4-homogeneous latin trades

Nicholas Cavenagh[1], Diane Donovan[2] & Aleš Drápal[3]

December 20, 2003

[1]DIMATIA, Charles University
Malostranské Naměsti 25,
11800 Praha 1,
Czech Republic

[3]Department of Mathematics, KA
Charles University
Sokolovská 83, 18675 Praha 8
Czech Republic

[2]Centre for Discrete Mathematics and Computing,
Department of Mathematics
The University of Queensland
Queensland 4072, Australia

Abstract: Let T be a partial latin square and L be a latin square with $T \subseteq L$. We say that T is a latin trade if there exists a partial latin square T' with $T' \cap T = \emptyset$ such that $(L \setminus T) \cup T'$ is a latin square. A k-homogeneous latin trade is one which intersects each row, each column and each entry either 0 or k times. In this paper, we construct 4-homogeneous latin trades from rectangular packings of the plane with circles.

Keywords: Latin square, latin trade.

1 Introduction

We start with basic definitions which allow us to state and prove our main results. Let $N(n) = \{0, 1, 2, \dots n-1\}$, $R(n) = \{r_i \mid i \in N(n)\}$, $C(n) = \{c_i \mid i \in N(n)\}$ and $E(n) = \{e_i \mid i \in N(n)\}$.

A *partial latin square* P of order n is a set of ordered triples of the form (r_i, c_j, e_k), where $r_i \in R(n)$, $c_j \in C(n)$ and $e_k \in E(n)$ with the following properties:

- if $(r_i, c_j, e_k) \in P$ and $(r_i, c_j, e_{k'}) \in P$ then $k = k'$,
- if $(r_i, c_j, e_k) \in P$ and $(r_i, c_{j'}, e_k) \in P$ then $j = j'$ and
- if $(r_i, c_j, e_k) \in P$ and $(r_{i'}, c_j, e_k) \in P$ then $i = i'$.

The third author wishes to acknowledge the support of: The Institutional Grant MSM113200007; The Grant Agency of Charles University Grant 269/2001/B-MAT/MFF; University of Queensland Raybould Fellowship.

We may also represent a partial latin square P as an $n \times n$ array with entries chosen from the set N such that if $(r_i, c_j, e_k) \in P$, the *entry* k occurs in cell (r_i, c_j). (In this sense, r stands for "row", c for "column" and e for "entry".)

A partial latin square has the property that each entry occurs at most once in each row and at most once in each column. If all the cells of the array are filled then the partial latin square is termed a latin square. That is, a *latin square* L of order n is an $n \times n$ array with entries chosen from the set $E(n) = \{e_0, e_1, e_2, \dots, e_{n-1}\}$ in such a way that each element of $E(n)$ occurs precisely once in each row and precisely once in each column of the array.

For a given partial latin square P the set of cells $\mathcal{S}_P = \{(r_i, c_j) \mid (r_i, c_j, e_k) \in P, \text{ for some } e_k \in E(n)\}$ is said to determine the *shape* of P and $|\mathcal{S}_P|$ is said to be the *size* of the partial latin square. That is, the size of P is the number of non-empty cells in the array. For each $r_i \in R(n)$, let $\mathcal{R}_P^i$ denote the set of entries occurring in row r_i of P. Formally, $\mathcal{R}_P^i = \{e_k \mid (r_i, c_j, e_k) \in P\}$. For each $c_j \in C(n)$, we define $\mathcal{C}_P^j = \{e_k \mid (r_i, c_j, e_k) \in P\}$. Finally, for each $e_k \in E(n)$, we define $\mathcal{E}_P^k = \{(r_i, c_j) \mid (r_i, c_j, e_k) \in P\}$.

A partial latin square T of order n is said to be a *latin trade* (or *latin interchange*) if $T \neq \emptyset$ and there exists a partial latin square T' (called a *disjoint mate* of T) of order n, such that

- $\mathcal{S}_T = \mathcal{S}_{T'}$,
- if $(r_i, c_j, e_k) \in T$ and $(r_i, c_j, e_{k'}) \in T'$, then $k \neq k'$,
- for each $r_i \in R(n)$, $\mathcal{R}_T^i = \mathcal{R}_{T'}^i$, (the row r_i is *balanced*) and
- for each $c_j \in C(n)$, $\mathcal{C}_T^j = \mathcal{C}_{T'}^j$ (the column c_j is *balanced*).

Observe that if T is a latin trade with a disjoint mate T', then T' is also a latin trade, with a disjoint mate given by T. A latin trade T_1 is said to be *minimal* if there exists no latin trade T_2 with $T_2 \subset T_1$.

Information on latin trades may be found in [12], [13], [14], [16] and of course in the "latin square bible" [11]. In [15] it is shown how to embed a minimal latin trade onto an orientable surface. We thus may associated with every minimal latin trade a genus.

A latin trade T of order n is said to be k-homogeneous ($k \geq 2$) if

- for each $r_i \in R(n)$, $|\mathcal{R}_T^i| = 0$ or k, and
- for each $c_j \in C(n)$, $|\mathcal{C}_T^j| = 0$ or k, and
- for each $e_k \in E(n)$, $|\mathcal{E}_T^k| = 0$ or k.

Clearly if T is k-homogeneous, its volume is equal to km for some integer m. A minimal 2-homogeneous latin trade is uniquely a 2×2 latin subsquare. Constructions for 3-homogeneous latin trades of size $3m$ for each integer $m \geq 3$ are given in [10].

As well as being interesting combinatorial objects with frequently intriguing geometrical constructions, k-homogeneous latin trades occur often in the multiplication tables for groups (see, for example, [9]), and there is some indication they yield information about critical sets for such latin squares, often of minimum size.

A *critical set* in a latin square L of order n is a partial latin square $P \subseteq L$ such that $|P \cap T| \neq \emptyset$ for each latin trade $T \subseteq L$, and P is minimal with respect to this property. Equivalently, a critical set is a minimal defining set for a latin square. If P is a critical set in L, then for each $(r_i, c_j, e_k) \in P$ there exists a minimal latin trade $T \subseteq L$ such that $T \cap P = \{(r_i, c_j, e_k)\}$. Thus the concepts of critical sets and minimal latin trades are strongly interconnected. (Indeed they occupy the same chapter in the *CRC Handbook of Combinatorial Designs* [19]).

Because k-homogeneous latin trades often have the property of being large in size (with respect to the size of the latin square) yet minimal, they often are related to critical sets of small size. It is known that using only 2-homogeneous and 3-homogeneous latin trades, we can determine minimum critical sets in the latin squares for both $((\mathbb{Z}_2)^2, +)$ (size 5) and $((\mathbb{Z}_2)^3, +)$ (size 25) ([18]). (The size of the smallest critical set in $(\mathbb{Z}_2)^4, +)$ is not known, but is no greater than 121 [5].) We conjecture that k-homogeneous latin trades with $k > 3$ are needed to locate minimum critical sets in $((\mathbb{Z}_2)^n, +)$ for larger values of n.

The best known lower bound for the size of a critical set in an arbitrary latin square of order n is $\lfloor (4n-8)/3 \rfloor$ [17]. This bound can be improved under certain restrictions; such as if the critical set has an empty row ($2n-4$, [7]), the critical set has a strongly forced completion ($\lfloor n^2/4 \rfloor$, [3]), or if the latin square is the multiplication table for the integers modulo n ($n^{4/3}/2$, [8]). It is conjectured in [4] that in fact $\lfloor n^2/4 \rfloor$ is the actual lower bound. This is known to be true for $n \leq 8$ ([1], [2], [6]).

2 A doubling construction of latin trades

In this section we give a construction that takes an arbitrary latin trade and doubles the number of rows and the number of columns, whilst preserving the number of entries and the number of occurrences of each row, column and entry. We will show that if the original latin trade trade is minimal then the resultant latin trade is also minimal. This construction will be applied in the next section to obtain minimal 4-homogeneous latin trades.

DEFINITION 1 *Let T be a non-empty partial latin square of order m. Assume that $(r_0, c_0, e_0) \in T$. (Note that rows, columns and entries may be relabelled to make this assumption hold.) We define $T \bowtie T$ to be the following partial latin square of order $2m$:*

$$\{(r_0, c_1, e_0), (r_1, c_0, e_0)\} \cup \{(r_{2i}, c_{2j}, e_k), (r_{2i+1}, c_{2j+1}, e_k) \mid (r_i, c_j, e_k) \in T, (i,j) \neq (0,0)\}.$$

LEMMA 2 *Suppose that T is a latin trade of order m with disjoint mate T', and without loss of generality suppose that $(r_0, c_0, e_0) \in T$. Then $T \bowtie T$ is a latin trade of order $2m$ with a disjoint mate $T' \bowtie T'$.*

Proof: The fact that T and T' are disjoint and have the same shape implies that $T \bowtie T$ and $T' \bowtie T'$ are disjoint and have the same shape.

