3-homogeneous latin trades

Nicholas J. Cavenagh
DIMATIA, Charles University
Malostranské Naměsti 25
11800 Praha 1
Czech Republic

Aleš Drápal *
Department of Mathematics, KA, Charles University
Sokolovská 83, 18675 Praha 8
Czech Republic

Diane Donovan
Centre for Discrete Mathematics and Computing
Department of Mathematics
The University of Queensland
Queensland 4072
Australia

Abstract: Let T be a partial latin square and L be a latin square with $T \subseteq L$. We say that T is a latin trade if there exists a partial latin square T' with $T' \cap T = \emptyset$ such that $(L \setminus T) \cup T'$ is a latin square. A k-homogeneous latin trade is one which intersects each row, each column and each entry either 0 or k times. In this paper, we construct 3-homogeneous latin trades from hexagonal packings of the plane with circles. We show that 3-homogeneous latin trades of size $3m$ exist for each $m \geq 3$.

*This author wishes to acknowledge the support of a 2003 Ethel Raybould Visiting Fellowships from the School of Physical Sciences, The University of Queensland. This work was also supported by institutional grant MSM 113200007 and by Grant Agency of Charles Univ., grant number 269/2001/B-MAT/MFF.

Keywords: Latin square, latin trade, circle packing.

1 Introduction

A *trade* is subset of a combinatorial configuration which may be "exchanged" without changing overall parameters in the configuration. Surveys on trades in general combinatorial structures may be found in [19], [14], and [18]. The combinatorial configuration may be a design, a graph, a matrix or in the case of this paper, a latin square.

Specifically, a *latin trade* is a subset of a latin square which may be exchanged by a *disjoint mate* to obtain a new latin square. Some of what we know about latin trades is as follows. Latin trades of small size (up to 11) are classified in [7]. Some basic properties of latin trades are given in [7] and [11]. (In the latter paper latin trades are referred to as *exchangeable partial groupoids.*) In [8], a construction is given that decomposes any latin trade into a linear-algebraic "sum" of intercalates (latin trades of size 4).

Our bank of knowledge on latin trades is probably strongest when considering latin trades that occur in the back circulant square B_n, the latin square based on the addition table for the integers modulo n. Here we can exploit the cyclic nature of the latin square, as well as the algebraic properties of the underlying group $(\mathbb{Z}_n, +)$. It has been shown ([13], or [3] for an alternative proof), that the size of a latin trade in B_n is at least $O(\log p)$, where p is the least prime that divides n. In [9], it is shown that certain decompositions of equilateral triangles into smaller, equilateral triangles can be used to construct latin trades in B_n. These constructions give latin trades of size $O((\log n)^2)$ in B_n for any integer n. Other constructions of latin trades in B_n are given in [5], where they are used to show the existence of minimal defining sets in B_n.

The concept of latin trades is closely intertwined with the concept of *critical sets* (minimal defining sets) in latin squares. This is because a critical set must intersect every latin trade within a latin square. Indeed, results on lower bounds for the sizes of critical sets often rely on constructions of latin trades. In [2], it is shown that if a critical set of order n has an empty row, its size must be at least $2n-4$. The proof in [2] relies on latin trades that occur within sets of three rows of a latin square. A more general result is given in [15], where it is shown that the size of a critical set in a latin square of order n is at least $\lfloor (4n-8)/3 \rfloor$. Their proof relies on the construction of latin trades in the union of three rows three columns. In [4], it is shown that a critical set in the back circulant latin square B_n must have size at least $O(n^{4/3})$.

We are still a long way from the conjectured lower bound on the size of a critical set in a latin square of order n: $\lfloor n^2/4 \rfloor$ [1]. One way to go forward is to learn more about latin trades that intersect a *large* number of rows, a *large* number of columns and a *large* number of entries (*large* meaning with respect to n), yet are still relatively small in size.

One way to find such latin trades is to look at k-homogeneous latin trades. A k-homogeneous latin trade is one which intersects each row, each column and each entry either 0 or k times. In this paper, we construct 3-homogeneous latin trades from hexagonal packings of the plane with circles.

2 Definitions

We start with basic definitions which allow us to state and prove our main results.

Let $N = N(n) = \{0, 1, 2, \ldots, n-1\}$. A *partial latin square* P of order n is a set of ordered triples of the form (i, j, k), where $i, j, k \in N$ with the following properties:

- if $(i, j, k) \in P$ and $(i, j, k') \in P$ then $k = k'$,
- if $(i, j, k) \in P$ and $(i, j', k) \in P$ then $j = j'$ and
- if $(i, j, k) \in P$ and $(i', j, k) \in P$ then $i = i'$.

We may also represent a partial latin square P as an $n \times n$ array with entries chosen from the set N such that if $(i, j, k) \in P$, the *entry* k occurs in cell (i, j). We sometimes refer to the entry k in cell (i, j) of a partial latin square P as $(i, j)_P$.

A partial latin square has the property that each entry occurs at most once in each row and at most once in each column. If all the cells of the array are filled then the partial latin square is termed a latin square. That is, a *latin square* L of order n is an $n \times n$ array with entries chosen from the set $N = N(n) = \{0, 1, 2, \ldots, n-1\}$ in such a way that each element of N occurs precisely once in each row and precisely once in each column of the array.

For a given partial latin square P the set of cells $\mathcal{S}_P = \{(i, j) \mid (i, j, k) \in P, \text{ for some } k \in N\}$ is said to determine the *shape* of P and $|\mathcal{S}_P|$ is said to be the *size* of the partial latin square. That is, the size of P is the number of non-empty cells in the array. For each r, $1 \leq r \leq n$, let $\mathcal{R}_P^r$ denote the set of entries occurring in row r of P. Formally, $\mathcal{R}_P^r = \{k \mid (r, j, k) \in P\}$. For each c, $1 \leq c \leq n$, we define $\mathcal{C}_P^c = \{k \mid (i, c, k) \in P\}$. Finally, for each e, $1 \leq e \leq n$, we define $\mathcal{E}_P^e = \{(i, j) \mid (i, j, e) \in P\}$.

