \in \mathbf{S}(a)$ denote by $\kappa_x(a)$ the least positive integer such that $a = xu_2 \dots u_n$ for some terms $u_2, \dots, u_n$. For a term a and two positive integers n, m denote by $\mu_{n,m}(a)$ the (total) number of occurrences of the variables x in a such that $\nu_x(a) \ge n$ and $\kappa_x(a) \le m$, i.e.,

$$\mu_{n,m}(a) = \sum\{\nu_x(a) : \nu_x(a) \ge n,\ \kappa_x(a) \le m\}.$$

2.7. Lemma. (1) *Let a be a term and $x, y \in \mathbf{S}(a)$ be two distinct variables. Then $\mu_{n,m}(a) \le \mu_{n,m}(\sigma^x_y(a))$ for any n, m.*

(2) *Let a be a term, $x \in \mathbf{S}(a)$ be a variable with k occurrences in a, $y \in X - \mathbf{S}(a)$ and n, m be two positive integers. If $k < n$ then $\mu_{n,m}(\sigma^x_{xy}(a)) = \mu_{n,m}(a)$. If $k = n$ and $m = \kappa_x(a)$ then $\mu_{n,m}(\sigma^x_{xy}(a)) = \mu_{n,m}(a) - n < \mu_{n,m}(a)$.*

Proof. (1) The variables different from x and y contribute the same numbers to both sums. Since $\nu_y(\sigma^x_y(a)) = \nu_x(a) + \nu_y(a)$ and $\kappa_y(\sigma^x_y(a)) = \min(\kappa_x(a), \kappa_y(a))$, if one of x, y contributes to the sum for a then the contribution of y to the sum for $\sigma^x_y(a)$ is $\nu_x(a) + \nu_y(a)$.

(2) If $k < n$ then x does not contribute to the sum for a and neither x nor y contributes to the sum for $\sigma^x_{xy}(a)$; the other variables contribute the same

numbers. If $k = n$ and $m = \kappa_x(a)$ then again the only variables that matter are x and y; the contribution of x to the sum for a is $\nu_x(a)$, while neither x nor y contributes to the sum for $\sigma^x_{xy}(a)$, since $\kappa_x(\sigma^x_{xy}(a)) = \kappa_y(\sigma^x_{xy}(a)) = m + 1 > m$. $\square$

2.8. **Lemma.** *If $a \prec_2 b$ then $a \prec b$.*

Proof. Let $b = \sigma^x_{xy}(a)$ where $x \in \mathbf{S}(a)$ and $y \in X - \mathbf{S}(a)$ and suppose that a is not covered by b. Put $n = \nu_x(a)$. It follows from 2.4 that $n \geq 2$. In particular, a is not linear. It follows from 2.2 and 2.5 that whenever $a \leq u \prec v \leq \sigma^x_{xy}(a)$ then either $u \prec_2 v$ or $u \prec_3 v$. Consequently, applying 2.6 we conclude that $\sigma^x_{xy}(a) \sim c$ where

$$c = \sigma^{x_1}_{y_1} \dots \sigma^{x_p}_{y_p} \sigma^{z_1}_{z_1 u_1} \dots \sigma^{z_q}_{z_q u_q}(a)$$

for some p, q (and some x_i, y_i, z_j, u_j) such that $p + q > 1$. For $j = 1, \dots, q$ put

$$n_j = \nu_{z_j}(\sigma^{z_{j+1}}_{z_{j+1} u_{j+1}} \dots \sigma^{z_q}_{z_q u_q}(a)).$$

Clearly, $\lambda(c) = \lambda(a) + n_1 + \dots + n_q$ and $\lambda(\sigma^x_{xy}(a)) = \lambda(a) + n$, so that $n = n_1 + \dots n_q$. On the other hand, we have $\mathrm{Card}(\mathbf{S}(b)) = \mathrm{Card}(\mathbf{S}(a)) + q - p$ and $\mathrm{Card}(\mathbf{S}(\sigma^x_{xy}(a))) = \mathrm{Card}(\mathbf{S}(a)) + 1$, so that $p = q - 1$. It follows that $q \geq 2$ and $n_j < n$ for all j. Put $m = \kappa_x(a)$. By 2.7, $\mu_{n,m}(\sigma^x_{xy}(a)) < \mu_{n,m}(a)$ while $\mu_{n,m}(c) \geq \mu_{n,m}(a)$. But $\mu_{n,m}$ must give the same result when applied to two similar terms and we have obtained a contradiction. $\square$

2.9. **Lemma.** *Let a, b be two linear terms such that $a \prec_1 b$ and $a \prec_2 b$ at the same time. Then a, b are both slim.*

Proof. By induction on the length of a. Let $a = a_1 a_2$. We have $b \sim ax \sim \sigma^y_{yz}(a)$ for some variables x, y, z. Then either $a \sim \sigma^y_{yz}(a_1)$ and $x \sim \sigma^y_{yz}(a_2)$, or vice versa; we can assume without loss of generality that this first case takes place. Then a_2 is a variable different from y and not occurring in a_1. We have $a_1 \prec_1 a$ and $a_1 \prec_2 a$, so that, by induction, a_1 is slim. Since $a_2 \in X$, it follows that a is slim. Since $a \prec_1 b$, also b is slim. $\square$

2.10. **Lemma.** *Let a, b be two terms such that $a \prec_1 b$ and $a \prec_2 b$ at the same time. Then a, b are both slim and linear.*

Proof. We have $a' \prec_1 b'$ and $a' \prec_2 b'$ where $a' \sim \mathbf{lh}(a)$ and $b' \sim \mathbf{lh}(b)$. By 2.9, a' and b' are slim. But then a and b are slim. We have $a = x_1 \dots x_n$ for some variables $x_1, \dots, x_n$. Since $a \prec_1 b$, $b \sim x_1 \dots x_n x_{n+1}$ for a variable $x_{n+1} \notin \{x_1, \dots, x_n\}$. Since $a \prec_2 b$, either x_1 or x_2 has a single occurrence in a and b is similar to either $x_1 x_{n+1} x_2 \dots x_n$ or $x_2 x_{n+1} x_1 x_3 \dots x_n$. This implies that $x_1, \dots, x_{n+1}$ are pairwise different variables. $\square$

If $a \prec_i b$, then we say that b is a cover of a of type i.

