GROUP RINGS THAT ARE UJ RINGS

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ABSTRACT. The set $\Delta(R)$ of all elements r of a ring R such that $1 + ru$ is a unit for every unit u extends the Jacobson radical $J(R)$. R is a UJ ring (ΔU ring, respectively) if its units are of the form $1 + J(R)$ ($1+\Delta(R)$, respectively). Using a local characterization of ΔU rings, we describe structure of group rings that are UJ rings; if RG is a UJ group ring, then R is a UJ ring, G is a 2-group and, for every nontrivial finitely generated subgroup H of G, the commutator subgroup of H is proper subgroup of H. Conversely, if R is a UJ ring and G a locally finite 2-group, then RG is a UJ ring. In particular, if G is solvable, RG is a UJ ring if and only if R is UJ and G is a 2-group.

1. Introduction

It is well known that the Jacobson radical $J(R)$ of a unital associative ring R can be characterized as the set of all elements $j \in R$ such that $1 + jr$ is a unit for every $r \in R$ (see e.g. [1, Theorem 15.3]). From this fact immediately follows an observation that the set $1 + J(R)$ forms a normal subgroup of the group of all units $U(R)$. Rings over which the groups $U(R)$ and $1 + J(R)$ coincide are called UJ rings in this paper (cf. [6]). Structure of UJ rings and possibility of their application in various questions of non-commutative ring theory were studied in several recent works [3, 6, 7, 9].

The recalled criterion for elements of the Jacobson radical offers a natural extension of the Jacobson radical, which is the set

$$\Delta(R) = \{r \in R \mid \forall u \in U(R) : 1 + ru \in U(R)\}.$$

However $\Delta(R)$ is not necessarily an ideal in general, it forms a non-unital subring of R (see [9, Lemma 1]), and $1 + \Delta(R)$ is a normal subgroup of $U(R)$ containing $1 + J(R)$. A ring R satisfying the condition $U(R) = 1 + \Delta(R)$ is said to be a ΔU ring (cf. [7]). Note that every UJ-ring is a ΔU ring and the inclusion is strict by [7, Example 2.2]. ΔU rings and the set $\Delta(R)$ in general are studied in papers

2010 *Mathematics Subject Classification.* Primary 16D40, 16D50, 16D60, 16S34.

Key words and phrases. Unit, Jacobson radical, UJ-rings, group ring, trivial Morita context, solvable group, commutator subgroup, locally finite 2-group.

[7, 9] and structural knowledge of both the notions seems to be useful for further research of UJ rings as it is shown below.

The present paper has two main objectives: to give a local characterization of ΔU rings and, as a consequence, to describe structure of UJ group rings. The main result of the section 2 is Theorem 2.11 which characterizes ΔU rings R using the notion of a rationally closed subring. If RG is a UJ group ring, we prove that the ring R is necessarilly a UJ ring, G is a 2-group, and a commutator subgroup of any nontrivial finitely generated subgroup of G is proper (Theorem 3.2(3)). Conversely, Theorem 3.7 shows that RG is a UJ ring if R is a UJ ring and G a locally finite 2-group. As a consequence, we obtain a necessary and sufficient condition for RG to be a UJ-ring when G is a solvable group (Corollary 3.9).

In the sequel, R is an associative ring with unity and G be a group. Fro non-explained terminology we refer to [10] for ring theory, [12] for group rings and [13] for group theory.

2. ΔU rings

We begin with recalling the basic description and properties of $\Delta(R)$

$$\begin{aligned}\Delta(R) &= \{r \in R \mid \forall u \in U(R) : r + u \in U(R)\}\\ &= \{r \in R \mid \forall u \in U(R) : 1 + ru \in U(R)\}\\ &= \{r \in R \mid \forall u \in U(R) : 1 + ur \in U(R)\}\end{aligned}$$

by [9, Lemma 1, Corollary 9]:

Lemma 2.1. *For any ring R, we have:*

(1) $\Delta(R)$ *is a non-unital subring of* R.
(2) $\Delta(R)$ *is an ideal of* R *if and only if* $\Delta(R) = J(R)$.
(3) $ur, ru \in \Delta(R)$ *for any* $r \in \Delta(R)$ *and* $u \in U(R)$.
(4) $\Delta(\prod_{i\in I} R_i) = \prod_{i\in I} \Delta(R_i)$ *for any system of rings* $R_i, i \in I$.
(5) $\Delta(R[x]/(x^n)) = \Delta(R)[x]/(x^n)$.
(6) $\Delta(R[[x]]) = \Delta(R)[[x]]$.

