ON THE SCHRÖDER-BERNSTEIN PROPERTY FOR ABELIAN GROUPS

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Abstract. A right R-module M satisfies the Schröder-Bernstein property, if whenever direct summands, say N and K, of M are d-subisomorphic to each other (i.e. if N is isomorphic to a direct summand of K and K is isomorphic to a direct summand of N), then $N \cong K$. The module M is said to be ADS (Absolute Direct Summand) if for every decomposition $M = S \oplus T$ and every complement A of S, we have $M = S \oplus A$. We primarily show that the question, whether ADS abelian groups satisfying the Schröder-Bernstein property, has a positive answer. Then we consider a related problem on the property C2 (a group G is C2 if whenever A is a summand of G and B is a subgroup of G isomorphic to A, then B is also a summand of G) and we present several sufficient conditions of C2 abelian groups to satisfy the Schröder-Bernstein property.

1. Introduction

In the set theory, the Schröder-Bernstein theorem states that if there exist injective functions $A \to B$ and $B \to A$ between the sets A and B, then there exists a bijective function $A \to B$. This has been investigated in some branches of mathematics: In the module theory, Bumby [3] proved that the Schröder-Bernstein problem has a positive solution for homomorphism of modules which are invariant under endomorphisms of their injective envelopes. In [7], Dehghani et al. studied the Schröder-Bernstein property for direct summands. Two R-modules N and K are said to be *direct summand subisomorphic* to each other (or *d-subisomorphic*) if N is isomorphic to a direct summand of K and K is isomorphic to a direct summand of N, and a module M satisfies the *Schröder-Bernstein property*, or the "SB property" for short, if whenever direct summands N and K of M are d-subisomorphic to each other, then $N \cong K$ ([7, Definitions 1.5 and 1.6]). They proved that over a Noetherian ring R, every extending module (defined by the property that every submodule of the module is essential in a direct summand of it) satisfies the Schröder-Bernstein problem property. In the theory of abelian groups, the following question was raised by Kaplansky [11] (known as Kaplansky's First Test Problem):

If G and H are abelian groups such that G is isomorphic to a direct summand of H and H is isomorphic to a direct summand of G, are G and H necessarily isomorphic?

For more results on this direction, we refer to the papers [6], [8], [14].

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The notion of the *absolute direct summand* was introduced by Fuchs in [9]. In [1] and [4], the authors introduced and studied the module-theoretical version of the absolute direct summand. A right R-module M is said to be *Absolute Direct Summand* (ADS) if for every decomposition $M = S \oplus T$ and every complement A of S, we have $M = S \oplus A$. Let $\mathbb{P}$ denote the set of all prime numbers, A be an abelian group, and $p \in \mathbb{P}$. Following the terminology of [5] we say that A is *p-automorphic* if the map $a \to pa$ is an automorphism of A, and A is called *homococyclic* if there exist a cardinal λ, a value $k \in \mathbb{N} \cup \{\infty\}0$ and $p \in \mathbb{P}$ such that $A \cong \mathbb{Z}_{p^k}^{\lambda}$. In the recent paper [12], the authors studied ADS abelian groups and it is shown that

Theorem 1.1. [12, Theorem 3.1] *An abelian group is ADS if and only if*

(1) *either it is divisible,*
(2) *or it is a direct sum of an indecomposable torsion-free group and a divisible torsion group,*
(3) *or it is a torsion group such that p-component are homococyclic for all $p \in \mathbb{P}$.*

In view of the studies on the Schröder-Bernstein property in the theory of abelian groups, our main aim is to study the following problem.

Problem 1.2. *Characterize ADS abelian groups satisfying the Schröder-Bernstein property.*

We will answer Problem 1.2 in Section 2. Precisely, we first prove the following.

Theorem 1.3. *If A and B are d-subisomorphic ADS abelian groups, then A and B are isomorphic.*

Since each direct summand of an ADS module is ADS, we obtain the following direct consequence.

Corollary 1.4. *Every ADS abelian group satisfies the Schröder–Bernstein property.*

A group is *reduced* if it contains no nonzero divisible subgroup. Recall that every abelian group A contains a maximal divisible subgroup, say D, and a reduced subgroup, say R, such that $A = D \oplus R$.

A/an (abelian) group G is *C2* if whenever A is a direct summand of G and B is a subgroup of G isomorphic to A, then B is also a direct summand of G [5]. Since, by [5];

(i) every divisible group is injective (so quasi-injective) hence C2,

(ii) a torsion-free group is C2 iff it is divisible,

(iii) the only indecomposable C2 groups are the cocyclic groups and $\mathbb{Q}$,

(iv) a torsion group is C2 iff it has homococyclic,

it is natural to raise the following problem.