A partial latin square T of order n is said to be a *latin trade* (or *latin interchange*) if $T \neq \emptyset$ and there exists a partial latin square T' (called a *disjoint mate* of T) of order n, such that

- $\mathcal{S}_T = \mathcal{S}_{T'}$,
- if $(i, j, k) \in T$ and $(i, j, k') \in T'$ then $k \neq k'$,
- for each r, $0 \leq r \leq n-1$, $\mathcal{R}_T^r = \mathcal{R}_{T'}^r$,

- for each c, $0 \le c \le n-1$, $\mathcal{C}_T^c = \mathcal{C}_{T'}^c$.

A latin trade T of order n is said to be k-homogeneous if

- for each r, $0 \le r \le n-1$, $|\mathcal{R}_T^r| = 0$ or k, and
- for each c, $0 \le c \le n-1$, $|\mathcal{C}_T^c| = 0$ or k, and
- for each e, $0 \le e \le n-1$, $|\mathcal{E}_T^e| = 0$ or k.

Clearly if T is k-homogeneous, its size is equal to km for some integer m, where $m \ge k$.

A *critical set* in a latin square L (of order n) is a partial latin square $P \subseteq L$, such that

(1) L is the only latin square of order n which has element k in cell (i,j) for each $(i,j,k) \in P$; and

(2) no proper subset of P satisfies (1).

Let T be a partial latin square that is a subset of a latin square L. Observe that T is a latin trade if and only if there exists a disjoint mate T', with $T' \cap T = \emptyset$, such that $(L \setminus T) \cup T'$ is a latin square. It follows that a critical set P in a latin square L must intersect every latin trade in L; and is minimal with respect to this property.

If P is a partial latin square, we define a *parastrophy* (or *conjugate*) of P to be one of the partial latin squares given by:

$$\{(j,k,i) \mid (i,j,k) \in P\}, \{(k,i,j) \mid (i,j,k) \in P\}, \{(i,k,j) \mid (i,j,k) \in P\},$$
$$\{(k,j,i) \mid (i,j,k) \in P\}, \{(j,i,k) \mid (i,j,k) \in P\}, \{(i,j,k) \mid (i,j,k) \in P\}.$$

An observation which will be used later in the paper is that if a latin trade T is k-homogeneous, any parastrophy of T will also be k-homogeneous.

3 Hexagonal constructions

Definition 1 *Let S be the following set of co-ordinates in $\mathbb{R} \times \mathbb{R}$:*

$$S = \{(2\sqrt{3}i, 2j), ((2i+1)\sqrt{3}, 2(j+1)) \mid i,j \in \mathbb{Z}\}.$$

Let C be the set of unit circles in $\mathbb{R} \times \mathbb{R}$ whose centres are the elements of S. (This is the well-known hexagonal lattice in the plane. See Chapter 1 of [6] for more details on lattices and spherical packings.)

If $d \in C$ is a circle with centre (α, β), then let:

- *$u_1(d)$ be the circle with centre $(\alpha, \beta+2)$,*

- $u_2(d)$ *be the circle with centre* $(\alpha+\sqrt{3}, \beta+1)$,
- $u_3(d)$ *be the circle with centre* $(\alpha+\sqrt{3}, \beta-1)$,
- $u_4(d)$ *be the circle with centre* $(\alpha, \beta-2)$,
- $u_5(d)$ *be the circle with centre* $(\alpha-\sqrt{3}, \beta-1)$,
- $u_6(d)$ *be the circle with centre* $(\alpha-\sqrt{3}, \beta+1)$.

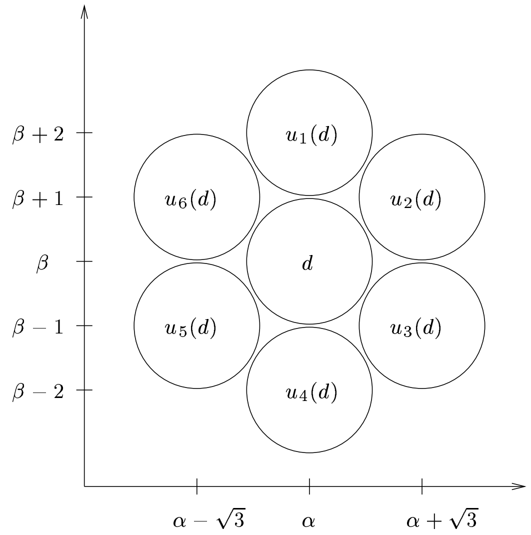

Figure 1: The neighbours of $d \in C$.

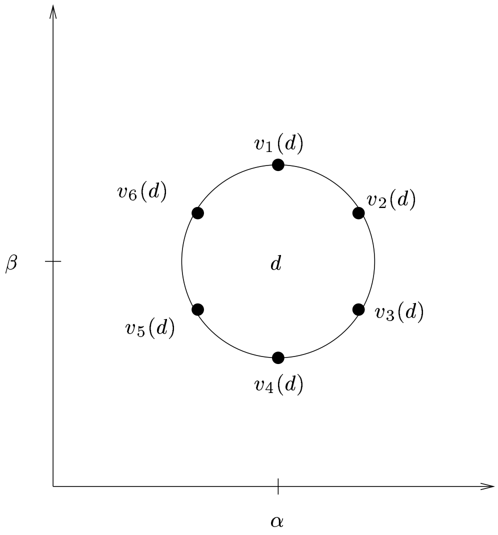

Figure 2: The vertices on $d \in C$.