Now, we define S and S' to be the following partial latin squares:

$$S = \{(r_{2i}, c_{2i}, e_k), (r_{2i+1}, c_{2i+1}, e_k) \mid (r_i, c_j, e_k) \in T\},$$

and

$$S' = \{(r_{2i}, c_{2i}, e_k), (r_{2i+1}, c_{2i+1}, e_k) \mid (r_i, c_j, e_k) \in T'\},$$

Clearly S is a latin trade with disjoint mate S'. In fact S (S') is just equal to two copies of T (T').

Next, observe that for each $r_i \in R(2m)$, $\mathcal{R}^i_{T \bowtie T} = \mathcal{R}^i_S$ and $\mathcal{R}^i_{T' \bowtie T'} = \mathcal{R}^i_{S'}$. But since S' is a disjoint mate of S, $\mathcal{R}^i_S = \mathcal{R}^i_{S'}$. Thus for each $r_i \in R(2m)$, $\mathcal{R}^i_{T \bowtie T} = \mathcal{R}^i_{T' \bowtie T'}$.

Similarly, for each $c_j \in C(2m)$, $\mathcal{R}^j_{T \bowtie T} = \mathcal{R}^j_{T' \bowtie T'}$. Thus $T \bowtie T$ is a latin trade with disjoint mate $T' \bowtie T'$. $\square$

The next two lemmata will enable us to show that if T is a minimal latin trade, then $T \bowtie T$ is also minimal.

LEMMA 3 *Let T be a latin trade of order m. Suppose in addition that there exist $R_1, R_2 \subset R(m)$ and $C_1, C_2 \subset C(m)$ such that all of the following conditions hold.*

1. $|R_1|, |R_2|, |C_1|, |C_2| \geq 1$;
2. $R_1 \cup R_2 = R(m)$; $R_1 \cap R_2 = \emptyset$;
3. $C_1 \cup C_2 = C(m)$; $C_1 \cap C_2 = \emptyset$;
4. (r_i, c_j) *is empty in T if either $r_i \in R_1$ and $c_j \in C_2$ or $r_i \in R_2$ and $c_j \in C_1$;*
5. *there exists a non-empty cell $(r_i, c_j) \in T$ such that $r_i \in R_1$ and $c_j \in C_1$; and*
6. *there exists a non-empty cell $(r_i, c_j) \in T$ such that $r_i \in R_2$ and $c_j \in C_2$.*

Then T is not minimal.

Proof: Let $S_1 \subset T$ be the partial latin square $\{(r_i, c_j, e_k) \mid (r_i, c_j, e_k) \in T, r_i \in R_1 \text{ and } c_i \in C_1\}$. Let T' be a fixed disjoint mate of T, and let $S'_1 \subset T'$ be the partial latin square $\{(r_i, c_j, e_k) \mid (r_i, c_j, e_k) \in T', r_i \in R_1 \text{ and } c_i \in C_1\}$.

Clearly S_1 and S'_1 have the same shape and are disjoint. Let $(r_i, c_j, e_k) \in S_1$. (Such a triple exists because of Condition 5 (6) above.) Then $(r_i, c_{j'}, e_k) \in T'$ for some $j \neq j'$. But $c_{j'} \in C_1$, otherwise Condition 4 is violated. Thus $(r_i, c_{j'}, e_k) \in S'_1$. Similarly, $(r_{i'}, c_j, e_k) \in S'_1$ for some $i' \neq i$. Therefore S_1 is a latin trade with disjoint mate S'_1.

Similarly, let $S_2 \subset T$ be the partial latin square $\{(r_i, c_j, e_k) \mid (r_i, c_j, e_k) \in T, r_i \in R_2 \text{ and } c_i \in C_2\}$, and let $S'_2 \subset T'$ be the partial latin square $\{(r_i, c_j, e_k) \mid (r_i, c_j, e_k) \in T', r_i \in R_2 \text{ and } c_i \in C_2\}$. Then by similar reasoning as above, S_2 is a latin trade with disjoint mate S'_2.

But $S_1 \cup S_2 = T$, so T is not minimal. $\square$

LEMMA 4 *Let T be a partial latin square of order m. Suppose in addition that there exist $R_1, R_2 \subset R(m)$, $C_1, C_2 \subset C(m)$, $e_K \in E(m)$ such that all of the following conditions hold.*

1. $|R_1|, |R_2|, |C_1|, |C_2| \geq 1$*;*
2. $R_1 \cup R_2 = R(m)$*;* $R_1 \cap R_2 = \emptyset$*;*
3. $C_1 \cup C_2 = C(m)$*;* $C_1 \cap C_2 = \emptyset$*;*
4. *there is exactly one non-empty cell* $(r_I, c_J) \in S_T$ *such that* $r_I \in R_1$ *and* $c_J \in C_2$*, and this cell contains the entry* e_K*;*
5. *there is no cell* $(r_i, c_j) \in S_T$ *with* $r_i \in R_2$ *and* $c_j \in C_1$ *containing the entry* e_K*.*

Then T is not a latin trade.

Proof: Suppose, for the sake of contradiction, that T is a latin trade with disjoint mate T'. Let l be the number of occurences of the entry e_K in the latin trade T within the $R_1 \times C_1$ subarray. Then, considering the set of columns C_1, the fact that the columns of T and T' are balanced and Condition 5, e_K occurs at most l times in T' in the $R_1 \times C_1$ subarray. However, as $r_I \in R_1$, and the rows of T and T' are balanced, then entry e_K must occur exactly $l+1$ times in T' in the subarray $R_1 \cap C_1$, a contradiction. □

THEOREM 5 *If T is a minimal latin trade then $T \bowtie T$ is a minimal latin trade.*

Proof: Assume that $T \bowtie T$ is not a minimal latin trade; that is $S \subset T \bowtie T$ for some latin trade S. Without loss of generality, there are three possiblities: $(r_0, c_1, e_0) \notin S$ and $(r_1, c_0, e_0) \notin S$; or $(r_0, c_1, e_0) \in S$ and $(r_1, c_0, e_0) \notin S$; or $(r_0, c_1, e_0), (r_1, c_0, e_0) \in S$.

In the first case let $S = S_1 \cup S_2$, where S_1 contains the cells selected from even rows and even columns, while S_2 contains the cells selected from odd rows and odd columns. Since S is a latin trade, S is non-empty, so one of S_1 or S_2 is non-empty. If S_1 is non-empty, then T contains an isomorphic copy of S_1, contradicting the fact that T is a minimal latin trade. If S_2 is non-empty, we obtain the same contradiction.

The second case is ruled out by Lemma 4. Otherwise, consider the third case, and let S' be a disjoint mate of S. If the cells (r_0, c_1) and (r_1, c_0) contain different entries in S', then by considering S' to be a latin trade with disjoint mate S, we again get a contradiction from Lemma 4. So we can assume that $(r_0, c_1, e_K), (r_1, c_0, e_K) \in S'$ for some entry e_K. Next, let S_3 be the partial latin square

$$(S \setminus \{(r_0, c_1, e_0), (r_1, c_0, e_0)\}) \cup \{(r_0, c_0, e_0), (r_1, c_1, e_0)\}.$$

Then S_3 is row and column balanced with S, and is thus a latin trade with disjoint mate

$$(S' \setminus \{(r_0, c_1, e_K), (r_1, c_0, e_K)\}) \cup \{(r_0, c_0, e_K), (r_1, c_1, e_K)\}.$$

But from Lemma 3, S_3 is not minimal. So let $S_3 = S_4 \cup S_5$, where S_4 is the latin trade from the even rows and even columns, while S_5 is the latin trade from the odd rows and odd columns. Since S is strictly a subset of T, taking the isomorphic copy of at least one of S_4, S_5 in T implies that T is not minimal, a contradiction. □

3 Constructing latin trades from circle packings

In what follows let m be a positive integer greater than or equal to 4 and let $\mathbb{E} = \{2e \mid e \in \mathbb{Z}\}$ be the set of even integers.

