Some Trends in Algebra '05

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Baer modules are projective

Lidia Angeleri Hügel

This is a report on joint work with Silvana Bazzoni and Dolors Herbera.

A module B over a commutative domain R is called a Baer module in case $\mathrm{Ext}^1_R(B,T) = 0$ for every torsion R-module T. This definition goes back to 1936 when R. Baer posed the question of characterizing the abelian groups G that satisfy $\mathrm{Ext}^1_{\mathbb{Z}}(G,T) = 0$ for all torsion groups. In 1962 Kaplansky asked the same question for modules over commutative domains.

The first answer was given by Griffith in 1968 who proved that the only Baer groups are the free groups. This result was generalized to modules over Dedekind domains by Grimaldi in 1972, and to modules over valuations domains by Eklof and Fuchs in 1988. The proof of Eklof and Fuchs relies on set theoretic methods. In 1990, Eklof, Fuchs and Shelah extended these methods to arbitrary domains and obtained an important reduction result: Baer modules are projective iff every countably generated Baer module is projective.

Aim of this talk is to show that all countably generated Baer modules over a commutative domain are projective. This implies that all Baer modules are projective and solves the problem raised by Kaplansky.

Our proof uses a relative version of the notion of a Mittag-Leffler module introduced by Rainaud and Gruson in 1971.

On self-duality of Harada rings

Yoshitomo Baba

An R-module M is called *non-small* if M is not small in $E(M)$. In 1979 M. Harada studied a perfect ring satisfying the following condition:

(*) Every non-small left R-module contains a non-zero injective submodule.

K. Oshiro named the ring a left Harada ring. The class of left Harada rings contains both QF-rings and serial rings. And both QF-rings and serial rings have self-dualities. So we naturally have a question: Whether left Harada rings have self-dualities or not. This problem was first considered by J. Kado and K. Oshiro in 1999, and solved negatively by K. Koike in 2001. So we have another question: What kind of left Harada rings have self-dualities. In 1999 J. Kado and K. Oshiro showed that every left Harada ring of homogeneous type has a weakly symmetric self-duality. But in general serial rings are not left Harada rings of homogeneous type. Now we define *Harada rings of a component type* which have weakly symmetric self-dualities and the class of which contains indecomposable serial rings.

In my talk, we give basic results of Harada rings and show Harada rings of a component type have weakly symmetric self-dualities.

Finitistic n-costar modules

Simion Breaz

In this talk we consider a unital associative ring R, a right R-module Q and S the endomorphism ring of Q. Then 1_S induces on Q a left S-module structure and we have a pair of adjoint contravariant functors

$$\Delta = \mathrm{Hom}_R(-, Q) : \text{Mod-}R \rightleftarrows S\text{-Mod} : \mathrm{Hom}_S(-, Q) = \Delta$$

with arrows of adjunction $\delta : 1 \to \Delta^2$. There exists maximal classes

$$\mathrm{Refl}(Q_R) = \{M \in \text{Mod-}R \mid \delta_M \text{ is an isomorphism}\}$$

and

$$\mathrm{Refl}({}_SQ) = \{A \in S\text{-Mod} \mid \delta_A \text{ is an isomorphism}\}$$

such that the restrictions $\Delta : \mathrm{Refl}(Q_R) \rightleftarrows \mathrm{Refl}({}_SQ) : \Delta$ are dualities. Such a module is called *reflexive*. If M (respectively A) is a right (left) R-module (S-module) such that δ_M (δ_A) is monic, then it is a *torsion less* module. We recall that a a module is torsion less if and only if it is *Q-cogenerated* (which means that it can be embedded is a power of A). The class of all torsion less left S-modules is denoted by $\mathrm{TL}(S_Q)$.

The general problem in this context is to establish connections between properties of Q as an R-module, properties of Q as an S-module and properties of some classes $\mathcal{C} \subseteq \mathrm{Refl}(Q_R)$ and $\mathcal{D} \subseteq \mathrm{Refl}({}_SQ)$ such that $\Delta : \mathcal{C} \rightleftarrows \mathcal{D} : \Delta$ are dualities. An important result in this direction was obtained by Colby and Fuller. They introduced the notion of *costar* module as an R-module Q such that the functors Δ induce a duality between the class of all Q-torsion less R-modules whose dual are finitely generated and the class of all finitely generated Q-torsion less S-modules. They characterize this situation with some properties of Q as an R-module: Q is costar if and only if every Q-module with finitely generated dual is semi-finitely copresented by Q (which means that there exists an exact sequence $0 \to A \to Q^X \to Q^Y$ with X a finite set) and Q is injective relative to all short exact sequences $0 \to L \to Q^X \to M \to 0$ in Mod-R where X is a finite set and M is an Q-cogenerated R-module (Q is *w-Π_f-quasi injective*).