For a given circle $d \in C$ *with centre* (α, β), *let* $v_1(d)$, $v_2(d), \ldots, v_6(d)$ *be the following vertices on circle* d:

- $v_1(d) = (\alpha, \beta+1)$,
- $v_2(d) = (\alpha+\sqrt{3}/2, \beta+1/2)$,
- $v_3(d) = (\alpha+\sqrt{3}/2, \beta-1/2)$,
- $v_4(d) = (\alpha, \beta-1)$,
- $v_5(d) = (\alpha-\sqrt{3}/2, \beta-1/2)$,
- $v_6(d) = (\alpha-\sqrt{3}/2, \beta+1/2)$.

Lemma 2 *For each* i, $1 \leq i \leq 6$, *each circle* $d \in C$ *intersects* $u_i(d)$ *at vertex* $v_i(d)$.

Definition 3 *Let $d \in C$ with centre (α, β). We define x_d, y_d and z_d to be arcs on the circumference of d given by:*

- $x_d = \{(\alpha + \cos\theta, \beta + \sin\theta) \mid 7\pi/6 \leq \theta \leq 11\pi/6\}$,
- $y_d = \{(\alpha + \cos\theta, \beta + \sin\theta) \mid -\pi/6 \leq \theta \leq \pi/2\}$,
- $z_d = \{(\alpha + \cos\theta, \beta + \sin\theta) \mid \pi/2 \leq \theta \leq 7\pi/6\}$.

(See Figure 3.)

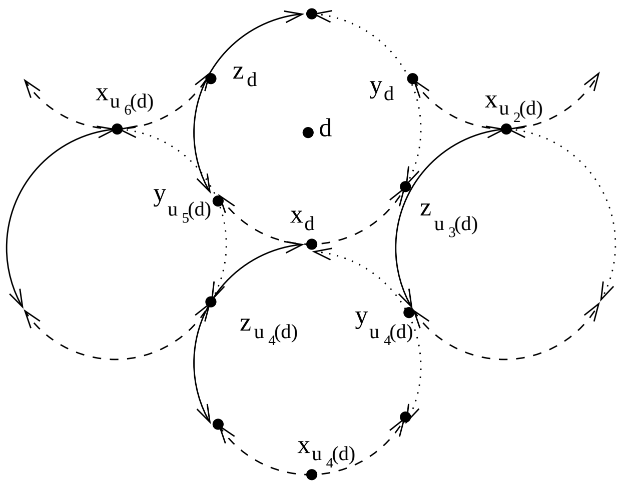

Figure 3

Definition 4 *Let $f : C \to N(m)$, where $m \geq 3$. We say that f is* coherent *if for each i, $1 \leq i \leq 6$, $f(d) = f(d')$ implies $f(u_i(d)) = f(u_i(d'))$, for each $d, d' \in C$. (In other words, if two circles d and d' have the same labelling, then $u_i(d)$ and $u_i(d')$ have the same labelling, for each $1 \leq i \leq 6$.)*

Definition 5 *Let $f : C \to N(m)$, where $m \geq 3$. We say that f is proper if for each i, $1 \leq i \leq 6$, $f(d) \neq f(u_i(d))$. (In other words, a coherent labelling is proper if we do not apply the same label to any pair of adjacent circles.)*

For the next definition, let $R(m) = \{r_i \mid 1 \leq i \leq m\}$, $C(m) = \{c_i \mid 1 \leq i \leq m\}$ and $E(m) = \{e_i \mid 1 \leq i \leq m\}$.

Definition 6 *Let f be a coherent, proper, onto labelling $f : C \to N(m)$. Let $\star$ be a binary operation $R(m) \times C(m) \to E(m)$ defined as follows: $r_i \star c_j = e_k$ if there exist circles $d, d', d'' \in C$ such that $f(d) = i$, $f(d') = j$, $f(d'') = k$, $x_d \cap y_{d'} \cap z_{d''} = \emptyset$, but $x_d \cap y_{d'} = \{(w_1, w_1')\}$, $x_d \cap z_{d''} = \{(w_2, w_2')\}$, and $y_{d'} \cap z_{d''} = \{(w_3, w_3')\}$, where $(w_1, w_1'), (w_2, w_2'), (w_3, w_3') \in \mathbb{R} \times \mathbb{R}$. Otherwise, $r_i \star c_j$ is undefined.*

The following lemma is verified on inspection of Figure 3.

Lemma 7 *Let f be a coherent, proper, onto labelling $f : C \to N(m)$ and let $\star$ be defined as in Definition 6. Then $r_i \star c_j = e_k$ if and only if there exist circles d, d' and d''such that $f(d) = i$, $f(d') = j$, $f(d'') = k$ and either:*

Case 1. *$d = d' = d''$, or*

Case 2. *$d' = u_4(d)$ and $d'' = u_3(d)$, or*

Case 3. *$d' = u_5(d)$ and $d'' = u_4(d)$.*

Definition 8 *Let f be a coherent, proper, onto labelling $f : C \to N(m)$. Let $\circ : R(m) \times C(m) \to E(m)$ be a binary operation defined as follows: $r_i \circ c_j = e_k$ if there exists $(w, w') \in \mathbb{R} \times \mathbb{R}$, and circles $d, d', d'' \in C$, such that $f(d) = i$, $f(d') = j$ and $f(d'') = k$ and $x_d \cap y_{d'} \cap z_{d''} = \{(w, w')\}$. Otherwise, $r_i \circ c_j$ is undefined.*

The following lemma is also verified on inspection of Figure 3.