Let $\mathcal{C}$ be the set of unit circles in $\mathbb{R} \times \mathbb{R}$ with centres given by the points (x, y), where $x, y \in \mathbb{E}$. For $x, y \in \mathbb{E}$, we will refer to the circle centred at the point (x, y) as $C(x, y)$. The set $\mathcal{C}$ can be partitioned into two subsets $\mathcal{C}_1$ and $\mathcal{C}_2$, where

$$\begin{aligned} \mathcal{C}_1 &= \{C(x, y) \mid x + y \equiv 0 \pmod 4\}, \text{ and} \\ \mathcal{C}_2 &= \{C(x, y) \mid x + y \equiv 2 \pmod 4\}. \end{aligned}$$

LEMMA 6 *For each pair of distinct circles $C(x, y), C(r, s) \in \mathcal{C}$ such that $C(x, y) \cap C(r, s) \neq \emptyset$, we have $|C(x, y) \cap C(r, s)| = 1$, $C(x, y) \cap C(r, s) \subseteq \{(x, y+1), (x+1, y), (x, y-1), (x-1, y)\}$, and $(r, s) \in \{(x, y+2), (x+2, y), (x, y-2), (x-2, y)\}$ and so $C(x, y) \in \mathcal{C}_1$ if and only if $C(r, s) \in \mathcal{C}_2$.*

DEFINITION 7 For any given circle $C(x, y) \in \mathcal{C}$ we define the *neighbourhood*, $\mathcal{N}[C(x, y)]$, of $C(x, y)$ to be a set of eight circles, where $C(r, s) \in \mathcal{N}[C(x, y)]$ if and only if

$$\begin{aligned} (r, s) \in \{&(x, y + 4), (x, y - 4), (x + 2, y + 2), (x + 2, y - 2), \\ &(x - 4, y), (x + 4, y), (x - 2, y + 2), (x - 2, y - 2)\}. \end{aligned}$$

LEMMA 8 *For each $C(x, y) \in \mathcal{C}_1$, if $C(r, s) \in \mathcal{N}[C(x, y)]$, then $C(r, s) \in \mathcal{C}_1$.*

DEFINITION 9 For $m \geq 4$, let f be an onto labelling such that $f : \mathcal{C}_1 \to R(m)$. We say that f is a *proper* labelling if, for all $C(x, y), C(r, s) \in \mathcal{C}_1$, whenever $C(r, s) \in \mathcal{N}[C(x, y)]$, then $f(C(r, s)) \neq f(C(x, y))$. Further, a proper labelling f is *coherent* if whenever $C(x, y), C(r, s) \in \mathcal{C}_1$ and $f(C(x, y)) = f(C(r, s))$, then $f(C(x + a, y + b)) = f(C(r + a, s + b))$ for all $a, b \in \mathbb{E}$ with $a + b \equiv 0 \pmod 4$.

DEFINITION 10 Let $f : \mathcal{C}_1 \to R(m)$, where $m \geq 4$, be a proper coherent labelling of $\mathcal{C}_1$. We define g to be the induced labelling, $g : \mathcal{C}_2 \to C(m)$, where $g(C(x, y)) = c_j$ if and only if $f(C(x, y - 2)) = r_j$. (It is noted that since f is onto $R(m)$, g is onto $C(m)$.)

LEMMA 11 *Let g be as in Definition 10. Then for all $C(x, y), C(r, s) \in \mathcal{C}_2$, if $g(C(x, y)) = g(C(r, s))$, then $g(C(x + a, y + b)) = g(C(r + a, s + b))$, for all $a, b \in \mathbb{E}$.*

Proof: This follows from the definition of g and the fact that f is a proper, coherent labelling. □

LEMMA 12 *Let $f : \mathcal{C}_1 \to R(m)$, where $m \geq 4$, be a proper coherent labelling of $\mathcal{C}_1$. For all $x, y \in \mathbb{E}$ such that $x + y \equiv 0 (mod\ 4)$, $g(C(x + a, y + b)) \neq g(C(x + c, y + d))$, where $(a, b), (c, d) \in \{(0, 2), (2, 0), (0, -2), (-2, 0)\}$ and $(a, b) \neq (c, d)$.*

Proof: Assume $g(C(x+a,y+b)) = g(C(x+c,y+d))$, where $(a,b),(c,d)$ are fixed but distinct elements of $\{(0,2),(2,0),(0,-2),(-2,0)\}$. Then $f(C(x+a,y-2+b)) = f(C(x+c,y-2+d))$. However, $C(x+a,y-2+b) \in \mathcal{N}[C(x+c,y-2+d)]$, and so $f(C(x+a,y-2+b)) = f(C(x+c,y-2+d))$ contradicts the fact that f is a proper labelling. The result now follows. $\square$

LEMMA 13 *Let $f : \mathcal{C}_1 \to R(m)$, where $m \geq 4$, be a proper coherent labelling of $\mathcal{C}_1$ and let g be as in Definition 10. For all $C(x,y) \in C_2$, $f(C(x+a,y+b)) \neq f(C(x+c,y+d))$, where $(a,b),(c,d) \in \{(0,2),(2,0),(0,-2),(-2,0)\}$ and $(a,b) \neq (c,d)$.*

Proof: The proof of this result follows from Lemma 12 and Definition 10. $\square$

DEFINITION 14 For all $x,y \in \mathbb{E}$, such that $x+y \equiv 0 (\mathrm{mod}\ 4)$, define a collection $\mathcal{S}$ of line segments $H(x,y)$, $V(x,y)$ as follows:

$$\begin{aligned} H(x,y) &= \{(x', y+1) \mid x' \in [x, x+2]\}, \\ V(x,y) &= \{(x+1, y') \mid y' \in [y, y-2]\}, \\ \mathcal{S} &= \{H(x,y), V(x,y) \mid x,y \in \mathbb{E}, x+y \equiv 0 (\mathrm{mod}\ 4)\}. \end{aligned}$$

LEMMA 15 *For each pair of distinct circles $C(x,y), C(r,s) \in \mathcal{C}$ such that $C(x,y) \cap C(r,s) \neq \emptyset$, there exists a unique line segment $S \in \mathcal{S}$, satisfying $C(x,y) \cap C(r,s) \cap S \neq \emptyset$. Further, without loss of generality we may assume that $C(x,y) \in \mathcal{C}_1$ and precisely one of the following holds:*

1. *$C(r,s) = C(x,y+2)$, $S = H(x,y)$ and $C(x,y) \cap C(r,s) \cap S = \{(x,y+1)\}$;*
2. *$C(r,s) = C(x,y-2)$, $S = H(x-2,y-2)$ and $C(x,y) \cap C(r,s) \cap S = \{(x,y-1)\}$;*
3. *$C(r,s) = C(x+2,y)$, $S = V(x,y)$ and $C(x,y) \cap C(r,s) \cap S = \{(x+1,y)\}$;*
4. *$C(r,s) = C(x-2,y)$, $S = V(x-2,y+2)$ and $C(x,y) \cap C(r,s) \cap S = \{(x-1,y)\}$.*

Proof: The result follows from observation of Figure 1. $\square$

DEFINITION 16 Let $f : \mathcal{C}_1 \to R(m)$, where $m \geq 4$, be a proper coherent labelling of $\mathcal{C}_1$. Define h to be the induced labelling $h : \mathcal{S} \to E(2m)$, where

$$h(S) = \begin{cases} e_i, & \text{if and only if } S = H(x,y) \text{ and } f(C(x,y)) = r_i, \text{ or} \\ e_{i+m}, & \text{if and only if } S = V(x,y) \text{ and } f(C(x,y)) = r_i. \end{cases}$$

LEMMA 17 *The labelling h defined in Definition 16 is a well defined function from $\mathcal{S}$ onto $E(2m)$.*