3. Definability of linear terms

For every positive integer n we denote by C_n the (up to similarity) only slim linear term of length n.

For every $n \geq 2$ we denote by D_n the term $x_1x_2 \dots x_n$ where $x_1, \dots, x_{n-1}$ are pairwise distinct variables and $x_n = x_1$. It is also determined uniquely by n up to similarity.

A term a is said to be thin if $O(a)$ is a chain, i.e., if $b \leq a$ and $c \leq a$ imply that either $b \leq c$ or $c \leq b$.

3.1. **Proposition.** *A term a is thin if and only if one of the following four cases takes place:*

(1) *a is a slim linear term, i.e., $a \sim C_n$ for some $n \geq 1$;*
(2) *$a \sim D_n$ for some $n \geq 2$;*
(3) *$a = xy \cdot zu$ where x, y, z, u are four distinct variables;*
(4) *$a = xy \cdot xz$ where x, y, z are three distinct variables.*

Proof. If (1) takes place then it follows from 1.2 that a is thin. If (2) takes place then it follows from 2.1 that a has, up to similarity, precisely one lower cover, namely, the slim linear term of the same length; since this lower cover is thin, it follows that a is thin. One can easily check that $O(a)$ is the four-element chain if (3) take place, and the five-element chain if (4) takes place.

Conversely, let a be a thin term. Since $xyzu$ and $xy \cdot zu$ are two incomparable terms both less than each of the terms $xyz \cdot uv$ and $(xy \cdot zu)v$, we have $xyz \cdot uv \not\leq a$ and $(xy \cdot zu)v \not\leq a$. Since xx and $xy \cdot z$ are two incomparable terms both less than each of the terms $xx \cdot yz$ and $xy \cdot xy$, we have $xx \cdot yz \not\leq a$ and $xy \cdot xy \not\leq a$. From this it follows that if $xy \cdot zu \leq a$ then $a = x_1x_2 \cdot x_3x_4$ for some variables x_1, x_2, x_3, x_4 such that $x_1 \neq x_2$, $x_3 \neq x_4$ and $x_1x_2 \neq x_3x_4$. But then, either (3) or (4) takes place.

It remains to consider the case when $xy \cdot zu \not\leq a$. Then $a = x_1x_2 \dots x_n$ for some variables $x_1, \dots, x_n$ ($n \geq 1$). If $n \leq 2$ then it is clear that either (1) or (2) takes place. Let $n \geq 3$. If $x_1, \dots, x_n$ are pairwise distinct, then (1) takes place. So, let a be not linear. Take a variable $x_{n+1} \notin \{x_1, \dots, x_n\}$ and for every $i = 1, \dots, n$ denote by b_i the term $x_1 \dots x_{i-1}x_{n+1}x_{i+1} \dots x_n$. If $x_1 = x_2$ then xx and C_n are two incomparable terms less than a, a contradiction. Hence $x_1 \neq x_2$. If $\{x_1, x_2\}$ is disjoint with $\{x_3, \dots, x_n\}$ then $x_i = x_j$ for some $3 \leq i < j \leq n$ and $x_1x_3 \dots x_n$, b_j are two incomparable terms both less than a, a contradiction. So, we have either $x_1 = x_p$ or $x_2 = x_p$ for some $p \geq 3$; since $x_1x_2 = x_2x_1$, without loss of generality $x_1 = x_p$. If also $x_2 = x_q$ for some $q \geq 3$, then b_p, b_q are two incomparable terms less then a. So, x_2 has a single occurrence in a. If $x_i = x_j$ for some $3 \leq i < j \leq n$ then b_p, b_j are two incomparable terms less than a. If $x_n \notin \{x_1, \dots, x_{n-1}\}$ then $x_1 \dots x_{n-1}$ and b_i are two incomparable terms less than a, a contradiction. We see that the variables $x_1, \dots, x_n$ are pairwise distinct with the only exception $x_1 = x_n$, so that (2) takes place. □

3.2. **Proposition.**

(1) *The set of thin terms is definable.*
(2) *The set of the terms similar to* C_n *for some* n*, i.e., the set of slim linear terms, is definable.*
(3) *The set of the terms similar to* D_n *for some* $n \geq 2$ *is definable.*
(4) *Every (pattern of a) thin term is definable.*

Proof. Clearly, the set of thin terms is definable. It follows from 3.1 that a term a is a slim linear term if and only if there exist thin terms b, c with $a \prec b \prec c$. Consequently, the set of slim linear terms is definable and every slim linear term is definable. The term $xy \cdot zu$ is, up to similarity, the only thin term that is not slim and linear and has a thin cover; the term $xy \cdot xz$ is (again up to similarity) its unique thin cover. For $n \geq 2$, D_n is the only one of the remaining thin terms that is above C_n but not above C_{n+1}. □

By a 1-special term we mean a term a satisfying these conditions:

(1) whenever $b \prec a$ and $c \prec a$ then $b \sim c$;
(2) $xy \cdot zu \leq a$;
(3) $(xy \cdot zu)v \nleq a$;
(4) $D_n \nleq a$ for all $n \geq 2$.

3.3. **Lemma.** *A term* a *can be written as* $a = (xy_1 \dots y_n)(xz_1 \dots z_m)$ *for some* $n, m \geq 1$ *and pairwise distinct variables* x, y_i, z_j *if and only if it is 1-special, there is no 1-special term larger than* a*, and there exists a 1-special term* $b < a$ *such that all the terms* t *with* $b \leq t \leq a$ *are comparable with each other.*

Proof. For $i = 1, 2, 3$ denote by V_i the set of the terms that can be written as $(x_1x_2 \dots x_n)(y_1y_2 \dots y_m)$ where $x_1, \dots, x_n$ are pairwise distinct variables, $y_1, \dots, y_m$ are pairwise distinct variables and (respectively)

(V_1) $n = m \geq 2$ and $x_i \neq y_j$ for all i, j;
(V_2) $n = m \geq 3$ and there is an index $k \geq 3$ such that $x_i = y_j$ if and only if either $(i, j) = (1, k)$ or $(i, j) = (k, 1)$;
(V_3) $n \geq 2$, $m \geq 2$ and $x_i = y_j$ if and only if $i = j = 1$.

One can easily see that every term belonging to $V_1 \cup V_2 \cup V_3$ is 1-special. We are going to prove first that there are no other 1-special terms.