Lemma 9 *Let f be a coherent, proper, onto labelling $f : C \to N(m)$ and let $\circ$ be defined as in Definition 6. Then $r_i \circ c_j = e_k$ if and only if there exist circles d, d' and d''such that $f(d) = i$, $f(d') = j$, $f(d'') = k$ and either:*

Case 1. *$d = d'$ and $d'' = u_3(d)$, or*

Case 2. *$d' = d'' = u_4(d)$, or*

Case 3. *$d' = u_5(d)$ and $d'' = d$.*

Lemma 10 *Let $\star$ and $\circ$ be the operations given in Definitions 6 and 8. The operation $r_i \star c_j$ is defined if and only if $r_i \circ c_j$ is defined. Moreover, if $r_i \circ c_j$ is defined, there is a unique k such that $r_i \circ c_j = e_k$ and a unique $k' \neq k$ such that $r_i \star c_j = e_{k'}$.*

Proof: Assume first that $r_i \circ c_j$ is defined, and let $e_k = r_i \circ c_j$. Then there exists circles d, d' and d'' such that $f(d) = i$, $f(d') = j$, $f(d'') = k$ and either Case 1, Case 2 or Case 3 of Lemma 9 is satisfied. From the coherency of f, in each case $f(d'')$ is uniquely determined.

For Case 1, we have $d = d'$, so $i = j$. $r_i \circ c_i = e_k$, where $f(d) = i$ and $f(u_3(d)) = k$. We are required to show that there exists a $k' \neq k$ such that $r_i \star c_i = e'_k$. From Case 1 of Lemma 7, $r_{f(d)} \star c_{f(d)} = e_{f(d)}$, or equivalently $r_i \star c_i = e_i$. So let $k' = i$. Since f is a proper labelling and $f(d) \neq f(u_3(d))$, it follows that $k' \neq k$.

For Case 2, it is assumed that $j = k$ and that $r_i \circ c_j = r_{f(d)} \circ c_{f(u_4(d))} = e_{f(u_4(d))} = e_k$. From Case 2 of Lemma 7, $r_{f(d)} \star c_{f(u_4(d))} = e_{f(u_3(d))}$, or equivalently $r_i \star c_j = e_{f(u_3(d))}$. Therefore set $k' = f(u_3(d))$ and, since $u_3(d)$ and $u_4(d)$ are adjacent and f is a proper labelling, $f(u_3(d)) \neq f(u_4(d))$ and so $k' \neq k$.

Finally from Case 3, it is assumed that $i = k$ and that $r_i \circ c_j = r_{f(d)} \circ c_{f(u_5(d))} = e_{f(d)} = e_k$. From Case 3 of Lemma 7, $r_{f(d)} \star c_{f(u_5(d))} = e_{f(u_4(d))}$ and $r_i \star c_j = e_{f(u_4(d))}$. Therefore set $k' = f(u_4(d))$. Since $u_4(d)$ and d are adjacent and f is a proper labelling, $f(u_4(d)) \neq f(d)$ and so $k' \neq k$.

Similarly, we can show that if $r_i \star c_j$ is defined, then $r_i \circ c_j$ is also defined. ∎

Lemma 11 *Let $\star$ and $\circ$ be the operations given in Definitions 6 and 8. If $r_i \circ c_j = e_k$, then there is a unique j' such that $r_i \star c_{j'} = e_k$, and a unique i' such that $r_{i'} \star c_j = e_k$.*

Proof: Let $e_k = r_i \circ c_j$. Then there exists circles d, d' and d'' such that $f(d) = i$, $f(d') = j$, $f(d'') = k$ and either Case 1, Case 2 or Case 3 of Lemma 9 is satisfied.

For Case 1, we begin by showing that there exists $j' \neq j$, with $i \star j' = k$. It is assumed that $i = f(d)$, $j = f(d)$ and $k = f(u_3(d))$, and so $r_{f(d)} \circ c_{f(d)} = e_{f(u_3(d))}$. Thus let $j' = f(u_4(d))$ and we see from Case 2 of Lemma 7 that $r_i \star c_j = r_{f(d)} \star c_{f(u_4(d))} = e_{f(u_3(d))} = e_k$. Since f is proper, $f(u_4(d)) \neq f(d)$ and $j' \neq j$.

Next, we wish to show that there exists $i' \neq i$, with $r_{i'} \star c_j = e_k$. Let $r_{i'} = f(u_2(d))$. Now, $u_5(u_2(d)) = d$ and $u_4(u_2(d)) = u_3(d)$. So from Case 3 of Lemma 7, we have $r_{i'} \star c_j = r_{f(u_2(d))} \star c_{f(d)} = e_{f(u_3(d))} = e_k$. Since f is proper it follows that $f(u_2(d)) \neq f(d)$ and $i' \neq i$.

For Case 2 of Lemma 9, we first show that there exists j', with $r_i \star c_{j'} = e_k$. It is assumed that $i = f(d)$, $j = f(u_4(d))$ and $k = f(u_4(d))$, and so $r_{f(d)} \circ c_{f(u_4(d))} = e_{f(u_4(d))}$. Let $j' = f(u_5(d))$ and from Case 3 of Lemma 7 $r_i \star c_{j'} = r_{f(d)} \star c_{f(u_5(d))} = e_{f(u_4(d))} = e_k$. Since f is proper and $u_4(d)$ and $u_5(d)$ are adjacent circles $f(u_5(d)) \neq f(u_4(d))$, and $j' \neq j$.

Next, we wish to show that there exists i', with $r_{i'} \star c_j = e_k$. Let $i' = f(u_4(d))$ and from Case 1 of Lemma 7 (replacing d with $u_4(d)$, $r_{i'} \star c_j = r_{f(u_4(d))} \star c_{f(u_4(d))} = e_{f(u_4(d))} = e_k$. Since f is proper, $f(u_4(d)) \neq f(d)$ and $i' \neq i$.