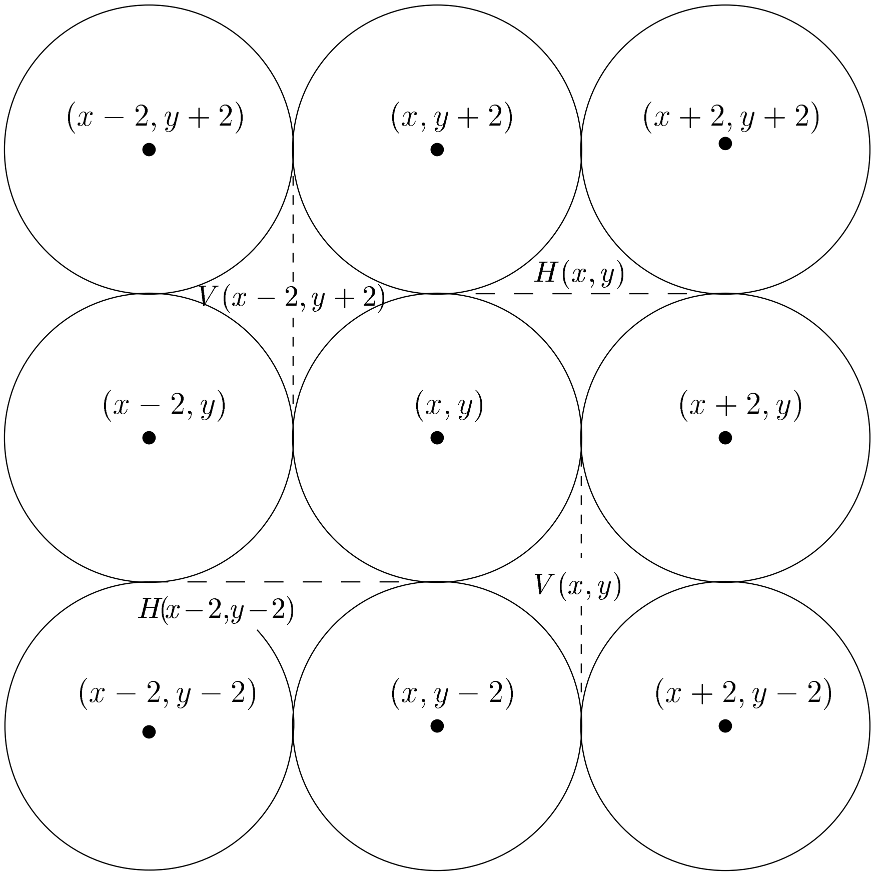

Figure 1: Some line segments from $\mathcal{S}$.

Proof: The fact that h is well-defined follows from the fact that f is a coherent labelling.

To show h is onto we let $e_i \in E(2m)$. If $0 \leq i \leq m-1$, then, since f is onto $R(m)$, there exists a circle $C(x_0, y_0) \in \mathcal{C}_1$ such that $f(C(x_0, y_0)) = r_i$. Further, $C(x_0, y_0) \cap C(x_0, y_0 + 2) \cap H(x_0, y_0) = \{(x_0, y_0+1)\}$ where $H(x_0, y_0) = \{(x', y_0+1) \mid x' \in [x_0, x_0+2]\}$. Thus there exists an $S \in \mathcal{S}$, namely $S = H(x_0, y_0)$ such that $h(S) = e_i$. If $m \leq i \leq 2m-1$, then, since f is onto $R(m)$, there exists a circle $C(x_0, y_0) \in \mathcal{C}_1$ such that $f(C(x_0, y_0)) = r_{i-m}$. Further, $C(x_0, y_0) \cap C(x_0+2, y_0) \cap V(x_0, y_0) = \{(x_0+1, y_0)\}$ where $V(x_0, y_0) = \{(x_0+1, y') \mid y' \in [y_0, y_0-2]\}$. Thus there exists an $S \in \mathcal{S}$, namely $S = V(x_0, y_0)$ such that $h(S) = e_i$. $\square$

DEFINITION 18 Let $f : \mathcal{C}_1 \to R(m)$, where $m \geq 4$, be a proper coherent labelling of $\mathcal{C}_1$, and let g and h be the induced labellings given in Definitions 10 and 16. Let $\circ : R(m) \times C(m) \to E(2m)$ be a binary operation where for each $r_i \in R(m)$ and each $c_j \in C(m)$, $r_i \circ c_j = e_k$, if there exists $C(x,y), C(r,s) \in \mathcal{C}, S \in \mathcal{S}$ such that $f(C(x,y)) = r_i, g(C(r,s)) = c_j, h(S) = e_k$ and $C(x,y) \cap C(r,s) \cap S \neq \emptyset$. Otherwise $r_i \circ c_j$ is undefined.

LEMMA 19 *The binary operation $\circ$ given in Definition 18 is well defined.*

Proof: This follows from Lemma 15 and the fact that f is coherent. $\square$

LEMMA 20 Let $f : \mathcal{C}_1 \to R(m)$, where $m \geq 4$, be a proper coherent labelling of $\mathcal{C}_1$. Let $C(x, y) \in \mathcal{C}_1$ be a circle such that $f(C(x, y)) = r_i$. (Such a circle exists because f is onto $R(m)$.) Then $r_i \circ c_j = e_k$ if and only if one of the following holds:

1. $g(C(x, y + 2)) = c_j$ and $h(H(x, y)) = e_k$;
2. $g(C(x, y - 2)) = c_j$ and $h(H(x - 2, y - 2)) = e_k$;
3. $g(C(x + 2, y)) = c_j$ and $h(V(x, y)) = e_k$;
4. $g(C(x - 2, y)) = c_j$ and $h(V(x - 2, y + 2)) = e_k$.

Proof: Cases 1 to 4 above follow from Cases 1 to 4 respectively from Lemma 15. $\square$

COROLLARY 21 Let $f : \mathcal{C}_1 \to R(m)$, where $m \geq 4$, be a proper coherent labelling of $\mathcal{C}_1$. Let $C(x', y') \in \mathcal{C}_2$ be a circle such that $g(C(x', y')) = c_j$. (Such a circle exists because g is onto $C(m)$.) Then $r_i \circ c_j = e_k$ if and only if one of the following holds:

1. $f(C(x', y' - 2)) = r_i$ and $h(H(x', y' - 2)) = e_k$;
2. $f(C(x', y' + 2)) = r_i$ and $h(H(x' - 2, y')) = e_k$;
3. $f(C(x' - 2, y')) = r_i$ and $h(V(x' - 2, y')) = e_k$;
4. $f(C(x' + 2, y')) = r_i$ and $h(V(x', y' + 2)) = e_k$.

Proof: Let $C(x', y') \in \mathcal{C}_2$ be a circle such that $g(C(x', y')) = c_j$. Then setting $x = x'$, $y = y' - 2$ in Case 1 of Lemma 20 gives Case 1 above. Similarly, letting $x = x'$ and $y = y' + 2$ in Case 2 of Lemma 20 gives Case 2 above. Next, let $x = x' - 2$ and $y = y'$ in Case 3 of Lemma 20 to obtain Case 3 above. Finally, letting $x = x' + 2$ and $y = y'$ in Case 4 of Lemma 20 gives Case 4 above. $\square$

COROLLARY 22 Let $f : \mathcal{C}_1 \to R(m)$, where $m \geq 4$, be a proper coherent labelling of $\mathcal{C}_1$. Let $H(x, y) \in \mathcal{S}$ be a line segment such that $h(H(x, y)) = e_k$, where $0 \leq k \leq m - 1$. (Such a circle exists because h is onto $E(2m)$.) Then $r_i \circ c_j = e_k$ if and only if one of the following holds:

1. $f(C(x, y)) = r_i$ and $g(C(x, y + 2)) = c_j$;
2. $f(C(x + 2, y + 2)) = r_i$ and $g(C(x + 2, y)) = c_j$.

Next, let $V(x, y) \in \mathcal{S}$ be a circle such that $h(V(x, y)) = e_k$, where $m \leq k \leq 2m - 1$. (Such a circle exists because h is onto $E(2m)$.) Then $r_i \circ c_j = e_k$ if and only if one of the following holds:

3. $f(C(x, y)) = r_i$ and $g(C(x + 2, y)) = c_j$;

4. $f(C(x+2,y-2)) = r_i$ and $g(C(x,y-2)) = c_j$.

Proof: The easiest proof comes by observation of Figure 1. $\square$

LEMMA 23 *For each $r_i \in R(m)$, there exists precisely four distinct entries $c_{ij} \in C(m)$, $j = 1,2,3,4$, for which $r_i \circ c_{ij}$ is defined. Further, $r_i \circ c_{ij} \neq r_i \circ c_{ij'}$, for distinct $j, j' \in \{1,2,3,4\}$.*

Proof: Let $C(x,y) \in \mathcal{C}_1$ be a circle such that $f(C(x,y)) = r_i$. Since f is a proper labelling and g is an induced labelling, Lemma 12 implies that the labels $g(C(x,y+2))$, $g(C(x+2,y))$, $g(C(x,y-2))$ and $g(C(x-2,y))$ are all distinct. Thus from Lemma 20, and since f is coherent, there exists precisely four distinct $c_j \in C(m)$ such that $r_i \circ c_j$ is defined.