Let a be a 1-special term. It follows from (2), (3) and (4) that $a = (x_1x_2 \dots x_n)(y_1y_2 \dots y_m)$ where $n, m \geq 2$, x_i are pairwise different variables and y_j are pairwise different variables. If a is linear and $n \neq m$ then $(x_2 \dots x_n)(y_1y_2 \dots y_m)$ and $(x_1x_2 \dots x_n)(y_2 \dots y_m)$ are two incomparable lower covers of a, a contradiction. Thus if a is linear, then $n = m$ and $a \in V_1$. Now let a be non-linear.

Suppose that each of x_1 and x_2 has a single occurrence in a. Since a is not linear, we have $x_i = y_j$ for some $i \geq 3$ and some j. Clearly, the term $(x_2 \dots x_n)(y_1y_2 \dots y_m)$ and the term obtained from a by differentiating x_i, y_j (i.e., by replacing one occurrence of this variable with a new variable) are two incomparable lower covers of a, a contradiction. Consequently, we can assume that $x_1 = y_k$ for some $k \neq 2$. Quite similarly, we can assume that $y_1 = x_p$ for some $p \neq 2$.

Let $k, p \geq 3$. If $k \neq p$ then a has two incomparable lower covers, one obtained by differentiating x_1, y_k and the other by differentiating y_1, x_p. Hence $k = p$. If also $x_i = y_j$ for some $i, j \neq k$ then a has again two incomparable lower covers, one obtained by differentiating x_1, y_k and the other by differentiating x_i, y_j. Hence there are no such indexes i, j and we get $a \in V_2$.

It remains to consider the case $x_1 = y_1$. If also $x_2 = y_2$, then a has two incomparable lower covers: the term $(x_2 \dots x_n)(y_2 \dots y_m)$ and the term obtained from a by differentiating x_1, y_1. Hence $x_2 \neq y_2$. If $x_i = y_j$ for some $i, j \geq 2$ then a has two incomparable lower covers, one obtained by differentiating x_1, y_1 and the other by differentiating x_i, y_j. We get $x_i \neq y_j$ whenever $(i, j) \neq (1, 1)$ and so $a \in V_3$.

Now when we have completed the description of the set of 1-special terms, one can easily see that a term is maximal among 1-special terms if and only if it belongs to $V_2 \cup V_3$. If a satisfies (V_3) and $n \leq m$, then the term b obtained from $(x_1 x_2 \dots x_n)(y_1 y_2 \dots y_n)$ by differentiating x_1, y_1 is a 1-special term and the interval restricted by b, a is a chain. On the other hand, for a term $a \in V_2$ and any 1-special term $b < a$, the interval is at least a four-element Boolean algebra. $\square$

3.4. **Theorem.** *The set of linear terms is definable. The binary relation* $b \sim \mathbf{lh}(a)$ *is definable.*

Proof. Denote by U the set of the terms $(xy_1 \dots y_n)(xz_1 \dots z_m)$ where $n, m \geq 1$ and x, y_i, z_j are pairwise distinct variables. One can easily see that a term a is linear if and only if $u \nleq a$ for all terms $u \in U$ and $D_n \nleq a$ for all $n \geq 2$. So, by 3.2, in order to prove that the set of linear terms is definable, it is sufficient to show that U is definable. By 3.3, the set U is definable if the set of 1-special terms is definable. By 3.2, definability of the set of 1-special terms according to the definition depends only on the definability of the term $(xy \cdot zu)v$. One can easily check that $(xy \cdot zu)v$ and $xyz \cdot uv$ are the only terms that are covers of both $xy \cdot zu$ and $xyzu$ (these two last terms are definable by 3.2). But (as it can be verified easily) $xyz \cdot uv$ has nine different upper covers, while $(xy \cdot zu)v$ has only six.

We have $b \sim \mathbf{lh}(a)$ if and only if b is a linear term, $b \leq a$ and $c \leq b$ for every linear term $c \leq a$. $\square$

3.5. **Theorem.** *The set of unary terms is definable. The binary relation* $b \sim \mathbf{uh}(a)$ *is definable.*

Proof. A term a is unary if and only if it is maximal among the terms b such that the linear hull of a is similar to the linear hull of b. $\square$

3.6. **Theorem.** *The set of slim terms is definable.*

Proof. A term is slim if and only if its linear hull is a slim linear term, so we can use 3.2. □

4. Definability of the types of covers

4.1. **Proposition.** *The binary relation $a \prec_3 b$ is definable.*

Proof. We have $a \prec_3 b$ if and only if $a \prec b$ and the linear hull of a is similar to the linear hull of b. □

4.2. **Lemma.** *Let a, b be two linear terms. Then $a \prec_1 b$ if and only if $a \prec b$ and $a' < b'$, where a' is the unary hull of a and b' is the unary hull of b.*

Proof. Let a', b' be the unary hulls such that $\mathbf{S}(a') = \mathbf{S}(b') = \{x\}$ for a variable x. If $a \prec_1 b$ then a' is a proper subterm of b', so that $a' < b'$. In order to prove the converse, let $a \prec b$ and $a' < b'$. Then $\lambda(a) = \lambda(a') < \lambda(b') = \lambda(b)$, so that (since a, b are linear) $\lambda(b) = \lambda(a) + 1$. Consequently, $\lambda(b') = \lambda(a') + 1$. But a', b' are unary, so this is possible only if $b' = a'x$. Then $b = ay$ for a variable y and we get $a \prec_1 b$. □

4.3. **Lemma.** *Denote by U_1 the set of the slim terms $a = x_1 \dots x_n$ such that $n \geq 3$, $x_1 \neq x_2$, $\{x_1, x_2\}$ is disjoint with $\{x_3, \dots, x_n\}$ and $x_n \in \{x_3, \dots, x_{n-1}\}$. A term a belongs to U_1 if and only if a is slim, a is nonlinear, $a \geq xy \cdot z$, every thin term below a is linear and a has, up to similarity, precisely one lower cover not of type 3. Consequently, the set U_1 is definable.*