Finally we are left with Case 3 of Lemma 9. We show first that there exists j', with $r_i \star c_{j'} = e_k$. It is assumed that $i = f(d)$, $j = f(u_5(d))$ and $k = f(d)$, and so $r_{f(d)} \circ c_{f(u_5(d))} = e_{f(d)}$. Let $j' = f(d)$, and from Case 1 of Lemma 7, $r_i \star c_{j'} = r_{f(d)} \star c_{f(d)} = e_{f(d)} = e_k$. Since f is proper, $f(u_5(d)) \neq f(d)$ and $j' \neq j$.

Next, we show that there exists i', with $i' \star j = k$. istinct points, $v_5(d)$, $v_6(d)$ and $v_1(u_5(d))$. Let $i' = f(u_6(d))$ and observe that $u_3(u_6(d)) = d$ and $u_4(u_6(d)) = u_5(d)$. So replacing d with $u_6(d)$ in Case 2 of Lemma 7, $r_{i'} \star c_j = r_{f(u_6(d))} \star c_{f(u_5(d))} = e_{f(d)} = e_k$. Since f is proper, $f(u_6(d)) \neq f(d)$ and $i' \neq i$. ∎

Theorem 12 *Let $\star$ and $\circ$ be the operations given in Definitions 6 and 8. The set of triples $T = \{(r_i, c_j, r_i \star e_j) \mid r_i \star c_j$ is defined $\}$ is a latin trade of size $3m$ with a disjoint mate given by $T' = \{(r_i, c_j, r_i \circ c_j) \mid r_i \circ c_j$ is defined $\}$. Moreover, this latin trade is 3-homogeneous.*

Proof: The fact that T is a latin trade with disjoint mate T' follows from Lemmas 10 and 11.

Next, consider i, $1 \leq i \leq m$. Since f is onto, there exists a circle $d \in C$ such that $f(d) = i$. The three cases from the proof Lemma 7 show that there are exactly three elements in row i of T. (Summing over the rows, the latin trade T has size $3m$.) To prove that each column of T has 3 entries, observe that we can obtain a parastrophe of T by rotating the original grid of circles by 120 degrees about the origin. Rotate the plane of circles again by 120 degrees about the origin to obtain the result that each entry occurs exactly three times.

Thus T is 3-homogeneous. ∎

4 The existence of 3-homogeneous latin trades of size $3m$, $m \geq 3$

Next we show the existence of an onto, coherent and proper mapping $f_n : C \to N(m)$ for each integer $m \geq 3$. (In fact, we will classify, up to isomorphism, all such mappings.) The following mapping partitions the set of circles C into parallelograms of dimension $m_1 \times m_2$, the circles of each parallelogram are labelled with $N(m_1 m_2)$.

Definition 13 *Let m_1, m_2 and k be integers such that $m_1 \geq 2$, $0 \leq k < m_1$, $m_2 \geq 1$ and if $m_2 = 1$, $m_1 \geq 3$ and $2 \leq k$. Then we define a mapping $f : C \to N(m_1 m_2)$ such that if $d \in C$ has centre (α, β), then $f(d) = \omega(d) + \eta(d)$, where*

$$\begin{aligned}
\delta(d) &= \alpha/\sqrt{3} \ (\textit{div } m_2), \\
\omega(d) &= \delta(d)(k-1) - (\beta + \alpha/\sqrt{3})/2 \ (\textit{mod } m_1) \textit{ and} \\
\eta(d) &= m_1 \alpha/\sqrt{3} \ (\textit{mod } m_1 m_2).
\end{aligned}$$

Lemma 14 *The mapping $f : C \to N(m_1m_2)$, as given in the previous definition, is an onto mapping.*

Proof: Let $i \in N(m_1m_2)$. Then we may write $i = xm_1 + y$, where x, y are integers such that $0 \le x \le m_2 - 1$ and $0 \le y \le m_1 - 1$. Let $d \in C$ be a circle with centre $(x\sqrt{3}, -2y - x)$. (The circle d is in C because $-2y - x$ and x have the same parity.) Then $\delta(d) = x$ (div m_2) $= 0$, $\eta(d) = m_1x$ (mod m_1m_2) $= m_1x$ and $\omega(d) = y$. Thus $f(d) = m_1x + y = i$. ∎

Lemma 15 *The mapping $f : C \to N(m_1m_2)$, as given in Definition* 13*, is a coherent mapping.*

Proof: Let $d, d' \in C$ be circles with centres $(x\sqrt{3}, y)$ and $(x'\sqrt{3}, y')$ respectively. From the definition of C, $x \equiv y$ (modulo 2) and $x' \equiv y'$ (modulo 2). Suppose that $f(d) = f(d')$. Now, $0 \le \omega(d), \omega(d') \le m_1 - 1$ and $\eta(d), \eta(d') \in \{0, m_1, 2m_1, \ldots, (m_2 - 1)m_1\}$. It follows that $\eta(d) = \eta(d')$ and $\omega(d) = \omega(d')$. The former implies that $x \equiv x'$ (modulo m_2).

Consider the circles $u_1(d)$ and $u_1(d')$. These have centres $(x\sqrt{3}, y+2)$ and $(x'\sqrt{3}, y'+2)$ respectively. Thus $\eta(u_1(d)) = \eta(u_1(d'))$ and $\omega(u_1(d)) = \omega(d) - 1$ (modulo m_1) $= \omega(d') - 1$ (modulo m_1) $= \omega(u_1(d'))$. Similarly we can prove that $f(u_4(d)) = f((u_4(d'))$.

Next consider the circles $u_2(d)$ and $u_2(d')$. These have centres $((x + 1)\sqrt{3}, y + 1)$ and $((x' + 1)\sqrt{3}, y' + 1)$ respectively.

First suppose that $x = (m_2 - 1)$. Then $\delta(u_2(d)) = \delta(d) + 1 = \delta(u_2(d'))$. It follows that $\omega(u_2(d)) = \omega(d) = \omega(d') = \omega(u_2(d'))$, and $\eta(u_2(d)) = 0 = \eta(u_2(d'))$.