Let these elements of $C(m)$ be c_{i1}, c_{i2}, c_{i3} and c_{i4} respectively for Cases 1 through to 4 of Lemma 20. The fact that f is a proper labelling ensures that $f(C(x,y)) \neq f(C(x-2,y-2))$, and from the definition of h, we have $h(C(x,y)) \neq h(C(x-2,y-2))$. Thus $r_i \circ c_{i1} \neq r_i \circ c_{i2}$. Similarly, $f(C(x,y)) \neq f(C(x-2,y+2))$, implying $r_i \circ c_{i3} \neq r_i \circ c_{i4}$. Finally $r_i \circ c_{i1}, r_i \circ c_{i2} \in \{e_i \mid 0 \leq i \leq m-1\}$, while $r_i \circ c_{i3}, r_i \circ c_{i4} \in \{e_i \mid m \leq i \leq 2m-1\}$. It follows that $r_i \circ c_{ij} \neq r_i \circ c_{ij'}$, for distinct $j, j' \in \{1,2,3,4\}$.

$\square$

LEMMA 24 *For each $c_j \in C(m)$, there exists precisely four distinct integers $r_{ji} \in R(m)$, $i \in \{1,2,3,4\}$, for which $r_{ji} \circ c_j$ is defined. Further, $r_{ji} \circ c_j \neq r_{ji'} \circ c_j$, for distinct $i, i' \in \{1,2,3,4\}$.*

Proof: Let $C(x',y') \in \mathcal{C}_2$ be a circle such that $g(C(x',y')) = c_j$. Since f is a proper labelling, from Lemma 13 the labels $f(C(x',y'+2))$, $f(C(x'+2,y'))$, $f(C(x',y'-2))$ and $f(C(x'-2,y'))$ are all distinct. Thus, since f is coherent, there exist precisely four distinct $r_i \in R(m)$ such that $r_i \circ c_j$ is defined.

Let these elements of $R(m)$ be r_{j1}, r_{j2}, r_{j3} and r_{j4} for Cases 1 through to 4 of Lemma 21, respectively. Lemma 13 tells us that $f(C(x',y'-2)) \neq f(C(x'-2,y'))$, which implies that $h(H(x',y'-2)) \neq h(H(x'-2,y'))$. Also, $f(C(x'-2,y')) \neq f(C(x',y'+2))$, implying $h(V(x'-2,y')) \neq h(V(x',y'+2))$. It follows that $r_{ji} \circ c_j \neq r_{ji'} \circ c_j$, for distinct $i, i' \in \{1,2,3,4\}$. $\square$

LEMMA 25 *For each $e_k \in E(2m)$ there exists precisely two distinct entries $r_{k1}, r_{k2} \in R(m)$ such that $r_{k1} \circ c_{k1} = e_k = r_{k2} \circ c_{k2}$, for some $c_{k1}, c_{k2} \in C(m)$, with $k1 \neq k2$.*

Proof: Fix $e_k \in E(2m)$.

First, suppose that $0 \leq \alpha \leq m-1$. Then since h is onto (from Lemma 17), $e_k = h(H(x,y))$, for some $H(x,y) \in S$. Since f is proper, $f(((C(x,y)) \neq f(C(x+2,y+2))$. Thus, since f is coherent, from Corollary 22, we have exactly two distinct values r_i such that $r_i \circ c_j = e_k$ for some $c_j \in C(m)$. So let $r_{k1} \circ c_{k1} = e_k = r_{k2} \circ c_{k2}$, where r_{k1} and c_{k1} come from Case 1 of Corollary 22 and r_{k2} and c_{k2} come from Case 2 of Corollary 22. From Lemma 12, $g(C(x,y+2)) \neq g(C(x+2,y))$. Thus $k1 \neq k2$.

Otherwise, $m \le \alpha \le 2m-1$. Then $e_k = h(V(x,y))$, for some $V(x,y) \in S$. Since f is proper, $f((C(x,y)) \neq f(C(x+2,y-2))$. Thus, since f is coherent, from Corollary 22, we have exactly two values r_i such that $r_i \circ c_j = e_k$ for some $c_j \in C(m)$. So let $r_{k1} \circ c_{k1} = e_k = r_{k2} \circ c_{k2}$, where r_{k1} and c_{k1} come from Case 3 of Corollary 22 and r_{k2} and c_{k2} come from Case 4 of Corollary 22. From Lemma 12, $g(C(x+2,y)) \neq g(C(x,y-2))$. Thus $k1 \neq k2$, as above. □

LEMMA 26 *Let $e_k \in E(2m)$. Let $r_{k1}, r_{k2} \in R(m)$, $c_{k1}, c_{k2} \in C(m)$, be the unique entries such that $r_{k1} \circ c_{k1} = e_k = r_{k2} \circ c_{k2}$ and $r_{k1} \neq r_{k2}$, $c_{k1} \neq c_{k2}$, as per Lemma 25. Then $r_{k1} \circ c_{k2}$ and $r_{k2} \circ c_{k1}$ are both defined, and moreover $r_{k1} \circ c_{k2}$, $r_{k2} \circ c_{k1}$ and e_k are pairwise distinct.*

Proof: There are two cases to consider, namely $0 \le k \le m-1$ and $m \le k \le 2m-1$.

Assume $0 \le k \le m-1$, then there exists an $x', y' \in \mathbb{E}$ such that $h(H(x',y')) = e_k$. From Corollary 22, $f(C(x',y')) = r_{k1}$, $g(C(x',y'+2)) = c_{k1}$, $f(C(x'+2,y'+2)) = r_{k2}$ and $g(C(x'+2,y')) = c_{k2}$.

So letting $x = x'$ and $y = y'$, from Case 3 of Lemma 20, $r_{k1} \circ c_{k2}$ is defined and is equal to $h(V(x',y'))$. Next, we let $x = x'+2$ and $y = y'+2$, and from Case 4 of Lemma 20, $r_{k2} \circ c_{k1}$ is defined and is equal to $h(V(x',y'+4))$. Since f is proper, $f(C(x',y')) \neq f(C(x',y'+4))$, which in turn implies that $h(V(x',y')) \neq h(V(x',y'+4))$. Also, since $0 \le k \le m-1$, $e_k \neq h(V(x',y'))$ and $e_k \neq h(V(x',y'+4))$.

Next let $m \le k \le 2m-1$. Then there exists an $x', y' \in \mathbb{E}$ such that $h(V(x',y')) = e_k$. From Corollary 22, $f(C(x',y')) = r_{k1}$, $g(C(x'+2,y')) = c_{k1}$, $f(C(x'+2,y'-2)) = r_{k2}$ and $g(C(x',y'-2)) = c_{k2}$.

So letting $x = x'$ and $y = y'$, from Case 2 of Lemma 20, $r_{k1} \circ c_{k2} = f(C(x',y')) \circ g(C(x',y'-2))$ is defined and is equal to $h(H(x'-2,y'-2))$. Next, we let $x = x'+2$ and $y = y'-2$, and from Case 1 of Lemma 20, $r_{k2} \circ c_{k1} = f(C(x'+2,y'-2)) \circ g(C(x'+2,y'))$ is defined and is equal to $h(H(x'+2,y'-2))$. Since f is proper, $f(C(x'-2,y'-2)) \neq f(C(x'+2,y'-2))$, which in turn implies that $h(H(x'-2,y'-2)) \neq h(H(x'+2,y'-2))$. Also, since $m \le k \le 2m-1$, $e_k \neq h(H(x'-2,y'-2))$ and $e_k \neq h(H(x'-2,y'+2))$. □

DEFINITION 27 Let m be a positive integer greater than 3, and let $\circ$ be the binary operation given in Definition 18. Define $(N(2m), \circ)$ to be a $2m \times 2m$ array, where cell (r_i, c_j) contains symbol k if, and only if, $r_i \circ c_j = e_k$.