Proof. If $a \in U_1$ then $x_1 x_3 \dots x_n$ is the only lower cover of a that is not of type 3. Conversely, let $a = x_1 \dots x_n$ be a slim term satisfying the conditions. Since $a \geq xy \cdot z$, we have $n \geq 3$. Since all the non-linear thin terms $y_1 \dots y_k y_1$ are not below a, we have $x_1 \neq x_2$ and $x_1, x_2 \notin \{x_3, \dots, x_n\}$. So, $x_1 x_3 \dots x_n$ is a lower cover of a and it is not of type 3. If $x_n \notin \{x_1, \dots, x_{n-1}\}$, then also $x_1 x_2 \dots x_{n-1}$ is a lower cover of a not of type 3; these two lower covers are not similar, a contradiction. Hence $x_n \in \{x_1, \dots, x_{n-1}\}$. □

4.4. **Proposition.** *The binary relation $a \prec_1 b$ is definable.*

Proof. By 4.2, this relation restricted to linear terms is definable. So, we will be done if we prove the following: $a \prec_1 b$ if and only if $a \prec b$, $a' \prec_1 b'$ where $a' \sim \mathbf{lh}(a)$ and $b' \sim \mathbf{lh}(b)$, and for every $u \in U_1$ we have $u \leq a$ if and only if $u \leq b$ (where U_1 was introduced in 4.3). The direct implication is easy. For the converse, suppose that $a \prec b$ and the above conditions are satisfied. Since $\lambda(b') = \lambda(a') + 1$, we have $\lambda(b) = \lambda(a) + 1$ and so it is sufficient to consider the case when $b \sim \sigma^x_{xy}(a)$ for a variable x with a single occurrence in a and a variable $y \notin \mathbf{S}(a)$. Then also $b' \sim \sigma^z_{zu}(a')$ for a variable $z \in \mathbf{S}(a')$ and a variable $u \notin \mathbf{S}(a')$. By 2.9, a' and b' are slim. But then also a and b are slim. We have $a = x_1 x_2 \dots x_n$ for some variables $x_1, \dots, x_n$ where (since b is slim) $x \in \{x_1, x_2\}$. Without loss of generality, $x = x_1$. Then $b \sim xyx_2 \dots x_n$. Suppose that a is nonlinear and denote by j the largest index such that $x_i = x_j$ for some $i < j$, so that $j \geq 3$. Then $xyx_2 \dots x_j \in U_1$

is below b, so it must be also below a, which is clearly impossible. Hence a is linear. Then b is also linear and we get $a \prec_1 b$. □

4.5. **Proposition.** *The binary relation $a \prec_2 b$ is definable.*

Proof. By 2.10, $a \prec_2 b$ if and only if $a \prec b$, $a \not\prec_3 b$ and either $a \not\prec_1 b$ or a, b are both slim and linear. □

5. Definability of the addition for codes of positive integers

Since for every positive integer n there is, up to similarity, precisely one slim linear term of length n, these slim linear terms C_n can serve as codes for positive integers.

The depth $\delta(a)$ of a term a can be also defined as the length of a maximal slim linear term b such that $b \leq a$. So, the binary relations expressing the facts that a, b are two terms with $\delta(a) = \delta(b)$, or $\delta(a) < \delta(b)$, or $\delta(b) = \delta(a)+1$, are definable. This makes it possible to speak freely about the depth of a term in statements serving to prove that a given relation is definable.

For slim terms, the depth is the same as the length. So, in the case of slim terms we can also speak freely about the length.

A term a is said to be 2-special if $a = x_1 x_2 \dots x_n$ where $n \geq 2$, x_1, x_2 are two distinct variables and $x_2 = x_3 = \dots = x_n$.

5.1. **Lemma.** *A term is 2-special if and only if it is slim, has a unary cover of type 3 and has a slim cover of type 2. Consequently, the set of 2-special terms is definable.*

Proof. If $a = xy \dots y$ is 2-special, then $xx \dots x$ is a unary cover of a of type 3 and $xzy \dots y$ is a slim cover of a of type 2. Conversely, let $a = x_1 x_2 \dots x_n$ be a slim term with a unary cover of type 3 and a slim cover of type 2. Since a has a unary cover of type 3, we have $n \geq 2$ and $\operatorname{Card} \mathbf{S}(a) = 2$. If each of the variables x_1, x_2 has more than one occurrence in a then a has no slim cover of type 2. So, without loss of generality, x_1 has a single occurrence in a. Then $x_2 = x_3 = \dots = x_n$. □

A term a is said to be 3-special if $a = x_1 x_2 \dots x_n$ where $n \geq 3$, x_1, x_2 are two distinct variables, $x_2 = x_3 = \dots = x_{n-1}$ and $x_1 = x_n$.

5.2. **Lemma.** *A term a is 3-special if and only if $b \prec_1 c \prec_3 a$ for a 2-special term b and some term c, there is no term d with $d \prec_1 a$, and either a is of length 3 or a is not 2-special. Consequently, the set of 3-special terms is definable.*

Proof. It is easy. □

5.3. **Lemma.** *Denote by R the set of the triples (a, b, c) such that $a \sim C_n$, $b \sim C_m$ and $c \sim C_k$ for some $4 \leq n \leq m$ and $k \geq n + m - 2$. The ternary relation R is definable.*

Proof. Let $a \sim C_n$, $b \sim C_m$ and $c \sim C_k$ where $4 \le n \le m$. Clearly, we will be done if we prove that $k \ge n + m - 2$ if and only if there exists a term t with the following properties:

(1) t is a slim term and $\lambda(t) \le k$;
(2) a 3-special term of length j is below t if and only if either $j = 3$ or $j = n$;
(3) the 2-special term of length m is below t.

First we are going to prove the direct implication. Let $k \ge n + m - 2$. Take two distinct variables x, y and put $t = x_1 y_2 \dots y_{n-1} x_n \dots x_{n+m-2}$ where $x_1 = x_n = \dots = x_{n+m-2} = x$ and $y_2 = \dots = y_{n-1} = y$. Clearly, t is a slim term of length $n + m - 2$, so $\lambda(t) \le k$. It is easy to check that t has also the properties (2) and (3).