Otherwise $x < (m_2 - 1)$, $\delta(u_2(d)) = \delta(d) = \delta(u_2(d'))$ and $\omega(u_2(d)) \equiv \omega(d) - 1$ (modulo n) $\equiv \omega(d') - 1 \equiv \omega(u_2(d'))$ (modulo n). Also $\eta(u_2(d)) \equiv= \eta(u_2(d'))$ (mod m_1m_2). So in both cases $f(u_2(d)) = f((u_2(d'))$.

Similarly it follows that $f(u_3(d)) = f((u_3(d'))$, $f(u_5(d)) = f((u_5(d'))$ and $f(u_6(d)) = f((u_6(d'))$. ∎

Lemma 16 *The mapping $f : C \to N(m_1m_2)$, as given in Definition* 13*, is a proper mapping.*

Proof: Suppose, for the sake of a contradiction, that f is not a proper mapping. Then we have $d, d' \in C$ such that $f(d) = f(d')$, where $d' = u_i(d)$, for some i, $1 \le i \le 6$. Let d, d' have centres $(x\sqrt{3}, y)$ and $(x'\sqrt{3}, y')$ respectively. Therefore $y' \in \{y + 2, y + 1, y - 1, y - 2\}$ and $x' \in \{x + 1, x, x - 1\}$.

As observed in the proof of the previous lemma, $\omega(d) = \omega(d')$ and $\eta(d) = \eta(d')$.

Suppose first that $m_2 = 1$. In this case $m_1 \ge 3$ and $(m_1 - 1) \ge k \ge 2$. In this case $\delta(d) = x$ and $\delta(d') = x'$. We have $\eta(d) = \eta(d') = 0$ for any values of x and x'. From $\omega(d) = \omega(d')$, we have

$$x(k - 1) - (x + y)/2 \equiv x'(k - 1) - (x' + y')/2 \text{ (mod } m_1).$$

So if $x' = x$, $(y - y')/2 \equiv 0 \pmod{m_1}$. However in this case $y - y'$ is equal to either 2 or -2, and $m_1 \geq 3$, so we have a contradiction.

If $x' = x + 1$, then we have $k \equiv (y' - y + 1)/2 \pmod{m_1}$. But in this case either $y' - y = 1$ or $y' - y = -1$, which gives $k \equiv 0$ or $1 \pmod{m_1}$, contradicting $(m_1 - 1) \geq k \geq 2$.

If $x' = x - 1$, then we have $k \equiv (y - y' + 1)/2 \pmod{m_1}$, leading to a similar contradiction.

Otherwise $m_2 > 1$. Then from $\eta(d) = \eta(d')$, we have $x = x'$. Thus $\delta(d) = \delta(d')$. Since $m_2 \geq 2$ and $|y - y'| = 2$, $\omega(d) = \omega(d')$ gives a contradiction. ∎

The next lemma follows from the three previous lemmas.

Lemma 17 *The function $f : C \rightarrow N(m_1 m_2)$, as given in Definition 13, is an onto, proper, coherent labelling.*

In fact, Definition 13 defines all onto, proper, coherent labellings of the form $f : C \rightarrow N(m)$ (with some of these mappings equivalent under isomorphism). We omit a proof of this observation.

Corollary 18 *For each integer $m \geq 3$, there exists a 3-homogeneous latin trade of size $3m$.*

Proof: Let $m_1 = m$, $m_2 = 1$ and $k = 2$ as in Definition 13. Then Lemma 17 states that there exists an onto, coherent, proper mapping $f : C \rightarrow N(m)$. Definitions 8, 6 and Theorem 12 from the previous section imply the existence of a 3-homogeneous latin trade of size $3m$. ∎

5 Examples

Example 19 *Here we construct a 3-homogeneous latin trade of size 9. Let $m_1 = 3$, $m_2 = 1$ and $k = 2$ as in Definition 13. So Lemma 17 tells us that f is a coherent, proper mapping onto $N(3)$. Let d, d' and d'' be circles in C with centres equal to $(0, 0)$, $(0, -2)$ and $(0, -4)$ respectively.*

Then, $\delta(d) = \delta(d') = \delta(d'') = \eta(d) = \eta(d') = \eta(d'') = 0$, $\omega(d) = 0$, $\omega(d') = 1$ and $\omega(d'') = 2$. Thus $f(d) = 0$, $f(d') = 1$ and $f(d'') = 2$. The circles $u_3(d)$, $u_5(d)$, $u_3(d')$, $u_5(d')$, $u_4(d'')$, $u_3(d'')$ and $u_5(d'')$ have centres $(\sqrt{3}, -1)$, $(-\sqrt{3}, -1)$, $(\sqrt{3}, -3)$, $(-\sqrt{3}, -3)$, $(0, -6)$, $(\sqrt{3}, -5)$ and $(-\sqrt{3}, -5)$. It follows that $f(u_3(d)) = f(u_5(d)) = 2$, $f(u_3(d') = f(u_5(d')) = f(u_4(d'')) = 0$ and $f(u_3(d'')) = f(u_5(d'')) = 1$. (See Figure 4.)