LEMMA 28 *Let m be a positive integer greater than 3. The array $(N(2m), \circ)$ constructed in Definition 27 is a partial latin square of order $2m$ which satisfies the following properties:*

1. *Each row contains either zero or four filled cells.*

2. *Each column contains either zero or four filled cells.*

3. *Each of the symbols $0, \ldots, 2m-1$ occurs in precisely two filled cells.*

4. *Let $e_k \in E(2m)$ and let (r_i, c_j) and $(r_{i'}, c_{j'})$ represent the two cells containing symbol e_k. Then there exists $e_{k'}, e_{k''} \in E(2m)$ such that $e_{k'}$ occurs in cell $(r_i, c_{j'})$ and symbol $e_{k''}$ occurs in cell $(r_{i'}, c_j)$. Further all three symbols $e_k, e_{k'}, e_{k''}$ are distinct.*

Proof: The definition of the binary operation $\circ$ and Lemmas 23 and 24 imply that each of the symbols $0, \ldots, 2m-1$ occurs at most once in every row and at most once in each column. Hence $(N(2m), \circ)$ is clearly a partial latin square of order $2m$. Points 1 and 2 follow from Lemmas 23 and 24. Lemmas 17 and 25 imply that each symbol occurs in two rows and two columns. Finally, Point 4 can be deduced directly from Lemma 26. $\square$

DEFINITION 29 Let $f : \mathcal{C}_1 \to R(m)$, where $m \geq 4$, be a proper coherent labelling of $\mathcal{C}_1$, and let g and h be the induced labellings given in Definitions 10 and 16. Let $* : R(m) \times C(m) \to E(2m)$ be a binary operation where for each $r_i \in R(m)$ and each $c_j \in C(m)$, $r_i * c_j = e_k$, if and only if $r_i \circ c_{j'} = e_k = r_{i'} \circ c_j$ for some $r_{i'} \in R(m)$ and $c_{j'} \in C(m)$, with $i \neq i'$ and $j \neq j'$. Otherwise $r_i * c_j$ is undefined.

Further, define $(N(2m), *)$ to be a $2m \times 2m$ array, where cell (i, l) contains symbol k if, and only if, $r_i * c_l = e_k$.

LEMMA 30 *The binary operation $*$ given in Definition 29 is well defined.*

Proof: Assume that this is not the case. That is, there exists $r_i \in R(m)$ and $c_j \in C(m)$ such that $r_i * c_j = e_\alpha$ and $r_i * c_j = e_\beta$, where α and β are distinct elements of $N(2m)$. Then there exists $r_{i'}, r_{i''} \in R(m)$, where $r_i, r_{i'}, r_{i''}$ are all distinct, and there exists $c_{j'}, c_{j''} \in C(m)$, where $c_j, c_{j'}, c_{j''}$ are all distinct, such that

$$r_i \circ c_{j'} = e_\alpha = r_{i'} \circ c_j, \quad \text{and,} \quad r_i \circ c_{j''} = e_\beta = r_{i''} \circ c_j.$$

First assume $0 \leq \alpha \leq m-1$ and so $e_\alpha = h(H(x, y))$ for some $x, y \in \mathbb{E}$. From Corollary 22, either:

(*A*) $f(C(x, y)) = r_i, g(C(x+2, y)) = c_j, f(C(x+2, y+2)) = r_{i'}, g(C(x, y+2)) = c_{j'}$;
(*B*) $f(C(x+2, y+2)) = r_i, g(C(x, y+2)) = c_j, f(C(x, y)) = r_{i'}, g(C(x+2, y)) = c_{j'}$.

First consider Case A. From Lemma 20 there are four possibilities for $c_{j''}$: $g(C(x, y+2))$, $g(C(x, y-2))$, $g(C(x+2, y))$ and $g(C(x-2, y))$. However the first case implies $j' = j''$ and the third case implies $j'' = j$. So either $e_\beta = h(H(x-2, y-2))$ or $e_\beta = h(V(x-2, y+2))$. However from analysis of Corollary 22 it is then impossible to find $r_{i''}$ such that $r_{i''} \circ c_j = e_\beta$.

Next consider Case B. From Lemma 20 there are four possibilities for $c_{j''}$: $g(C(x+2, y+4))$, $g(C(x+2, y))$, $g(C(x+4, y+2))$ and $g(C(x, y+2))$. However the second case implies $j'' = j'$ and the fourth case implies $j = j''$. So either $e_\beta = h(H(x+2, y+2))$ or $e_\beta = h(V(x+2, y+2))$. We want to find $r_{i''}$ such that $r_{i''} \circ c_j = e_\beta$, where $c_j = g(C(x, y+2))$. However Corollary 22 rules out the existence of such an entry $r_{i''}$.

Next let $m \leq \alpha \leq 2m-1$ and so $e_\alpha = h(V(x, y))$ for some $x, y \in \mathbb{E}$. From Corollary 22, either

(*C*) $f(C(x, y)) = r_i, g(C(x, y-2)) = c_j, f(C(x+2, y-2)) = r_{i'}, g(C(x+2, y)) = c_{j'}$;
(*D*) $f(C(x+2, y-2)) = r_i, g(C(x+2, y)) = c_j, f(C(x, y)) = r_{i'}, g(C(x, y-2)) = c_{j'}$.

First consider Case C. From Lemma 20 there are four possibilities for $c_{j''}$: $g(C(x,y+2))$, $g(C(x,y-2))$, $g(C(x+2,y))$ and $g(C(x-2,y))$. However the second case implies $j''=j$ and the third case implies $j''=j'$. So either $e_\beta = h(H(x,y))$ or $e_\beta = h(V(x-2,y+2))$. However from Corollary 22 it is then impossible to find $r_{i''}$ such that $r_{i''} \circ c_j = e_\beta$.

Finally consider Case D. From Lemma 20 there are four possibilities for $c_{j''}$: $g(C(x+2,y))$, $g(C(x+2,y-4))$, $g(C(x+4,y-2))$ and $g(C(x,y-2))$. However the first case implies $j''=j$ and the fourth case implies $j''=j'$. So either $e_\beta = h(H(x,y-4))$ or $e_\beta = h(V(x+2,y-2))$. We want to find $r_{i''}$ such that $r_{i''} \circ c_j = e_\beta$, where $c_j = g(C(x+2,y))$. Once again Corollary 22 rules out the existence of such an entry $r_{i''}$. □

LEMMA 31 *The array $(N(2m),*)$ is a partial latin square of order $2m$ which satisfies the following properties:*

1. *$(N(2m),*)$ has the same shape as $(N(2m),\circ)$;*
2. *$(N(2m),*)$ and $(N(2m),\circ)$ are disjoint;*
3. *$(N(2m),*)$ and $(N(2m),\circ)$ are row and column balanced.*

Proof: First we prove that the arrays definined in Definitions 18 and 29 have the same shape and are disjoint. Assume that there exists a cell (r_i, c_j) which contains symbol e_k in $(N(2m),*)$. The definition of $*$ implies that there exists $r_{i'} \in R(m)(i' \neq i)$ and $c_{j'} \in C(m)(j' \neq j)$ such that $r_i \circ c_{j'} = e_\alpha = r_{i'} \circ c_j$. Now Lemma 26 implies that cell (r_i, c_j) of $(N(2m),\circ)$ contains symbol e_{k1}, where $k1 \neq k$.

Next assume that there exists a cell (r_i, c_j) which contains symbol e_k in $(N(2m),\circ)$. Let $C(x,y) \in \mathcal{C}_1$ be a circle such that $f(C(x,y)) = r_i$. Thus from Lemma 20 there are four cases for c_j and e_k:

1. $g(C(x,y+2)) = c_j$ and $h(H(x,y)) = e_k$;
2. $g(C(x,y-2)) = c_j$ and $h(H(x-2,y-2)) = e_k$;
3. $g(C(x+2,y)) = c_j$ and $h(V(x,y)) = e_k$;
4. $g(C(x-2,y)) = c_j$ and $h(V(x-2,y+2)) = e_k$.

For Case 1, let $r_{i'} = f(C(x-2,y+2))$ and $c_{j'} = g(C(x-2,y))$. Since f is proper and from Lemma 12, $i \neq i'$ and $j \neq j'$. From Lemma 20, $r_i \circ c_{j'} = h(V(x-2,y+2)) = r_{i'} \circ c_j$. Moreover, $h(V(x-2,y+2)) \neq h(H(x,y)) = e_k$ by the definition of h. Hence by Definition 29 the cell (r_i, c_j) is filled in $(N(2m,\star)$.

For Case 2, let $r_{i'} = f(C(x+2,y-2))$ and $c_{j'} = g(C(x+2,y))$. Since f is proper and from Lemma 12, $i \neq i'$ and $j \neq j'$. From Lemma 20, $r_i \circ c_{j'} = h(V(x,y)) = r_{i'} \circ c_j$. Moreover $h(V(x,y)) \neq h(H(x-2,y-2)) = e_k$. Therefore by Definition 29 the cell (r_i, c_j) is filled in $N(2m,\star)$.