For the converse, let there exist a term t satisfying (1), (2) and (3). We have $t = x_1 x_2 \dots x_p$ for some variables $x_1, \dots, x_p$. It follows from (2) that $p \ge n$ and $x_1 x_2 \dots x_n$ is 3-special. So, without loss of generality, $x_1 \ne x_2$, $x_1 = x_n$ and $x_2 = \dots = x_{n-1}$. (We have $x_1 \ne x_2$ because, also by (2), $xx \not\le t$.) Denote by s the 2-special term $xy_1 \dots y_{m-1}$ where $x \ne y_1 = \dots = y_{m-1}$. Since $s \le t$, there is a substitution f such that $f(s) \subseteq t$. Clearly, $f(y_1)$ is a variable. If $f(x)$ is of length $j \le n - 2$, then $x_n = f(y_{n-j}) = f(y_{n-j-1}) = x_{n-1}$, a contradiction. Hence $\lambda(f(x)) \ge n - 1$. Then $\lambda(f(s)) \ge n + m - 2$, so that $\lambda(t) \ge n + m - 2$ and $k \ge n + m - 2$. □

5.4. **Theorem.** *The set of the triples (a, b, c) such that $a \sim C_n$, $b \sim C_m$ and $c \sim C_{n+m}$ for some $n, m \ge 1$ is a definable ternary relation.*

Proof. It follows easily from 5.3. □

6. Definability of substitution instances

6.1. **Proposition.** *The following relations are definable:*

$R_1(a, b, c)$: *a is a term, $b \sim ax_1 \dots x_n$ for some $n \ge 1$ and pairwise distinct variables $x_1, \dots, x_n \notin \mathbf{S}(a)$, and $c \sim C_n$.*

$R_2(a, b, c)$: *a is a term, $b \sim a(xy)x_1 \dots x_n$ and $c \sim C_n$ for some $n \ge 1$ and pairwise distinct variables $x, y, x_1, \dots, x_n \notin \mathbf{S}(a)$.*

$R_3(a, b, c)$: *a is a term, $b \sim ax_n x_1 \dots x_n$ and $c \sim C_n$ for some $n \ge 1$ and pairwise distinct variables $x_1, \dots, x_n \notin \mathbf{S}(a)$.*

$R_4(a, b)$: *a is a term and $b = axx$ for a variable $x \notin \mathbf{S}(a)$.*

$R_5(a, b, c, d)$: *a, b are two terms, $c \sim ax_1 \dots x_n x_n$ and $d \sim bx_1 \dots x_n x_n$ for some $n \ge 1$ and pairwise distinct variables $x_1, \dots, x_n \notin \mathbf{S}(a) \cup \mathbf{S}(b)$.*

$R_6(a, b)$: *a is a term and b is a substitution instance of a, i.e., $b = f(a)$ for some substitution f.*

Proof. Using 1.3, it is easy to prove that $R_1(a, b, c)$ if and only if $a < b$, $a \le d \prec e \le b$ implies $d \prec_1 e$, c is a slim linear term and $\delta(b) = \delta(a) + \lambda(c)$.

We have $R_2(a, b, c)$ if and only if there are terms d, e such that $a \prec_1 d \prec_2 e$, $R_1(e, b, c)$ and either $a \in X$ or there is no u with $u \prec_1 e$.

We have $R_3(a,b,c)$ if and only if there exist terms $d,\bar{c}$ such that $R_1(a,d,\bar{c})$, $c \prec \bar{c}$, $d \prec_3 b$, there is no u with $u \prec_1 b$, and either $a \in X$ and b is a nonlinear thin term or else there is no triple v,a',v' of terms with $b \prec_2 v$, $a' \sim \mathbf{lh}(a)$, $v' \sim \mathbf{lh}(v)$ and $R_2(a',v',c)$. For the proof of the direct implication put $d = ax_{n+1}x_1 \dots x_n$ and suppose that $a \notin X$ and there exists a triple v,a',v' as above, so that $v \sim \sigma^x_{xy}(ax_nx_1 \dots x_n)$ for some x,y. It follows from $R_2(a',v',c)$ that $v' \sim a'(xy)x_1 \dots x_n$. On the other hand, if $x \in \mathbf{S}(a)$ then $v' \sim a_1x_{n+1}x_1 \dots x_n$ for a term a_1 longer than a, so that $a'(xy) \sim a_1x_{n+1}$ and hence $a \in X$, a contradiction. If $x = x_n$, then $v' \sim a'(xy)x_1 \dots x_n \sim a'(xy)x_1 \dots x_{n-1}(xy)$, which is impossible. Finally, if $x \in \{x_2,\dots,x_{n-1}\}$, then $v' \sim a'(xy)x_1 \dots x_n \sim a'x_{n+1}x_1 \dots (xy) \dots x_n$, which is again impossible. It remains to prove the converse implication. Clearly, $b \sim \sigma^x_y(ax_{n+1}x_1 \dots x_n)$ for some variables $x \neq y$ from $\mathbf{S}(a) \cup \{x_1,\dots,x_{n+1}\}$. Since there is no u with $u \prec_1 b$, $x_n \in \{x,y\}$; without loss of generality, $x_n = x$. If $a \notin X$ and $y \neq x_{n+1}$, then we can put $v = \sigma^x_y(a(x_{n+1}x_{n+2})x_1 \dots x_n)$ to obtain a contradiction.

We have $R_4(a,b)$ if and only if $R_3(a,b,C_1)$.

The definability of R_5 follows easily from the definability of R_1 and R_4.

We have $R_6(a,b)$ if and only if whenever $R_5(a,b,c,d)$ then $c \le d$. Indeed, if $ax_1 \dots x_nx_n \le bx_1 \dots x_nx_n$ where n is very large and a is not a variable then $f(ax_1 \dots x_nx_n) \subseteq bx_1 \dots x_nx_n$ implies $f(x_n) = x_n$, $f(x_{n-1}) = x_{n-1}$, $\dots$, $f(a) = b$. □

7. Finite sequences of terms and code-terms

For every nonempty finite sequence $a_1,\dots,a_n$ of terms we denote by $H(a_1,\dots,a_n)$ the term $xa_1a_2 \dots a_nx$ where x is a variable not contained in $\mathbf{S}(a_1) \cup \dots \cup \mathbf{S}(a_n)$. This term (determined uniquely up to similarity) is called the code of the given sequence. Obviously, the sequence can be reconstructed from its code.

We have $H(a_1,\dots,a_n) \sim H(b_1,\dots,b_m)$ if and only if $n = m$ and there is an automorphism α of F such that $b_i = \alpha(a_i)$ for all $i = 1,\dots,n$. (This is stronger than just $a_i \sim b_i$ for all i.)

By a code-term we mean a term that is a code of some sequence. Obviously, a is a code-term if and only if $a = bx$ for a variable x and a term $b \notin X$ having precisely one occurrence of x.