Consider the following ordered triples of arcs.

$$\{(x_d, y_d, z_d), (x_d, y_{d'}, z_{u_3(d)}), (x_{d'}, y_{d'}, z_{d'}), (x_d, y_{u_5(d)}, z_{d'}), (x_{d'}, y_{d''}, z_{u_3(d')}),$$
$$(x_{d'}, y_{u_5(d')}, z_{d''}), (x_{d''}, y_{d''}, z_{d''}), (x_{d''}, y_{u_4(d'')}, z_{u_3(d'')}), (x_{d''}, y_{u_5(d'')}, z_{u_4(d'')})\}.$$

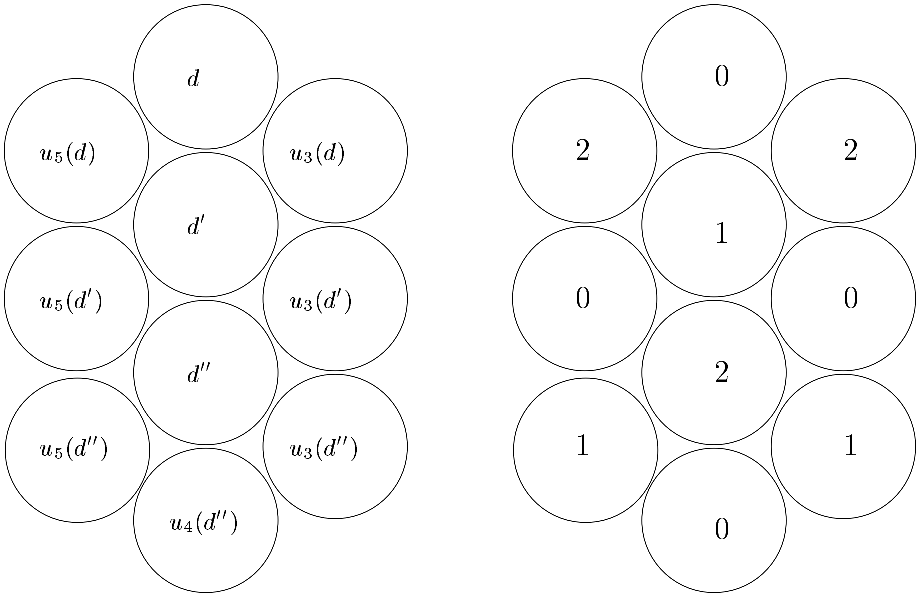

Figure 4: Circles in C with corresponding values of f.

From Definition 6*, Lemma* 7*, and Theorem* 12*, the following partial latin square is a* 3*-homogeneous latin trade of size* 9*:*

$$\{(r_0, c_0, e_0), (r_0, c_1, e_2), (r_1, c_1, e_1), (r_0, c_2, e_1), (r_1, c_2, e_0),$$
$$(r_1, c_0, e_2), (r_2, c_2, e_2), (r_2, c_0, e_1), (r_2, c_1, e_2)\}.$$

(In fact, this is simply a latin square of order 3*!)*

To construct the disjoint mate, consider the following set of arcs.

$$\{(x_d, y_d, z_{u_3(d)}), (x_d, y_{d'}, z_{d'}), (x_d, y_{u_5(d)}, z_d), (x_{d'}, y_{d'}, z_{u_3(d')}), (x_{d'}, y_{d''}, z_{d''}),$$
$$(x_{d'}, y_{u_5(d')}, z_{d'}), (x_{d''}, y_{d''}, z_{u_3(d'')}), (x_{d''}, y_{u_4(d'')}, z_{u_4(d'')}), (x_{d''}, y_{u_5(d'')}, z_{d''})\}.$$

From Definition 8*, Lemma* 9 *and Theorem* 12*, we have a disjoint mate T' as follows:*

$$\{(r_0, c_0, e_2), (r_0, c_1, e_1), (r_0, c_2, e_0), (r_1, c_1, e_0), (r_1, c_2, e_2),$$
$$(r_1, c_0, e_1), (r_2, c_2, e_1), (r_2, c_0, e_0), (r_2, c_1, e_2)\}.$$

The previous example is not a minimal latin trade. In a sense this is a degenerate example of our construction; in all other cases a minimal latin trade is obtained. The next example is minimal, and may be embedded into an abelian 2-group.

Example 20 *Here we construct a* 3*-homogeneous latin trade of size* 12*. Let $m_1 = 2$, $m_2 = 2$ and $k = 0$ as in Definition* 13*. Lemma* 17 *tells us that f is a coherent, proper* *mapping onto* $N(3)$. *Let* $d0$, $d1$, $d2$ *and* $d3$ *be circles in* C *with centres equal to* $(0,0)$, $(0,-2)$, $(\sqrt{3},-1)$ *and* $(\sqrt{3},-3)$ *respectively.*

Thus $f(di)=i$, *for each* i, $0\le i\le 3$. *The circles* $u_5(d0)$, $u_5(d1)$, $u_4(d1)$, $u_4(d3)$, $u_3(d3)$ *and* $u_3(d2)$ *have centres* $(-\sqrt{3},-1)$, $(-\sqrt{3},-3)$, $(0,-4)$, $(\sqrt{3},-5)$, $(2\sqrt{3},-4)$ *and* $(2\sqrt{3},-2)$ *respectively. It follows that* $f(u_5(d0))=3$, $f(u_5(d1))=2$, $f(u_4(d1))=0$, $f(u_4(d3))=2$, $f(u_3(d3))=1$ *and* $f(u_3(d2))=0$

Consider the following sets of arcs.

$$\{(x_{d0},y_{d0},z_{d0}),(x_{d0},y_{u_5(d0)},z_{d1}),(x_{d0},y_{d1},z_{d2}),(x_{d1},y_{d1},z_{d1}),\\(x_{d1},y_{u_4(d1)},z_{d3}),(x_{d1},y_{u_5(d1)},z_{u_4(d_1)}),(x_{d2},y_{d2},z_{d2}),(x_{d2},y_{d1},z_{d3}),\\(x_{d2},y_{d3},z_{u_3(d2)}),(x_{d3},y_{u_4(d1)},z_{u_4(d3)}),(x_{d3},y_{d3},z_{d3}),(x_{d3},y_{u_4(d3)},z_{u_3(d3)})\}.$$