The following, based on easy matrix computation and [9, Theorem 3], collects basic properties of the subring $T(R)$ of a ring R generated by all units $U(R)$.

Lemma 2.2. *For any ring R, we have:*

(1) $U(R) = U(T(R))$,
(2) $T(\mathbb{M}_n(R)) = \mathbb{M}_n(R)$ *for each* $n > 1$,
(3) $\Delta(R) = \Delta(T(R)) = J(T(R))$.

The following observation characterizes ΔU rings in the language of the subring $T(R)$.

Theorem 2.3. *The following conditions are equivalent for a ring R:*

(1) *R is a ΔU ring,*
(2) *$U(R) + U(R) = \Delta(R)$,*
(3) *$U(R) \cap (U(R) + U(R)) = \emptyset$ and $U(R) + U(R) + U(R) \subseteq U(R)$,*
(4) *$T(R)/J(T(R)) \cong \mathbb{F}_2$,*
(5) *$T(R)$ is a UJ ring.*

Proof. (1)⇔(2) This is proved in [7, Proposition 2.3].
(2)⇒(3) This is clear, since

$$U(R) \cap \Delta(R) = \emptyset,$$

$$1 + \Delta(R) = U(R)$$

and

$$u + \Delta(R) = u(1 + \Delta(R)) = uU(R) = U(R)$$

for each $u \in U(R)$.
(3)⇒(4) Put $D := U(R) + U(R)$. Then $D + U(R) \subseteq U(R)$ and $D \cap U(R) = \emptyset$ by the hypothesis. Moreover

$$U(R)U(R) = U(R),$$

$$DD = D + D \subseteq D,$$

$$U(R)D = DU(R) = D,$$

which implies that

$$T(R) = U(R) \cup D,$$

$D = T \setminus U(R)$ is the unique maximal ideal of $T(R)$ and

$$U(R) = 1 + (-1) + U(R) \subseteq 1 + D \subseteq U(R) + D \subseteq U(R).$$

Hence $T(R) = (1+D) \cup D$ is a local ring with $J(R) = D$ and $T(R)/J(T(R)) \cong \mathbb{F}_2$.
(4)⇒(5) Clearly, $1 + J(T(R)) \subseteq U(T(R))$. Conversely, if $a \in U(T(R)) = U(R)$, then $a + J(T(R)) \in U(T(R)/J(T(R))) = \{1 + J(T(R))\}$ by the hypothesis. Hence $a + J(T(R)) = 1 + J(T(R))$, which implies that $U(T(R)) = 1 + J(T(R))$.
(5)⇒(1) The equalities $U(R) = U(T(R)) = 1 + J(T(R)) = 1 + \Delta(R)$ follows immediately from the hypothesis and Lemma 2.2. □

The proof of (3)⇒(4) of Theorem 2.3 can be formulated as the following consequence (cf. [7, Example 2.2(2)]).

Corollary 2.4. *R is a ΔU ring if and only if $T(R)$ is a local ring such that $T(R)/J(T(R)) \cong \mathbb{F}_2$.*

Since $U(R) = U(R[x])$ and so $T(R) = T(R[x])$ for any domain R, we obtain another consequence of Theorem 2.3:

Corollary 2.5. *Let R be a domain. Then $R[x]$ is ΔU if and only if R is so.*

Proof. This follows immediately from Corollary 2.4 and Lemmas 2.2 and 2.1. □

By applying Lemma 2.2 we can significantly shorten the proof of [7, Theorem 2.5].

Corollary 2.6. *Let R be a ring. Then $\mathbb{M}_n(R)$ is a ΔU ring if and only if $n = 1$ and R is a ΔU ring.*

Proof. Let $n > 1$. By Lemma 2.2(2), $T(\mathbb{M}_n(R)) = \mathbb{M}_n(R)$. Now, we suppose that $\mathbb{M}_n(R)$ is a ΔU ring. Then it is local by Corollary 2.4, which contradicts to the hypothesis that $n > 1$. Thus $n = 1$ and $R \cong \mathbb{M}_1(R)$ is a ΔU ring.