If a is the code of a sequence $a_1,\dots,a_n$ then this sequence is called the decode of a, the number n is called the width of a, the terms a_i are called members of a and, for $i = 1,\dots,n$, we put $a[i] = a_i$.

7.1. Lemma. *Let $a = H(a_1,\dots,a_n)$ and $b = H(b_1,\dots,b_m)$ be such that $b = f(a)$ for a substitution f. Then $n = m$ and $b_i = f(a_i)$ for $i = 1,\dots,n$.*

Proof. It is obvious. □

7.2. Proposition. *The set of code-terms is definable.*

Proof. A term a is a code-term if and only if there exist terms b, c, d, e, a', d' with $b \notin X$, $b \prec_1 c \prec_3 a \prec_2 d$, $a' \sim \mathbf{lh}(a)$, $d' \sim \mathbf{lh}(d)$ and $a' \prec_2 e \prec_2 d'$, and there is no term u with $u \prec_1 d'$. Indeed, if $a = bx$ and $\nu_x(b) = 1$, then we can take $d = \sigma^x_{xy}(a)$ where $y \in X - \mathbf{S}(a)$. Conversely, it follows from $b \prec_1 c \prec_3 a$ that $a = a_1 x$ for a term a_1 and a variable x; since $a \prec_2 d$, $d \sim \sigma^z_{zy}(a)$ for some variable $z \in \mathbf{S}(a)$ and some variable $y \notin \mathbf{S}(a)$; since $a' \prec_2 e \prec_2 d'$, z has precisely two occurrences in a; since d' has no lower cover of type 1, $z = x$. □

For every $3 \le i < n$ denote by $E_{n,i}$ the term $x_1 x_2 \dots x_n$ where $x_1, \dots, x_{n-1}$ are pairwise distinct variables and $x_n = x_i$.

For every $2 \le i < j < n$ denote by $G_{n,i,j}$ the term $x_1 x_2 \dots x_n$ where $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}$ are pairwise distinct variables, $x_j = x_i$ and $x_n = x_1$.

7.3. **Lemma.**

(1) *We have $D_n \le E_{m,i}$ if and only if $i = m - n + 2$.*

(2) *We have $E_{m,k} \le G_{n,i,j}$ if and only if $m - k = j - i$ and $k \le i$.*

Proof. It is easy. □

7.4. **Lemma.** *The following relations are definable:*
$R_7(a, b, c)$*: $a \sim C_n$, $b \sim C_i$ and $c \sim E_{n,i}$ for some $3 \le i < n$.*
$R_8(a, b, c, d)$*: $a \sim C_n$, $b \sim C_i$, $c \sim C_j$, $d \sim G_{n,i,j}$ for some $2 \le i < j < n$.*

Proof. The definability of R_7 and R_8 follows from 7.3 and from the following two observations. Given an n, a term t is similar to $E_{n,i}$ for some i if and only if $C_n \prec_3 t$, t is not thin and there is no term u with $u \prec_1 t$. Given an n, a term t is similar to $G_{n,i,j}$ for some i, j if and only if $D_n \prec_3 t$, $xx \not\le t$ and there is no pair m, k with $R_6(E_{m,k}, t)$. □

7.5. **Proposition.** *The following relations are definable:*
$R_9(a, b)$*: for some n, $a \sim C_n$ and b is a code-term of width n.*
$R_{10}(a, b, c, d)$*: $a \sim C_n$, $b \sim C_i$, $c \sim C_j$ for some $1 \le i < j \le n$ and d is a code-term of width n such that $d[i] = d[j]$.*
$R_{11}(a, b, c)$*: for some n, $a \sim C_n$, b is a code-term of width n, $b[1] \sim c$ and $b[2], \dots, b[n]$ are pairwise distinct variables not occurring in $b[1]$.*
$R_{12}(a, b, c)$*: for some n, $a \sim C_n$, b is a code-term of width n and $c \sim b[1]$.*
$R_{13}(a, b, c, d)$*: $a \sim C_n$, $b \sim C_i$ for some $2 \le i \le n$, c is a code-term of width n such that $c[1] = c[i] \sim d$ and $c[2], \dots, c[i-1], c[i+1], \dots, c[n]$ are pairwise distinct variables not occurring in $c[1]$.*
$R_{14}(a, b, c, d)$*: $a \sim C_n$, $b \sim C_i$ for some $2 \le i \le n$ and c is a code term of width n such that $c[i] \sim d$ and $c[1], \dots, c[i-1], c[i+1], \dots, c[n]$ are pairwise distinct variables not occurring in $c[i]$.*
$R_{15}(a, b, c, d)$*: $a \sim C_n$, $b \sim C_i$ for some $1 \le i \le n$, c is a code-term of width n and $d \sim c[i]$.*

Proof. We have $R_9(C_n, b)$ if and only if b is a code-term and b is a substitution instance of D_{n+2}.

We have $R_{10}(C_n, C_i, C_j, d)$ if and only if $R_9(a, d)$ and d is a substitution instance of $G_{n+2,i+1,j+1}$.

We have $R_{11}(a, b, c)$ if and only if $R_3(c, b, a)$.

We have $R_{12}(a, b, c)$ if and only if there is a term b' with $R_{11}(a, b', c)$ such that b is a substitution instance of b' and whenever $R_{11}(a, u, v)$ and b is a substitution instance of u then c is a substitution instance of v.

We have $R_{13}(a, b, c, d)$ if and only if $R_{11}(a, c, d)$, $R_{10}(a, C_1, b, c)$ and every term t satisfying $R_{12}(a, t, d)$ and $R_{10}(a, C_1, b, t)$ is a substitution instance of c.

We have $R_{14}(C_n, C_i, c, d)$ $(2 \le i \le n)$ if and only if either d is a variable and c is a smallest term of width n, or else d is not a variable, c is a code-term of width n, $c[1]$ is a variable and there exists a term e with $R_{13}(a, b, e, d)$ such that e is a substitution instance of c and whenever e is a substitution instance of a code-term c' of width n with $c'[1] \in X$ then c is a substitution instance of c'.