From Definition 6, *Lemma* 7 *and Theorem* 12, *the following partial latin square is a* 3*-homogeneous latin trade* T *of size* 12*:*

$$\{(r_0,c_0,e_0),(r_0,c_3,e_1),(r_0,c_1,e_2),(r_1,c_1,e_1),(r_1,c_0,e_3),(r_1,c_2,e_0),\\(r_2,c_2,e_2),(r_2,c_1,e_3),(r_2,c_3,e_0),(r_3,c_0,e_2),(r_3,c_3,e_3),(r_3,c_2,e_1)\}.$$

There is a unique disjoint mate, given as follows:

$$\{(r_0,c_0,e_2),(r_0,c_3,e_0),(r_0,c_1,e_1),(r_1,c_1,e_3),(r_1,c_0,e_0),(r_1,c_2,e_1),\\(r_2,c_2,e_0),(r_2,c_1,e_2),(r_2,c_3,e_3),(r_3,c_0,e_3),(r_3,c_2,e_2),(r_3,c_3,e_1)\}.$$

In fact, this latin trade may be embedded into the group $(\mathbb{Z}_2)^2$, *if we let* $r_0=c_0=e_0=(0,0)$, $r_3=c_1=e_2=(0,1)$, $r_1=c_2=e_3=(1,0)$ *amd* $r_2=c_3=e_1=(1,1)$. *This is shown in the figure below.*

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$		$(1,1)$
$(0,1)$	$(0,1)$		$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	
$(1,1)$		$(1,0)$	$(0,1)$	$(0,0)$

Figure 5: The latin trade T *embedded in* $(\mathbb{Z}_2)^2$.

6 Future directions

We pose the following open problems.

Open Problem 21 *For what values of* k *and* m *do there exist a* k*-homogeneous trade of size* km*?*

We must have $k \geq 2$, since each row, column and entry in a latin trade occurs at least twice. Also $m \geq k$, as each row must have k distinct entries. If $k = 2$ then we must have $m = 2$, with the unique 2-homogeneous trade being a 2×2 latin square. If $m = k - 1$ and $m \notin \{2, 6\}$, we can use a pair of mutually orthogonal latin squares of order m. The authors have made progress on the above open problem in the case where $k = 4$; we are currently in the process of writing up this result.

Example 20 gives a 3-homogeneous latin trade that embeds into an abelian 2-group. In fact many 3-homogeneous latin trades have this property. We pose the following question.

Open Problem 22 *Which 3-homogeneous latin trades embed into $(\mathbb{Z}_2)^m$, for some integer m?*

Progress on such a problem could increase our knowledge about critical sets in the latin squares $(\mathbb{Z}_2)^m$.

Finally, we ask whether the construction given in this paper in fact gives every possible 3-homogeneous latin trade.

Open Problem 23 *Does there exist a 3-homogeneous latin trade T, which may not be constructed as in Definition 13 and Lemma 17?*

References

[1] J.A. Bate and G.H.J. van Rees, Minimal and near-minimal critical sets in back circulant latin squares, Australasian J. Combinatorics (to appear).

[2] N.J. Cavenagh, Latin trade algorithms and the smallest critical set in a latin square, J. Automata, Languages and Combinatorics (to appear).

[3] N.J. Cavenagh, The size of the smallest latin trade in a back circulant latin square, Bulletin of the Institute of Combinatorics and Applications **38** (2003), 11–18.

[4] N.J. Cavenagh, The size of the smallest critical set in the back circulant latin square, submitted.

[5] N.J. Cavenagh, D. Donovan, A. Khodkar, On the spectrum of critical sets in back circulant latin squares, submitted.

[6] J.C. Conway and N.J. Sloane, Sphere packings, lattices and groups, (New York, Springer-Verlag, 1998).

[7] D. Donovan, A. Howse and P. Adams, A discussion of latin interchanges, J. Comb. Math. Comb. Comput. **23** (1997), 161–182.

[8] D. Donovan and E.S. Mahmoodian, An algorithm for writing any latin interchange as the sum of intercalates, Bulletin of the Inst. Comb. Appl. **34** (2002), 90–98.

[9] A. Drápal, On a planar construction of quasigroups, Czechoslovak Math. J. **41** (1991), 538–548.

[10] A. Drápal, Hamming distances of groups and quasi-groups, Discrete Math. **235** (2001), 189–197.

[11] A. Drápal and T. Kepka, Exchangeable Groupoids I, Acta Universitatis Carolinae - Mathematica et Physica **24** (1983), 57–72.

[12] A. Drápal and T. Kepka, Exchangeable Groupoids II, Acta Universitatis Carolinae - Mathematica et Physica **26** (1985), 3–9.

[13] A. Drápal and T. Kepka, On a distance of groups and latin squares, Comment. Math. Univ. Carolinae **30** (1989), 621–626.

[14] A.S. Hedayat, The theory of trade-off for t-designs, *in* "Coding theory and design theory, Part II," IMA Vol. Math. Appl. 21, Springer, NY, 1990.

[15] P. Horak, R. E. L. Aldred and H. Fleischner, Completing Latin squares: critical sets, J. Combinatorial Designs (to appear).

[16] A.D. Keedwell, Critical sets and critical partial latin squares, *in* "Proc. Third China-USA International Conf. on Graph Theory, Combinatorics, Algorithms and Applications," World Sci. Publishing, NJ, 1994.

[17] A.D. Keedwell, Critical sets for latin squares, graphs and block designs: A survey. *Congr. Numer.* **113** (1996), 231–245.

[18] G.B. Khosrovshahi, H.R. Maimani and R. Torabi, *On trades: an update*, Discrete Applied Mathematics **95** (1999), 361–376.

[19] A.P. Street, Trades and defining sets, in: C.J. Colbourn and J.H. Dinitz, ed., CRC Handbook of Combinatorial Designs (CRC Press, Boca Raton, FL., 1996), 474–478.