The converse is obvious. □

Recall that a Morita context is a 4-tuple $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where A and B are rings, ${}_AM_B$ and ${}_BN_A$ are bimodules, and there exist context products $M \times N \to A$ and $N \times M \to B$ written multiplicatively as $(w, z) = wz$ and $(z, w) = zw$, such that $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is an associative ring with the obvious matrix operations. A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is called trivial if the context products are trivial, i.e., $MN = 0$ and $NM = 0$ (see [11, p. 1993]). We have

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N),$$

where $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a trivial Morita context by [5].

Recall that a radical class $\mathfrak{R}$ is called hereditary if $R \in \mathfrak{R}$ implies $I \in \mathfrak{R}$ for arbitrary two sided ideal I of R. A radical, say Γ, is called left strong if $I \in \Gamma$ implies $IR^* \in \Gamma$ for arbitrary left ideal I of R, where the usual extension of a ring R obtained by adjoining unity is denoted by R^*. And a radical is called an N-radical if it contains all nilpotent rings and is left hereditary and left strong (see [4]).

Theorem 2.7. *Let* $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *be a Morita context. Then* R *is a* ΔU *ring if and only if* A, B *are* ΔU *rings,* $MN \subseteq J(A)$ *and* $NM \subseteq J(B)$*.*

Proof. Put $e := \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$. Note that e and $1-e$ are idempotents, and there are canonical ring isomorphisms $A \cong eRe$ and $B \cong (1-e)R(1-e)$.
(:⇒) Suppose that R is a ΔU ring. Then $A \cong eRe$ and $B \cong (1-e)R(1-e)$ are ΔU rings by [7, Proposition 2.6]. Since $\begin{pmatrix} 1_A & m \\ 0 & 1_B \end{pmatrix}, \begin{pmatrix} 1_A & 0 \\ n & 1_B \end{pmatrix} \in U(R)$ for each $m \in M$ and $n \in N$, it is easy to obtain that $\begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix} \subseteq \Delta(R)$ and so $I = \begin{pmatrix} MN & M \\ N & NM \end{pmatrix} \subseteq \Delta(R)$. Note that I is an ideal of R, hence $I \subseteq J(R)$. As $\begin{pmatrix} 1_A + x & 0 \\ 0 & 1_B + y \end{pmatrix} \in J(R)$ for each $x \in MN$ and $y \in NM$, we get that $1_A + x \in U(A)$ and $1_B + y \in J(B)$ hence $x \in J(A)$ and $y \in J(B)$.
(⇐:) Let A, B be ΔU rings, $MN \subseteq J(A)$ and $MN \subseteq J(B)$. Since the Jacobson radical is an N-radical by [4, Examples 3.6.1(iii), 3.18.6(i) and Theorem 3.18.12], the ideal $I = \begin{pmatrix} MN & M \\ N & NM \end{pmatrix}$ of the ring R is contained in $J(R)$ by [4, Theorem 3.18.14]. Hence R is a ΔU ring if and only if R/I is a ΔU ring by [7, Proposition 2.4(5)]. Since $R/I \cong A/MN \times B/NM$ where A/MN and B/NM are ΔU rings, the conclusion follows from [7, Proposition 2.4]. □

Let us formulate an easy consequence of [7, Example 2.2] and [9, Theorem 11].

Lemma 2.8. *Let* R *be a* ΔU *ring. Then* R *is a* UJ*-ring if and only if* $\Delta(R) = J(R)$*.*

A homomorphism of rings $S \to R$ is said to be local if it carries non-units to non-units, that is, the image of $S \setminus U(S)$ lies in $R \setminus U(R)$. A rationally closed subring of R is a subring S such that $U(S) = S \cap U(R)$, which is equivalent to the condition that the inclusion map $S \to R$ is a local homomorphism.

Lemma 2.9. *Let* R *be a ring.*

(1) *If* S *is a rationally closed subring of* R*, then* $\Delta(R) \cap S \subseteq \Delta(S)$*. Furthermore,* $\Delta(R) \cap Z(R) \subseteq \Delta(Z(R))$*, where* $Z(R)$ *is the center of* R*.*
(2) *Every rationally closed subring of a* ΔU *ring is a* ΔU *ring.*
(3) *Every rationally closed subring of a* UJ*-ring is a* UJ*-ring.*

(4) *If S_i, $i \in I$, are rationally closed subrings of R, then $\bigcap_{i\in I} S_i$ is a rationally closed subring of R.*