$R_{15}(C_n, C_i, c, d)$ can be definably reformulated by $R_{12}(a, c, d)$ if $i = 1$; if $i \ge 2$, we can use R_{14} in the same way as R_{11} was used in the reformulation of R_{12}. □

7.6. **Proposition.** *The following relations are definable:*

$R_{16}(a, b, c, d)$*: $a \sim C_n$, $b \sim C_i$, $c \sim C_j$ where $1 \le i, j \le n$ and d is a code-term of width n such that $\mathbf{S}(d[i]) \subseteq \mathbf{S}(d[j])$.*

$R_{17}(a, b, c, d, e)$*: $a \sim C_n$, $b \sim C_i$, $c \sim C_j$, $d \sim C_k$ where $1 \le i, j, k \le n$, $k \notin \{i, j\}$ and e is a code-term of width n such that $e[k] = e[i]e[j]$.*

Proof. We have $R_{16}(C_n, C_i, C_j, d)$ if and only if d is a code-term of width n and for every code-term e of width n that is a substitution instance of d, $d[j] \sim e[j]$ implies $d[i] \sim e[i]$.

We have $R_{17}(C_n, C_i, C_j, C_k, e)$ if and only if e is a code-term of width n and e is a substitution instance of a code-term u of width n such that $u[i]$ and $u[j]$ are variables, $u[i] \ne u[j]$ if $i \ne j$, and $u[k] = u[i]u[j]$; this equality can be expressed by saying that there is a cover v of a variable such that $v \sim u[k]$ if $i \ne j$ and $v \prec_3 u[k]$ if $i = j$, and using R_{16} to require that $\mathbf{S}(u[i]) \subseteq \mathbf{S}(u[k])$ and $\mathbf{S}(u[j]) \subseteq \mathbf{S}(u[k])$. □

7.7. **Proposition.** *The following relations are definable:*

$R_{18}(a, b, c)$*: $a \sim C_n$ for some $n \ge 1$, $b \sim H(a_1, \dots, a_n)$ is a code-term of width n and $c \sim H(a_1, \dots, a_n, a_{n+1})$ for some term a_{n+1}.*

$R_{19}(a, b, c, d)$*: $a \sim C_n$, $b \sim C_m$ where $1 \le n \le m$, c is a code-term of width n and d is a code-term of width m such that the decode of c is a beginning of the decode of d.*

$R_{20}(a, b, c, d)$*: $a \sim C_n$, $b \sim C_i$ where $1 \le i \le n$, $c \sim H(a_1, \dots, a_n)$ is a code-term of width n and $d \sim (H, a_i, a_1, \dots, a_n)$.*

$R_{21}(a, b, c, d, e)$*: $a \sim C_n$, $b \sim C_i$, $c \sim C_j$ where $1 \le i, j \le n$, $d \sim H(a_1, \dots, a_n)$ is a code-term of width n and $e \sim H(a_i, a_j)$.*

$R_{22}(a, b)$*: $a \sim H(p, q)$ is a code-term of width 2 and $b \sim H(py, qy)$ for a variable $y \notin \mathbf{S}(pq)$.*

$R_{23}(a,b)$: *$a \sim H(p,q)$ is a code-term of width 2 and $b \sim H(py_1\dots y_n, qy_1\dots y_n)$ for some pairwise distinct variables $y_1,\dots,y_n \notin \mathbf{S}(pq)$ $(n \geq 0)$.*

$R_{24}(a,b)$: *$a \sim H(p,q)$ and $b \sim H(u,v)$ are two code-terms of width 2 such that v can be obtained from u by replacing one occurrence of a subterm $f(p)$, for some substitution f, with the term $f(q)$.*

$R_{25}(a,b,c,d)$: *$a \sim C_n$, $b \sim C_i$, $c \sim C_j$ where $1 \leq i,j \leq n$ and d is a code-term of width n such that $d[i]$ is a subterm of $d[j]$.*

Proof. Perhaps we should start by explaining why the obvious proof for the definability of R_{18} does not work. One would be tempted to take the unique expression $b = xa_1\dots a_nx$ for the term b, delete the outer occurrence of x to obtain the term $xa_1\dots a_n$ and say that c is an arbitrary term obtained from the last one if it is multiplied first by an arbitrary term not containing x and then by x. The trouble is that if we delete the outer occurrence of x, much of the information about the sequence $a_1,\dots,a_n$ is lost; the variable x may not be the only variable in $xa_1\dots a_n$ with a single occurrence. For a working proof we can exploit the technique of code-terms in such a way that the variable x is stored together with the term b at a different place.

We have $R_{18}(C_n,b,c)$ if and only if b is a code-term of width n, c is a code-term of width $n+1$ and there exists a code-term $u \sim H(u_1,\dots,u_7)$ of width 7 such that $u_1 \sim b$, $u_2 \sim c$, u_3 is a variable, $u_1 = u_3u_4$, $u_2 = u_3u_5$, $u_5 = u_6u_7$, $u_3 \in \mathbf{S}(u_7)$ and $u_4 = u_7$. To see this, observe that if $u_1 = xa_1\dots a_nx$ and $u_2 = xb_1\dots b_nb_{n+1}x$ then necessarily $u_3 = x$, $u_4 = xa_1\dots a_n$, $u_5 = xb_1\dots b_nb_{n+1}$, $u_6 = b_{n+1}$ and $u_7 = xb_1\dots b_n$.

We have $R_{19}(C_n,C_m,c,d)$ if and only if c is a code-term of width n, d is a code term of width $m \geq n$ and there exists a code-term u of width $k = m-n+1$ such that $u[1] \sim c$, $u[k] \sim d$ and $R_{18}(u[i],u[i+1])$ for every $i < k$.

We have $R_{20}(C_n,C_i,c,d)$ if and only if c is a code-term of width n and there exists a code-term $u \sim H(u_1,\dots,u_{3n+4})$ of width $3n+4$ such that $u_1 \sim c$, $u_{3n+4} \sim d$, u_{n+2} is a variable, $u_1 = u_2u_{n+2}$, $u_j = u_{j+1}u_{n+j+1}$ for $2 \leq j \leq n+1$, $u_{2n+3} = u_{n+2}u_{2n-i+3}$ and $u_j = u_{j-1}u_{4n+6-j}$ for $2n+4 \leq j \leq 3n+4$. To see this, observe that if $u_1 = xa_1\dots a_nx$ then

$$u_2 = xa_1\dots a_n,\ u_3 = xa_1\dots a_{n-1},\dots,\ u_{n+1} = xa_1,\ u_{n+2} = x,$$

$$u_{n+3} = a_n,\ u_{n+4} = a_{n-1},\dots,\ u_{2n+2} = a_1,$$

$$u_{2n+3} = xa_i,\ u_{2n+4} = xa_ia_1, u_{2n+5} = xa_ia_1a_2,\dots,\ u_{3n+3} = xa_ia_1\dots a_n,$$

$$u_{3n+4} = xa_ia_1\dots a_nx.$$

We have $R_{21}(C_n,C_i,C_j,d,e)$ if and only if d is a code-term of width n, e is a code-term of width 2 and there exist two terms u,v such that $R_{20}(C_n,C_j,d,u)$, $R_{20}(C_{n+1},C_{i+1},u,v)$ and $R_{19}(C_2,C_{n+2},e,v)$.