We study modules which verify some conditions which have the same flavor as those which appear in the definition of costar modules and in some generalizations for the notion of $\star$-module. In this context it is possible to enunciate dual statements for some results proved for n-$\star$-modules by Wei, adding some finitistic conditions.

An R-module M is *n-finitely Q-cogenerated* if there exists an exact sequence

$$0 \to L \overset{\alpha_0}{\to} Q^{X_0} \overset{\alpha_1}{\to} Q^{X_1} \overset{\alpha_2}{\to} \dots \overset{\alpha_n}{\to} Q^{X_{n-1}}$$

such that X_i are finite sets for all $i \in \{1, \dots, n\}$. We denote by n-cogen(Q) the class of all n-finitely Q-cogenerated modules. If S is a ring we say that a left S-module A is *n-finitely presented* if there exists an exact sequence

$$S^{Y_{n-1}} \to \cdots \to S^{Y_0} \to B \to 0$$

such that Y_i are finite sets for all $i \in \{1, \dots, n\}$. We denote by n-pres(S) the class of all finitely n-presented modules. Of course, 1-pres(S) coincides to the class of all finitely generated left S-modules and 2-pres(S) is in fact the class of all finitely presented modules.

Let Q be a right R-module. A short exact sequence

$$0 \to K \to L \to M \to 0$$

is *Q-cobalanced* if the induced sequence

$$0 \to \Delta(M) \to \Delta(K) \to \Delta(L) \to 0$$

is exact. We say that Q is *n-w_f-quasi-injective* if every exact sequence $0 \to L \to Q^X \to M \to 0$ with $M \in n$-cogen(Q) is Q-cobalanced.

We say that a right R-module Q is a *finitistic n-costar* module if it is n-w_f-quasi-injective and n-cogen$(A) = (n+1)$-cogen(Q). Finitistic n-costar modules will be the main subject of this talk.

Purely extending modules

Septimiu Crivei

A module is called extending (respectively purely extending) if every submodule is essential in a direct summand (respectively pure submodule). The gap between direct summands and pure submodules have only allowed generalizations of few properties from extending modules to purely extending ones. We review these and present some new results, with emphasis on Σ-purely extending modules.

On flat comodules

Juan Cuadra

For a coalgebra C over a field K the category CComod of left C-comodules is a locally finitely presented Grothendieck category. According to Stenström, an object F in such a category is flat if every epimorphism $M \to F$ is pure. Using this notion, a left C-comodule is called flat if it is a flat object in CComod . Since in CComod the classes of finitely presented objects and finite dimensional objects coincide, $F \in C$Comod is flat if for every epimorphism $f : M \to F$ in CComod and every $N \in C$Comod of finite dimension the map $Hom_C(N, f) : Hom_C(N, M) \to Hom_C(N, F)$ is surjective.

In this talk we will study this notion of flat comodule. We will show several characterizations of these comodules. We will discuss the problem of the existence of enough flat comodules and the problem of the existence of flat covers in CComod . The so special features of the category of comodules (finiteness conditions, duality, etc) endow the class of flat comodules with certain distinguished characteristics, not present in other kind of categories. For instance, we will show that any flat left C-comodule may be written as a union of flat subcomodules of at most countable dimension, whenever CComod has enough projective objects. Finally, we will deal with the following question: Assume that CComod has enough flat comodules, has it enough projective comodules? We will give a partial answer, providing so a new characterization of one side semiperfect coalgebras. In this partial answer an interesting new class of comodules comes to the surface. The results to be presented in this talk are part of a joint work with Daniel Simson from the University of Toruń.

Relative injective modules and relative coherent rings

Nanqing Ding

Let m and n be fixed positive integers and M a right R-module. Recall that M is said to be (m,n)-injective if $Ext^1_R(P,M)=0$ for any (m,n)-presented right R-module P; M is called (m,n)-flat if $Tor^1_R(M,Q)=0$ for any (m,n)-presented left R-module Q. In this talk, M is defined to be (m,n)-projective (resp. (m,n)-cotorsion) if $Ext^1_R(M,N)=0$ (resp. $Ext^1_R(N,M)=0$) for any (m,n)-injective (resp. (m,n)-flat) right R-module N. These concepts are used to characterize von Neumann regular rings and (m,n)-coherent rings. Some known results are extended.