For Case 3, let $r_{i'} = f(C(x+2,y+2))$ and $c_{j'} = g(C(x,y+2))$. As in Cases 1 and 2, $i \neq i'$ and $j \neq j'$. From Lemma 20, $r_i \circ c_{j'} = h(H(x,y)) = r_{i'} \circ c_j$. Moreover $h(H(x,y)) \neq h(V(x,y)) = e_k$. So once again the cell (r_i, c_j) is filled in $N(2m,\star)$.

Finally for Case 4, let $r_{i'} = f(C(x-2, y-2))$ and $c_{j'} = g(C(x, y-2))$. As in Cases 1, 2 and 3, $i \neq i'$ and $j \neq j'$. From Lemma 20, $r_i \circ c_{j'} = h(H(x-2, y-2)) = r_{i'} \circ c_j$. Moreover $h(H(x-2, y-2)) \neq h(V(x-2, y+2)) = e_k$. Therefore the cell (r_i, c_j) is filled in $N(2m, \star)$.

So we have shown that $(N(2m), *)$ has the same shape as $(N(2m), \circ)$ and that $(N(2m), *)$ and $(N(2m), \circ)$ are disjoint.

Consider the occurrence of symbol e_k in the partial latin square $(N(2m), \circ)$. Lemma 25 implies that e_k occurs in precisely two cells, say (r_u, c_v) and (r_w, c_z), ($u \neq w$ and $v \neq z$) of $(N(2m), \circ)$. Definition 29 implies that e_k occurs in cells (r_u, c_z) and (r_w, c_v) of $(N(2m), *)$. Since $(N(2m), *)$ and $(N(2m), \circ)$ are the same shape and disjoint, it follows that the two partial latin squares are row and column balanced, in turn implying that $(N(2m), *)$ is a partial latin square. $\square$

COROLLARY 32 *The partial latin square $(N(2m), \circ)$ is a latin trade with disjoint mate $(N(2m), *)$.*

REMARK 33 *It should be noted that the partial latin square $(N(2m), *)$ may also be constructed directly from line segments in $\mathbb{R} \times \mathbb{R}$. To see this, we take the line segments of $\mathcal{S}$ and rotate them through 90^o about their midpoint. That is, for all $x, y \in \mathbb{E}$, such that $x + y \equiv 0 (mod\ 4)$, define a collection $\mathcal{S}'$ of line segments $H'(x, y)$ and $V'(x, y)$ in the plane $\mathbb{R} \times \mathbb{R}$, as follows:*

$$H'(x, y) = \{(x+1, y') \mid y' \in [y, y+2]\},$$
$$V'(x, y) = \{(x', y-1) \mid x' \in [x, x+2]\},$$
$$\mathcal{S}' = \{H(x, y)', V(x, y)' \mid x, y \in \mathbb{E}, x + y \equiv 0 (mod\ 4)\}.$$

Let $f : \mathcal{C}_1 \to R(m)$, where $m \geq 4$, be a proper coherent labelling of $\mathcal{C}_1$. Then define an induced labelling $h' : \mathcal{S}' \to E(2m)$, where

$$h'(S') = \begin{cases} e_i, \text{ if and only if } S' = H'(x, y) \text{ and } f(C(x, y)) = r_i, \text{ or} \\ e_{i+m}, \text{ if and only if } S' = V'(x, y) \text{ and } f(C(x, y)) = r_i. \end{cases}$$

Now let $\star : R(m) \times C(m) \to E(2m)$ be a binary operation where for each $r_i \in R(m)$ and each $c_j \in C(m)$, $r_i \star c_j = e_k$, if there exists $C(x, y), C(r, s) \in \mathcal{C}, S' \in \mathcal{S}'$ such that $f(C(x, y)) = r_i, g(C(r, s)) = c_j, h'(S') = e_k$ and $C(x, y) \cap C(r, s) \cap S \neq \emptyset$. Otherwise $r_i \circ c_j$ is undefined.

4 Primary 4-homogeneous latin trades

In this section we use the latin trades from the previous section to construct minimal 4-homogeneous latin trades.

LEMMA 34 *The latin trade $(N(2m), \circ)$ from Corollary 32 is minimal.*

Proof: Suppose that T is a latin trade with $T \subseteq (N(2m), \circ)$. It will be shown that $T = (N(2m), \circ)$.

Let $r_i \in R(m)$ be a non-empty row in T, and let $f(C(x, y)) = r_i$ for some circle $C(x, y) \in \mathcal{C}_1$. (Such a circle exists because f is onto.) Then by Lemma 20, the cell (r_i, c_j) must be filled in T for at least one c_j such that either:

1. $g(C(x, y + 2)) = c_j$;
2. $g(C(x, y - 2)) = c_j$;
3. $g(C(x + 2, y)) = c_j$; or
4. $g(C(x - 2, y)) = c_j$.

Suppose the first case is true. Then by Lemma 20, the entry $h(H(x, y))$ must occur in T. But if T is a latin trade, $h(H(x, y))$ must occur at least twice in T, so from Corollary 22, the cell $(f(C(x + 2, y + 2)), g(C(x + 2, y))$ is non-empty in T. Since the entry $h(H(x, y))$ occurs exactly twice in T, in a disjoint mate of T the cells $(f(C(x+2, y+2)), g(C(x, y+2)))$ and $(f(C(x, y)), g(C(x+2, y)))$ must be non-empty. Thus these cells are also non-empty in T. So the first case implies the third case. In addition, the first case implies that r_{i1} is non-empty, where $r_{i1} = f(C(x + 2, y + 2))$.

Let's jump to the third case. By Lemma 20, the entry $h(V(x, y))$ must occur in T. Again, $h(V(x, y))$ must occur twice in T, so from Corollary 22, the cell $(f(C(x + 2, y - 2)), g(C(x, y - 2))$ is non-empty in T. Since the entry $h(V(x, y))$ occurs exactly twice in T, in a disjoint mate of T the cells $(f(C(x + 2, y - 2)), g(C(x + 2, y)))$ and $(f(C(x, y)), g(C(x, y - 2)))$ must be non-empty. Thus these cells are also non-empty in T. So the third case implies both the second case and that r_{i2} is non-empty in T, where $r_{i2} = f(C(x + 2, y - 2))$.

Next assume the second case holds. By Lemma 20, the entry $h(H(x - 2, y - 2))$ must occur in T. Again, $h(H(x - 2, y - 2))$ must occur twice in T, so from Corollary 22, the cell $(f(C(x - 2, y - 2)), g(C(x - 2, y))$ is non-empty in T. As above, the cells $(f(C(x - 2, y - 2)), g(C(x, y - 2)))$ and $(f(C(x, y)), g(C(x - 2, y)))$ must be non-empty in T. So the second case implies both the fourth case and that r_{i3} is non-empty in T, where $r_{i3} = f(C(x - 2, y - 2))$.

In a similar fashion the fourth case implies both the first case and that r_{i4} is non-empty in T, where $r_{i4} = f(C(x - 2, y + 2))$.

So since at least one of the four cases are true, all of them are true. Moreover, rows r_{i1}, r_{i2}, r_{i3} and r_{i4} are all non-empty in T, where $r_{i1} = f(C(x + 2, y + 2))$, $r_{i2} = f(C(x + 2, y - 2))$, $r_{i3} = f(C(x - 2, y - 2))$ and $r_{i4} = f(C(x - 2, y + 2))$. Therefore, by recursion, we have that for any circle $C(x, y) \in \mathcal{C}_1$, r_i is non-empty in T, where $f(C(x, y)) = r_i$. Moreover, for each non-empty row r_i in T, (r_i, c_j) is filled in T for four different values of j. Since f is onto $R(m)$, it follows that $T = (N(2m), \circ)$. $\square$

It is easy to construct a 4 homogeneous latin trade of size $4m$ by taking two copies of $(N(2m), \circ)$. However we wish to construct 4-homogeneous latin trades that are *minimal*, namely latin trades that contain no smaller latin trades. For this reason we use the doubling construction from Section 2.

The previous lemma and Lemma 34 imply the following.

COROLLARY 35 *Let* $T = (N(2m), \circ)$ *be the latin trade constructed from the previous section. Then* $T \bowtie T$ *(as given in Definition* 1*) is a minimal,* 4*-homogeneous latin trade of size* $8m$.

Together with Remark 38 in the previous section this implies the existence of minimal, 4-homogeneous latin trades of size $8m$ for each integer $m \geq 4$.