Proof. (1) This is proved in [9, Proposition 6].
(2) Let S be a rationally closed subring of of a ΔU ring R. Since $U(R)+U(R) = \Delta(R)$ by Theorem 2.3 and $U(S) = U(R) \cap S$, we obtain that $U(S) + U(S) \subseteq \Delta(R) \cap S \subseteq \Delta(S)$ which implies that $U(S) \cap (U(S) + U(S)) = \emptyset$. Furthermore, $U(S)+U(S)+U(S) \subseteq U(R)\cap S = U(S)$, hence S is a ΔU ring by Theorem 2.3(3).
(3) This is proved in [6, Proposition 2.1]. It also follows directly from (1) and Lemma 2.8.
(4) Obviously, $U(\bigcap_i S_i) \subseteq \bigcap_i U(S_i) = \bigcap_i(U(R) \cap S_i) = U(R) \cap \bigcap_i S_i$. On the other hand, if $u \in \bigcap_i U(S_i) \subseteq U(R)$, then $u^{-1} \in S_i$ for all $i \in I$, and so $u \in U(\bigcap_i S_i)$. □

Corollary 2.10. *The center of a ΔU ring is a ΔU ring.*

Let S be a subring of a ring R and $F \subseteq U(R)$. We define

$$C_S(F) := \bigcap\{A \subseteq R \mid A \text{ is a rationally closed subring of } R \text{ with } S \cup F \subseteq A\}.$$

Note that R is a rationally closed subring of itself and that $C_S(F)$ forms a rationally closed subring of R by Lemma 2.9(4).

We finish the section by characterization of ΔU rings by its finitely generated rationally closed subrings.

Theorem 2.11. *Let R be a ring and S a rationally closed subring of R. The following conditions are equivalent:*

(1) *R is a ΔU ring,*
(2) *$C_S(F)$ is a ΔU ring for every finite set $F \subseteq U(R) \setminus S$,*
(3) *$C_S(\{u, v\})$ is a ΔU ring for every pair $u, v \in U(R) \setminus S$.*
(4) *For every pair $u, v \in U(R) \setminus S$, there exists a rationally closed subring A containing $S \cup \{u, v\}$ which is a ΔU ring.*

Proof. (1)⇒(2) Since $C_S(F)$ is rationally closed by Lemma 2.9(4) and R is a ΔU ring, we get that S and $C_S(F)$ are ΔU rings by Lemma 2.9(3).
(2)⇒(3) and (3)⇒(4) The implications are obvious.
(4)⇒(1) By Theorem 2.3 it is enough to show that $U(R) \cap (U(R) + U(R)) = \emptyset$ and that $U(R) + U(R) + U(R) \subseteq U(R)$.

Assume that there exists $u, v, w \in U(R)$ such that $u+v = w$. Note that $uw^{-1}+vw^{-1} = 1$. Let A be a rationally closed ΔU subring containing $S\cup\{uw^{-1}, vw^{-1}\}$. As $1, uw^{-1}, vw^{-1} \in U(A)$, we get that $1 = uw^{-1}+vw^{-1} \in U(A)\cap(U(A)+U(A))$, which contradicts to the hypothesis that A is a ΔU ring by Theorem 2.3.

Now assume that there exists $u, v, w \in U(R)$ such that $u + v + w \notin U(R)$. Let A be a rationally closed ΔU subring containing $S \cup \{uw^{-1}, vw^{-1}\}$ Then $uw^{-1} + vw^{-1} + 1 \notin U(R)$, a contradiction (with the fact that the ΔU ring A satisfies $uw^{-1} + vw^{-1} + 1 \in U(A) + U(A) + U(A) = U(A) \subset U(R)$). □

3. Group rings over UJ and ΔU rings

Given a ring R and a group G, we denote the group ring of G over R by RG. An arbitrary element of RG, say $\alpha \in RG$, is of the form $\alpha = \sum_{g\in G} r_g g$ where $r_g \in R$ and $\{g \in G|\ r_g \neq 0\}$ is finite.

First, recall a well-known observation on rationally closed subrings of a group ring.