Gorestein Categories Tate Cohomology on Projective Schemes

Edgar E. Enochs

We define Gorenstein categories and argue that such categories always admit Tate cohomological functors. We show that categories of quasi-coherent sheaves on nice projective schemes are always such categories.

On big partial tilting modules with a small orthogonal class

Gabriella D'Este

We investigate bounded complexes T , with projective components, corresponding to partial tilting modules, say X , big enough to inherit from tilting modules a functorial condition on the kernels of all Hom and Ext functors. As a consequence, these modules X have the following property:

(+) Every simple module S occurs as the epimorphic image of a submodule M of X .

Under suitable assumptions, concerning

- the orthogonal class associated to X ;
- possible complements of X , as a direct summand of a tilting module;

property (+) implies that X satisfies the functorial Hom - Ext condition verified by tilting modules.

Finally, we construct more or less complicated indecomposable bounded complexes C , with projective components, such that any morphism from T to any shift complex $C[i]$ is homotopic to zero.

Quasi-tilted rings and generalizations

Enrico Gregorio

The notion of tilted algebra introduced by Happel and Ringel has proved very useful in the representation theory of artin algebras.

More recently, Happel, Reiten and Smal introduced the notion of quasi-tilted algebra, and characterized them in various ways.

Colpi and Fuller (2004) started an investigation about generalizing this notion to arbitrary rings.

A quasi-tilted ring is a ring S with a torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-S such that

- S belongs to $\mathcal{Y}$,
- every module in Y has projective dimension at most 1.

An abelian category $\mathcal{A}$ is called an *abelian category with a tilting object* if there exists an object P such that

- arbitrary coproducts of copies of P exist in $\mathcal{A}$,
- P is self-small,
- $\mathrm{Gen}(P) = \ker \mathrm{Ext}^1_{\mathcal{A}}(P, -)$,
- $\mathrm{Gen}(P)$ cogenerates $\mathcal{A}$.

The category $\mathcal{A}$ is *hereditary* if the second (Yoneda) Ext functor vanishes on $\mathcal{A}$.

A ring S is a *quasi-tilted ring* if and only if there exist a hereditary abelian category $\mathcal{A}$ with tilting object P such that $\mathrm{End}(P)$ is isomorphic to S and the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ induces an equivalence between $\mathrm{Gen}(P)$ and a torsion-free class $\mathcal{Y}$ in Mod-S.

The basic ingredient for one of the implications is a subcategory of the bounded derived category of Mod-S called briefly “Heart”.

We will show that any two categories which “tilt” to the same torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-S are equivalent, so that the Heart is as good as any other construction.

Then we will study generalizations of this situation, by dropping the conditions on projective dimension. In particular we will study the AB-type properties of the Heart, starting from a faithful torsion pair in Mod-S.

The talk is partly a review and partly an exposition of recent results from a joint work with R. Colpi and F. Mantese.

On (n, r)-coherence

Josef Jirásko

Let r be a hereditary radical. We characterize right (n, r)-coherent rings.

A construction of tilting modules

Otto Kerner

If R is a ring, freuqently a module X is called a *partial tilting module*, if there exists a tilting module $T = X \oplus Y$ (of projective dimension at most one), such that $\{X\}^\perp = \{T\}^\perp$, where for a set $\mathcal{C}$ of R-modules

$$\mathcal{C}^\perp = \{M \mid \operatorname{Ext}^1_R(\mathcal{C}, M) = 0\}.$$

In a joint work with Jan Trilifaj a more general problem is considered:

When does there exist for a module X a module Y, such that $X \oplus Y$ is a tilting module. Restricting to hereditary Artin algebras, our results say

Theorem. *Let H be a hereditary Artin algebra and X a product-complete H-module with $\operatorname{Ext}^1_H(X, X) = 0$. Then there exists an H-module Y, such that $X \oplus Y$ is a tilting H-module.*

Recall that X is called *product complete*, if Add X is closed und products.