Problem 1.5. *Characterize C2 abelian groups satisfying the Schröder-Bernstein property.*

We will partially answer Problem 1.5 in Sections 3 and 4. In particular, we formulate several structural conditions under which is a group C2 and satisfies the Schröder-Bernstein property.

Throughout this paper, R is an associative ring with unity and all modules over R are unitary right modules. $r_R(x)$ denotes a right annihilator of an element x over a ring R. We also write M_R to indicate that M is a right R-module. For a submodule N of M, we use $N \leq M$ to mean that N is a submodule of M. We write $\mathbb{Z}$ and $\mathbb{N}$ for the ring of integers and for the set of all positive integer numbers, respectively. For any group G, as usually $X \subseteq G$ shows X is a subset of G but $X \leq G$ is used only for a subgroup X of G. For unexplained notions and results, we refer the reader to [9].

2. Problem 1.2

Let us formulate a well-known observation about fully invariant modules and its easy consequence.

Lemma 2.1. *Let A be a fully invariant submodule of a module M and B a direct summand of M. Then $B \cap A$ is a direct summand of A and $(B + A)/A$ is a direct summand of M/A .*

Proof. By the hypothesis, the natural projection $M \to B$ can be represented as an idempotent $\epsilon \in \mathrm{End}(M)$ satisfying $\epsilon(M) = B$ and $(1 - \epsilon)(M) \oplus \epsilon(M) = M$. Since A is fully invariant, both images $\epsilon(A)$ and $(1 - \epsilon)(A)$ are submodules of A. Thus $\epsilon(A) = A \cap B$ and $A = \epsilon(A) \oplus (1 - \epsilon)(A)$. Similarly, $\tilde{\epsilon}(m + A) = \epsilon(m) + A$ presents a correctly defined idempotent endomorphism of the module M/A, hence $M/A = \tilde{\epsilon}(M/A) \oplus (1 - \tilde{\epsilon})(M/A)$ with $\tilde{\epsilon}(M/A) = B + A/A$. □

Lemma 2.2. *Let A be a d-subisomorphic to an abelian group B, let E, F be maximal divisible subgroups of A and B respectively, and $S \subseteq \mathbb{P}$. If $A_S = \bigoplus_{p\in S} A_p$ and $B_S = \bigoplus_{p\in S} B_p$, then*

(1) *A_S is d-subisomorphic to B_S,*
(2) *A/A_S is d-subisomorphic to B/B_S,*
(3) *E is d-subisomorphic to F,*
(4) *A/E is d-subisomorphic to B/F.*

Proof. Let us denote by $C_S = \bigoplus_{p\in S} C_p$ for an arbitrary abelian group and remark that C_S is a fully invariant submodule of C. Suppose that D is a direct summand of B which is isomorphic to A.
(1) Since $A_p \cong D_p = D \cap B_p$, it is easy to see that $A_S \cong D_S = D \cap B_S$, which is a direct summand of B_S by Lemma 2.1.
(2) Note that $A/A_S \cong D/D_S = D/(D \cap B_S) \cong D + B_S/B_S$ by the hypothesis. Then the conclusion follows since $D+B_S/B_S$ is a direct summand of B/B_S by Lemma 2.1.
(3) Denote by G the maximal divisible subgroup G of D. Since G is a direct summand of B and it is isomorphic to E, the assertion is clear.
(4) Similarly as in (2), we get $A/E \cong D/G = D/(D \cap F) \cong D + F/F$ by Lemma 2.1, where the last group is a direct summand of B/F, as F is fully invariant. □

Lemma 2.3. *If A and B are d-subisomorphic homococyclic abelian groups, then $A \cong B$.*

Proof. By the hypothesis there exists $k, l \in \mathbb{N} \cup \{\infty\}$ and cardinals κ, λ such that $A \cong \mathbb{Z}_{p^k}^{(\kappa)}$ and $B \cong \mathbb{Z}_{p^l}^{(\lambda)}$. Since A is d-subisomorphic to B we get that $k = l$ and $\kappa \leq \lambda$. The symmetric argument says that $\kappa = \lambda$. □

We are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2.

Assume that A and B are d-subisomorphic ADS abelian groups. Then they are either divisible, or a direct sum of an indecomposable torsion-free group and a divisible torsion group, or torsion groups such that each p-component is homococyclic by Theorem 1.1. We show that $A \cong B$ in all these cases.