We have $R_{22}(a,b)$ if and only if a,b are code-terms of width 2 and there exists a code-term $u = H(u_1,\dots,u_{12})$ of width 12 such that $u_1 \sim a$, $u_{12} \sim b$ and, where $u_1 = xpqx$, we have $u_2 = x$, $u_3 = xpq$, $u_4 = xp$, $u_5 = p$,

$u_6 = q$, $u_7 = y$ for a variable $y \notin \mathbf{S}(u_1)$, $y_8 = py$, $u_9 = qy$, $u_{10} = x(py)$, $u_{11} = x(py)(qy)$ and $u_{12} = x(py)(qy)$. (Each step should be reformulated using the previous relations.)

We have $R_{23}(a, b)$ if and only if a, b are two code-terms of width 2 and there exists a code-term u of some width n such that $u[1] \sim a$, $u[n] \sim b$ and $R_{22}(u[i], u[i+1])$ for all $i < n$.

We have $R_{24}(a, b)$ if and only if a, b are two code-terms of width 2 and b is a substitution instance of some code-term u of width 2 such that $R_{23}(a, u)$.

We have $R_{25}(C_n, C_i, C_j, d)$ if and only if d is a code-term of width n and there exist a code-term u of some width m and a number k with $n < k \leq m$ such that $R_{19}(C_n, C_m, d, u)$, $u[k] = u[i]$, $u[m] = u[j]$ and whenever $k \leq l < m$ then $R_{17}(C_m, C_l, C_p, C_{l+1}, u)$ for some $p < l$. □

8. Main results

8.1. **Theorem.** *Every term pattern is definable.*

Proof. By a C-sequence we will mean a finite sequence $c_1, \dots, c_n$ $(n \geq 1)$ such that for every $i = 1, \dots, n$ either c_i is a variable or c_i is an ordered pair of positive integers, both of them less than i. Given such a C-sequence, for every $i = 1, \dots, n$ we define a term t_i by induction as follows: if c_i is a variable, then $t_i = c_i$; if $c_i = (p, q)$ then $t_i = t_p t_q$. The term t_n is called the value of the given C-sequence. It is easy to see (prove it by induction on the length of t) that every term t is the value of some C-sequence. Now if t is the value of a C-sequence $c_1, \dots, c_n$, then t is, up to similarity, the only term u for which there exists a code-term v of width n such that $v[n] \sim u$, whenever c_i is a variable then $v[i]$ is a variable, whenever c_i and c_j are two distinct variables then $v[i] \neq v[j]$, and whenever $c_i = (p, q)$ then $v[i] = v[p]v[q]$. □

8.2. **Corollary.** *The ordered set of term patterns has no automorphisms except the identity.*

8.3. **Theorem.** *The set of the pairs (a, b) such that a is similar to a subterm of b is a definable binary relation.*

Proof. A term a is similar to a subterm of b if and only if there exists a code-term u of width 2 such that $u[1] \sim a$, $u[2] \sim b$ and $R_{25}(C_2, C_1, C_2, u)$. We did not succeed to find a more straightforward proof, not relying so heavily on the technique of code-terms. □

8.4. **Theorem.** *The following binary relation $S(a, b)$ is definable: $S(a, b)$ if and only if $a \sim H(H(p_1, q_1), \dots, H(p_n, q_n))$ and $b \sim H(u, v)$ for some $n \geq 1$ and some equations (i.e., ordered pairs of terms) (p_i, q_i) and (u, v) such that (u, v) is a consequence of $\{(p_1, q_1), \dots, (p_n, q_n)\}$.*

Proof. We have $S(a, b)$ if and only if a is a code-term of some width n, $a[i]$ is a code-term of width 2 for every $1 \leq i \leq n$ and there exists a code-term u of some width m such that $R_{21}(C_m, C_1, C_m, u, b)$ and for every $1 \leq i < m$ there exist a number j with $1 \leq j \leq n$ and a code-term v of width 2 such that $R_{24}(a[j], v)$, and either $R_{21}(C_m, C_i, C_{i+1}, v)$ or $R_{21}(C_m, C_{i+1}, C_i, v)$. $\square$

9. Concluding remarks

Theorem 8.4 may not seem to be a suitable candidate for the list of main results, but it is here because it is the result that will be used most often in a next paper on definability in the lattice of equational theories of commutative groupoids, a continuation of the present paper. We will later also rely on the definability of some similar relations, involving a more detailed syntactic structure of an equation. We hope that in all cases it would be apparent how to use the above presented technique in a similar way to obtain the desired concern.

In [2] we did not succed to obtain an analog of Theorem 8.1 and its corollary 8.2. After proving an analog of Theorem 2.1 and a few auxiliary results, it seemed difficult to continue working in the ordered set of (general) term patterns and so we escaped from the ordered set to a larger lattice of full sets of terms (sets U such that $a \geq b \in U$ implies $a \in U$). For applications to equational theories, this escape did not matter. However, the investigation of definability in the ordered set of term patterns may be interesting in itself, so there remained a gap. We still do not know if the ordered set of (noncommutative) groupoid terms has automorphisms other than the identity and the second obvious one. Perhaps this gap could now be filled. It would also answer the fourth of the open problems formulated in [3].

We hope that the present paper also shows that the structure of commutative terms, although in many respects similar to that of general terms, can be a subject of independent interest. There are questions with trivial answers for general terms but difficult to answer in the commutative case. The author was not able to decide whether the following is true: If a, b are two (commutative) terms such that $f(a) \sim f(b)$ for all substitutions f, then $a = b$.

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