Breaking into B(2)-groups

Claudia Metelli

This is a work in progress in collaboration with Clorinda De Vivo. A completely decomposable group is a direct sum of torsionfree Abelian groups of rank 1. A finite rank Butler group is a torsionfree quotient of a completely decomposable group of finite rank. If the rank of the denominator is n, we have the class of B(n)-groups. B(1)-groups have been amply studied; they give rise to a variety of structures, both linar and lattice-theoretical. B(2)-groups have only now started to be investigated: up to now only degenerate B(2)-groups (i.e. direct sums of two B(1)-groups) have been characterized. We study a subclass of B(2)-groups that already presents the essential difficulties of the main class; the tools are the most varied, adding to the linear and the lattice-theoretical ones also a geometrical side. Rather than listing our results (determination of the typeset, decomposition theorems), I will try to illustrate the interwoven structures that make this such an unusual and interesting field in Algebra.

Projective modules over the endomorphism ring of a uniserial module

Pavel Příhoda

A module U is said to be uniserial if the lattice of its submodules is a chain. We shall try to classify projective right modules over the endomorphism ring of U and compare these projectives with objects of $Add(U)$.

Semiprime Preradicals and Semiprime Submodules

Francisco Raggi

This talk is concern with the study of semiprime elements in the big lattice of preradicals of module categories. In particularwe are interested in the module theoretic characteriztions of the property. Since preradicals are closed related to fully invarint submodules the resultes are different to those for semiprimeness of submodules of a given module.

Lattices of two-sided ideals of unit-regular rings and ideal lattices of dimension groups

Pavel Růžička

A dimension group is a partially ordered directed, unperforated abelian group with the interpolation property. Directed convex subgroups (called ideals) of a dimension group form an algebraic distributive lattice isomorphic to the ideal lattice of the maximal semilattice quotient of its positive cone.

The Grothendieck group $K_0(R)$ of a unit-regular ring R is a dimension group and the lattice of two-sided ideals of the ring R corresponds to the lattice of ideals of $K_0(R)$.

Thus representations of algebraic distributive lattices in ideal lattices of dimension groups, resp. in lattices of two-sided ideals of unit-regular rings are closely connected. We will study these representation problems.

Zero Inversive Rings

Phill Schultz

A unital ring is called 0-inversive if every non-zero left ideal contains a non-zero idempotent. The concept, as well as the name, comes from Semigroup Theory, and it was introduced into Ring Theory by Kasch, 20 years ago. The property is a generalization of regularity and it has important applications to endomorphism rings of modules. I will present some structure theorems for 0-inversive rings, classify the abelian groups whose endomorphism ring is 0-inversive, and describe various classes of modules whose endomorphism ring is 0-inversive.

Closure properties of (co-)tilting classes

Jan Šťovíček

I will present a joint work with Silvana Bazzoni and Jan Trlifaj.

Let R be any (associative, unital) ring and n a natural number. Then any n-tilting as well as any n-cotilting class within the class of all (right R-) modules is *definable*, that is, closed under direct products, direct limits and pure submodules. Furthermore, 2-cotilting classes are characterized as special covering reflective classes, and the situation when a cotilting class is a direct limit of its finitely presented modules is characterized in terms of approximations.

In the tilting case, the notions of a class of finite and countable type are used, [2]. The proof involves two steps. First, any n-tilting class $\mathcal{T}$ is proved to be of countable type using set-theoretic methods, [6]. Next, $\mathcal{T}$ is proved to be of finite type, generalizing results in [3], which already implies the definability of $\mathcal{T}$.

In the cotilting case, it turned out to be enough to prove the following statement:

If U is a module and ${}^{\perp_1}U = \operatorname{Ker}\operatorname{Ext}^1_R(-,U)$ is closed under direct products and pure submodules, then ${}^{\perp_1}U$ is closed under direct limits.

The proof of the assertion is given by analyzing the cokernel of $M \hookrightarrow PE(M)$, where M is a module and $PE(M)$ is a pure-injective hull of M.

References

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The number of QF rings with radical cube zero

Hiroshi Yoshimura

For a given (not necessarily commutative) field k, how many local quasi-Frobenius (QF) rings Λ such that

$$(*) \quad \Lambda/Rad\Lambda \cong k \text{ and } (Rad\Lambda)^3 = 0$$

are there ? In this talk, we consider this problem in case k is a commutative field and Λ are algebras over k. We present the canonical forms of local QF k-algebras Λ with the condition $(*)$ and classify, up to k-isomorphism, those algebras in low dimensional case.

On self-small modules

Jan Žemlička

A module M is called self-small if the functor $Hom(M, -)$ commutes with direct sums of copies of M. However several sufficient conditions on the endomorphism ring $End(M)$ which ensure M is self-small are known, self-small modules cannot be characterized using their ring of endomorphisms in general. We discuss the question over which rings there exists no infinitely generated self-small module.