Lemma 3.1. *Let R be a ring, G a group and H a subgroup of G. Then RH is a rationally closed subring of the group ring RG.*

Let R be a ring, G a group, and H a subgroup of G. We will denote by $\Delta(H, G)$ the left ideal of RG generated by the set $\{1 - h|\ |\ h \in H\}$. Put $\Delta(G) = \Delta(G, G)$ and recall that $\Delta(G, H)$ is finitely generated whenever H is a finitely generated left ideal [12, Lemma 3.3.2]. Moreover, if H is a normal subgroup of G, then $\Delta(G, H)$ is a two-sided ideal and $R(G/H) \cong RG/\Delta(G, H)$ by [12, Corollary 3.3.5].

For every group H we will denote by H' the commutator subgroup of H, i.e. the subgroup generated by all elements of the form $x^{-1}y^{-1}xy$. Note that H' forms a fully invariant subgroup of H such that H/H' is commutative.

Let us formulate necessary conditions for group ΔU and UJ rings:

Theorem 3.2. *Let R be a ring and G a group. The following holds for a group ring RG:*

1. *Let H be a subgroup and N be a normal subgroup of G. If RG is a UJ ring, then RH and $R(G/N)$ are UJ rings.*
2. *If RG is a ΔU ring, then R is a ΔU ring and G is a 2-group.*
3. *If RG is a UJ ring, then R is a UJ ring, G is a 2-group and, for every nontrivial finitely generated subgroup H of G, $H' \neq H$ where H' is a commutator subgroup of H.*

Proof. (1) By Lemmas 2.9(3) and 3.1, we obtain that RH is a UJ ring. Since $N\Delta(G, N) \subseteq \Delta(G) \subseteq J(RG)$, we have $R(G/N) \cong RG/\Delta(G, N)$ is a UJ ring by [12, Corollary 3.3.5] and [6, Proposition 1.3(5)].

(2) Let $g \in G$. Then $R\langle g\rangle$ and $R \cong R\{1_G\}$ are rationally closed subrings of RG by Lemma 3.1. By Lemma 2.9(2), both are ΔU rings.

If $\langle g\rangle$ is an infinite cyclic group, then

$$1+g+g^2 \in U(G)+U(G)+U(G)=U(G).$$

Hence there exist integers $a \leq b$ and $c_i \in R$ with $c_a \neq 0 \neq c_b$ such that $1 = \sum_{i=a}^{b} c_i g^i (1+g+g^2) = c_a g^a + \sum_{i=a+1}^{b+1} d_i g^i + c_b g^{b+2}$ for suitable $d_i \in R$, $i \in \mathbb{Z}$, a contradiction. If there exists an element of G such that its order is divisible by an odd prime, say p, then there exists an element g of G with $o(g) = p$. Since $\sum_{i=0}^{p-1} g^i \in U(RG)$ by Theorem 2.3(3) and $(1-g)\cdot \sum_{i=0}^{p-1} g^i = 0$, we get that $1-g=0$, a contradiction.

(3) By (1), the rings $R \cong R\{1_G\}$ and RH are UJ rings. As RG is a ΔU ring, we get that G is a 2-group by (2).

Let H be a nontrivial finitely generated subgroup of G. Note that $H \subseteq U(RH)$ and $\Delta(H)$ is an ideal of the UJ ring RH which is finitely generated as a left ideal of the UJ ring RH by [12, Lemma 3.3.2]. It implies that

$$J(RH)\Delta(H) \subseteq J(\Delta(H)) \subsetneq \Delta(H)$$

and

$$\Delta(H) \subseteq U(RH)+U(RH) = J(RH)$$

by Theorem 2.3 and Lemma 2.8. Furthermore

$$1-x^{-1}y^{-1}xy = x^{-1}y^{-1}[(1-y)(1-x)-(1-x)(1-y)] \in \Delta(H)^2$$

for every $x, y \in H$, which implies that $\Delta(H') \subseteq \Delta(H)^2$. We have shown that

$$\Delta(H') \subseteq \Delta(H)^2 \subseteq J(RH)\Delta(H) \subsetneq \Delta(H).$$

Thus $H' \neq H$. □

Example 3.3. Let G be a finitely generated simple 2-group which is infinite (for example, a simple factor of a minimal finite index subgroup of an infinite Burnside 2-group). Then $G' = G$, hence the group ring $\mathbb{F}_2 G$ is not a UJ ring by Theorem 3.2(3).

Question 3.4. Does the converse of Theorem 3.2(3)?

Recall an observation on the Jacobson radical of a group ring which will appear useful in the sequel.

Lemma 3.5. [2, Lemma 4] *If R is a ring and G a locally finite group, then $J(R) \subseteq J(R)G \subseteq J(RG)$.*

Now we are able to formulate a criterion for UJ group rings over finite 2-groups.