5 An example

DEFINITION 36 *Let* $f : \mathcal{C}_1 \to R(4)$ *be defined as follows.*

$$f(C(x,y)) = \begin{cases} r_0, \text{if } x+y \equiv 0 \pmod 8 \text{ and } x \equiv y \equiv 0 \pmod 4 \\ r_1, \text{if } x+y \equiv 0 \pmod 8 \text{ and } x \equiv y \equiv 2 \pmod 4 \\ r_2, \text{if } x+y \equiv 4 \pmod 8 \text{ and } x \equiv y \equiv 2 \pmod 4 \\ r_3, \text{if } x+y \equiv 4 \pmod 8 \text{ and } x \equiv y \equiv 0 \pmod 4). \end{cases}$$

The following lemma is straightforward to check.

LEMMA 37 *The map* $f : \mathcal{C}_1 \to R(4)$*, as given in the previous definition, is onto, coherent and proper as per Definition* 9*.*

REMARK 38 *In fact for each integer* $m \geq 4$*, there exists an onto, coherent, proper map* $f : \mathcal{C}_1 \to R(m)$*. We omit a proof of this statement, but remark that the "proper" condition is equivalent to colouring the vertices of the integer grid* $(\mathbb{Z})^2$ *with* m *colours so that no vertices at distance* 1 *or* $\sqrt{2}$ *have the same colour.*

LEMMA 39 *The following partial latin square of order* 8 *and size* 16 *is a minimal latin trade.*

$$\{(r_0,c_0,e_0),(r_0,c_1,e_4),(r_0,c_2,e_5),(r_0,c_3,e_2),(r_1,c_0,e_5),(r_1,c_1,e_1),(r_1,c_2,e_3),(r_1,c_3,e_4),$$
$$(r_2,c_0,e_7),(r_2,c_1,e_0),(r_2,c_2,e_2),(r_2,c_3,e_6),(r_3,c_0,e_1),(r_3,c_1,e_6),(r_3,c_2,e_7),(r_3,c_3,e_3)\}.$$

Proof: Apply Definition 18, Lemma 20 and Corollary 32 to the function given above in Definition 36. □

Now we apply the doubling technique from the previous section to the example from Lemma 39 to obtain, finally, an actual example of a 4-homogeneous latin trade.

LEMMA 40 *The following is a minimal,* 4*-homogeneous latin trade of order* 8 *and size* 32*.*

$$\{(r_0,c_1,e_0),(r_0,c_2,e_4),(r_0,c_4,e_5),(r_0,c_6,e_2),(r_1,c_0,e_0),(r_1,c_3,e_4),(r_1,c_5,e_5),(r_1,c_7,e_2),$$
$$(r_2,c_0,e_5),(r_2,c_2,e_2),(r_2,c_4,e_3),(r_2,c_6,e_4),(r_3,c_1,e_5),(r_3,c_3,e_2),(r_3,c_5,e_3),(r_3,c_7,e_4),$$
$$(r_4,c_0,e_7),(r_4,c_2,e_0),(r_4,c_4,e_2),(r_4,c_6,e_6),(r_5,c_1,e_7),(r_5,c_3,e_0),(r_5,c_5,e_2),(r_5,c_7,e_6),$$
$$(r_6,c_0,e_1),(r_6,c_2,e_6),(r_6,c_4,e_7),(r_6,c_6,e_3),(r_7,c_1,e_1),(r_7,c_3,e_6),(r_7,c_5,e_7),(r_7,c_7,e_3)\}.$$

6 An interesting 4-homogeneous latin trade in $(\mathbb{Z}_2)^3$

Consider the following 4-homogeneous latin trade in the addition table for $(\mathbb{Z}_2)^3$. It can be verified that this latin trade is minimal.

+	000	001	010	011	100	101	110	111
000	000		010		100			111
001		000		010		100	111	
010		011	000		110			101
011	011			000		110	101	
100	100			111		001		010
101		100	111		001		010	
110	110			100		011		001
111		110	100		011		001	

Figure 2: A minimal latin trade in $(\mathbb{Z}_2)^3$ of size 32.

(Note: this latin trade is almost the same as the one given in Lemma 40. We had to make a few adjustments so that it would embed in $(\mathbb{Z}_2)^3$.)

The above latin trade has a number of interesting properties. Firstly, it seems uncommon for a minimal latin trade to take up as much as half the elements of a latin square. In fact, the latin square for $(\mathbb{Z}_2)^3$ can be partitioned into exactly two disjoint copies of this latin trade. Remarkably both copies of the latin trade intersect each of the 112 2×2 subsquares in the latin square. Furthermore if we replace one of the latin trades with its unique disjoint mate we obtain a latin square containing no 2×2 subsquares. In fact, R. Bean verified the following by computer:

THEOREM 41 *The minimum distance between the latin square for $(\mathbb{Z}_2)^3$ and a latin square of order 8 containing no 2×2 subsquares is 32.*

Finally, if we add just one more entry to the latin trade we obtain a critical set in the latin square for $(\mathbb{Z}_2)^3$ of size 33. (The size of the smallest critical set in $(\mathbb{Z}_2)^3$ is 25 [18]).

Future directions for the research in this paper include:

- finding general ways to embed 4-homogeneous latin trades into $(\mathbb{Z}_2)^n$ and other groups;
- using such embeddings to find new, small, critical sets in the latin squares for these groups;
- determining classes of minimal 4-homogeneous latin trades which cannot be constructed as in this paper (in particular ones of size $4m$ where m is odd);
- generalizing the results from Section 2 - that is, finding methods to treble, quadruple, etc minimal latin trades whilst preserving minimality;
- constructing minimal k-homogeneous latin trades for $k > 4$, if possible using nice geometrical methods.

Acknowledgments

The authors would like to acknowledge the helpful work of Richard Bean who verified Theorem 41 by computer.

References

[1] P. Adams, R. Bean and A. Khodkar, A census of critical sets in the latin squares of order at most six, Ars Combinatoria, to appear.

[2] P. Adams and A. Khodkar, Smallest critical sets for the latin squares of orders six and seven, J. Combin. Math. Combin. Comput. **37** (2001), 225–237.

[3] J.A. Bate and G.H.J. van Rees, The size of the smallest strong critical set in a latin square, Ars Combin., **53** (1999), 73–83.

[4] J.A. Bate and G.H.J. van Rees, Minimal and near-minimal critical sets in back circulant latin squares, Australasian J. Combinatorics (to appear).

[5] R. Bean, Critical sets in the elementary abelian 2- and 3-groups, Utilitas Math., (to appear).

[6] R. Bean, The size of the smallest uniquely completable set in order 8 latin squares, submitted.

[7] N.J. Cavenagh, Latin trade algorithms and the smallest critical set in a latin square, Journal of Automata, Languages and Combinatorics (to appear).

[8] N.J. Cavenagh, The size of the smallest critical set in the back circulant latin square, submitted.

[9] N.J. Cavenagh, Embedding 3-homogeneous latin trades into abelian 2-groups, submitted.

[10] N.J. Cavenagh, D. Donovan, A. Drápal, 3-homogeneous latin trades, submitted.

[11] J. Dénes and A.D. Keedwell, Latin squares and their applications, (London, English Universities Press, 1974).

[12] D. Donovan, A. Howse and P. Adams, A discussion of latin interchanges, J. Comb. Math. Comb. Comput. **23** (1997), 161–182.

[13] D. Donovan and E.S. Mahmoodian, An algorithm for writing any latin interchange as the sum of intercalates, Bulletin of the Inst. Comb. Appl. **34** (2002), 90–98.

[14] A. Drápal, On a planar construction of quasigroups, Czechoslovak Math. J. **41** (1991), 538–548.

[15] A. Drápal, Hamming distances of groups and quasi-groups, Discrete Math. **235** (2001), 189–197.

[16] A. Drápal and T. Kepka, Exchangeable Groupoids I, Acta Universitatis Carolinae - Mathematica et Physica **24** (1983), 57–72.

[17] P. Horak, R.E.L. Aldred and H. Fleischner, Completing Latin squares: critical sets, J. Combinatorial Designs **10** (2002), 419–432.

[18] A. Khodkar, On smallest critical sets for the elementary abelian 2-group, Utilitas Math. **54** (1998), 45–50.

[19] A.P. Street, Trades and defining sets in: C.J. Colbourn and J.H. Dinitz, ed., CRC Handbook of Cobminatorial Designs (CRC Press, Boca Raton, FL., 1996), 474–478.