Proposition 3.6. *If R is a UJ ring and G is a locally finite 2-group, then RG is a UJ group ring.*

Proof. First, we will prove that RH is a UJ ring for every finitely generated subgroup H of G.

Let R be a UJ ring and H finitely generated subgroup of the locally finite 2-group G. Then H is a finite 2-group and $2 \in J(R) \subseteq J(RH)$ by [6, Proposition 1.3(1)] and Lemma 3.5. Hence RH is UJ if and only if $RH/2RH \cong (R/2R)H$ is UJ ring by [6, Proposition 1.3(5)]. By factoring $2R$ if necessary, we may assume that characteristic of R is 2.

Suppose that H is of order 2^k and we will prove by induction on k that RH is a UJ ring.

If $k = 0$, there is nothing to prove. Let us suppose that the assertion is true for $k-1$. It is well known that any finite 2-group has a non-trivial centre and a central subgroup is always normal (cf. e.g. [13, 1.6.13]), hence the group H contains a central subgroup $\langle g\rangle$ of order 2. Then $1-g$ is a central nilpotent element, because $(1-g)^2 = 2(1-g) = 0$, so $1-g$ belongs to the Jacobson radical $J(RH)$. Thus RH is UJ if and only if $RH/((1-g)RH)$ is UJ by [6, Proposition 1.3(5)]. Since $RH/((1-g)RH) \cong R(H/\langle g\rangle)$ by [12, Corollary 3.3.5], where the group $H/\langle g\rangle$ is of order 2^{k-1}, $RH/((1-g)RH)$ is a UJ ring by the induction hypothesis.

Now, we show that RG is a ΔU ring. By Theorem 2.11 and Lemma 3.1 it is enough to prove that for every pair $u, v \in U(RG) \setminus R$ there exists a ΔU rationally closed subring containing $R \cup \{u, v\}$. Since for every $u, v \in U(RG)$ there exists a finite subgroup H of G such that $u, v \in U(RG) \cap RH = U(RH)$, the ring RH, which is a UJ ring by the first part of the proof, is a ΔU ring. Then Theorem 2.11(4) implies that RG is a ΔU ring.

Finally, denote by $\mathcal{F}$ the set of all finite subgroups of G. Then $U(RG) = \bigcup_{H\in\mathcal{F}} U(RH)$, and hence

$$\begin{aligned}\Delta(RG) &= U(RG) + U(RG)\\ &= \textstyle\bigcup_{H\in\mathcal{F}}(U(RH) + U(RH))\\ &= \textstyle\bigcup_{H\in\mathcal{F}} \Delta(RH)\\ &= \textstyle\bigcup_{H\in\mathcal{F}} J(RH)\end{aligned}$$

by Lemma 2.8. It is easy to see that $\Delta(RG) = \bigcup_{H\in\mathcal{F}} J(RH)$ is an ideal, which implies that $\Delta(RG) = J(RG)$ by [9, Lemma 1(4)]. Thus RG is a UJ ring by applying Lemma 2.8 again. □

Let us formulate the main result of the paper:

Theorem 3.7. *Let G be a locally finite 2-group. Then RG is a UJ ring if and only if R is a UJ ring.*

Proof. The direct implication is proved by Proposition 3.6 and the converse follows from Theorem 3.2(3). □

Example 3.8. Let R be $\mathbb{F}_2$ or $\mathbb{F}_2[[x]]$ or the trivial extension $T(\mathbb{F}_2, \mathbb{F}_2)$. Then R is UJ by [6, Lemma 1.1, Example 1.2, Corollary 1.5 and Theorem 2.8]. If G is an elementary abelian 2-group, then RG is a UJ ring by Theorem 3.7.

Note that a solvable 2-group is locally finite by [13, 5.4.11]. We have the following corollary which generalizes [3, Theorem 5.3] and answers [3, Problem 2].

Corollary 3.9. *Let R be a ring and G a solvable group. Then RG is UJ if and only if R is UJ and G is a 2-group.*

Proof. (:⇒) This follows immediately from Theorem 3.2(3).
(⇐:) Since any subgroup of a solvable group is solvable and every finitely generated solvable 2-group is finite by [13, 5.4.13], every solvable 2-group is locally finite. Thus the assertion follows from Theorem 3.7. □

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