If A is divisible, then B is divisible as well, and so A and B are isomorphic by [3, Theorem]. Suppose that $A = F \oplus D$ for a nonzero indecomposable torsion-free group F and a divisible torsion group D. Since A is d-subisomorphic to B, the group B is a proper mixed ADS group, hence it is of the same form $B = \tilde{F} \oplus \tilde{D}$ where $\tilde{F}$ is a nonzero indecomposable torsion-free and $\tilde{D}$ is a divisible torsion group. As $D = \bigoplus_{p \in \mathbb{P}} A_p$ and $\tilde{D} = \bigoplus_{p \in \mathbb{P}} B_p$, both subgroups D, $\tilde{D}$ are fully invariant and D and $\tilde{D}$ are d-subisomorphic by Lemma 2.2(1). Thus D and $\tilde{D}$ are isomorphic by the argument of the first part of the proof. Similarly, $F \cong A/D$ and $\tilde{F} \cong A/\tilde{D}$ are d-subisomorphic pairs of groups by Lemma 2.2(2). Hence F and $\tilde{F}$ are isomorphic because F contains no proper direct summand.

Finally, let $A = \bigoplus_{p \in \mathbb{P}} A_p$ and $B = \bigoplus_{p \in \mathbb{P}} B_p$ be sums of homococyclic p-components. Then A_p and B_p are d-subisomorphic by Lemma 2.2(1) and so are isomorphic by Lemma 2.3. This proves that A and B are isomorphic.

Recall that a ring R is called right *pure-semisimple* if every right R-module is a direct sum of finitely generated R-modules.

Example 2.4. *Let R be a Dedekind domain and I a nonzero ideal of R. Then R/I is a commutative Artinian principal ideal ring by* [2, Theorems 9.3 and 8.5, Exercise 9.7, p. 99]*, and so it is pure-semisimple by* [10, Theorem 4.3]*. By* [7, Theorem 4.2]*, every right R-module has the SB-property. But it is not ADS by* [13, Theorem 2.4].

3. Problem 1.5

Example 3.1. *$\mathbb{Z}$ as a $\mathbb{Z}$-module is ADS (since $\mathbb{Z}$ is indecomposable) which does not satisfy C2.*

Example 3.2. *Let p be a prime integer and let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$. Then M is not an ADS module. However, M satisfies C2.*

We list some basic properties of reduced C2 abelian groups.

Lemma 3.3. *Let A be a reduced C2 abelian group and $E = \mathrm{End}(A)$. Then*

(1) *for each $p \in \mathbb{P}$, there exist $n_p \in \mathbb{N}$, a cardinal κ_p and a central idempotent $e_p \in E$ such that $e_p(A) = A_p \cong \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$ and $(1 - e_p)(A)$ is p-divisible,*
(2) *$A/t(A)$ is torsion free divisible,*
(3) *the map $\varepsilon : E \to \prod_{p\in\mathbb{P}} e_pE$ given by $\varepsilon(r) = (e_pr)_{p\in\mathbb{P}}$ is a ring embedding and $\bigoplus_{p\in\mathbb{P}} e_pE$ is an ideal of the ring $\varepsilon(E)$,*
(4) *if $s, e \in E$ and $a \in A$ such that e is an idempotent, $se(a) = 0$ and $e(a) \neq 0$, then there exists $g \in E$ for which $seg = 0$ and $eg \neq 0$*

Proof. (1) By [5, Theorem 8], A_p is homococyclic and there exists a p-divisible subgroup, say D_p, of A such that $A = A_p \oplus D_p$. This implies the existence of an idempotent, say $e_p \in E$, with $e_p(A) = A_p$ and $(1-e_p)(A) = D_p$, which is central since $\mathrm{Hom}(A_p, D_p) = 0 = \mathrm{Hom}(D_p, A_p)$. Finally, as A is reduced and A_p is homococyclic, there exist $n_p \in \mathbb{N}$ and a cardinal κ_p for which$A_p \cong \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$.
(2) Clearly, $A/t(A) = A/\bigoplus_{p\in\mathbb{P}} A_p$ is torsion free and it is p-divisible for each $p \in \mathbb{P}$ by (1).
(3) It is easy to see that ε is a ring homomorphism, so it is enough to show that it is injective. Let $\varepsilon(f) = 0$. Then $f(A_p) = 0$ for each $p \in \mathbb{P}$, and hence $f(t(A)) = 0$. Note that $A/t(A)$ is divisible by (2). Now $f(A)$ is isomorphic to a factor divisible group. Therefore $f(A)$ is a divisible subgroup of a reduced group A which implies $f(A) = 0$.
(4) As there exists $p \in \mathbb{P}$ satisfying $e_p(e(a\mathbb{Z})) \neq 0$ by (3) and $e_p(A) \cong \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$ is a free module over the ring $\mathbb{Z}_{p^{n_p}}$ we may chose $g \in e_pE$ for which $g(A) = e_p(e(a\mathbb{Z}))$. Now $eg \neq 0$ since $eg(A) = ee_pe(a\mathbb{Z}) = e_pe(a\mathbb{Z}) \neq 0$. Similarly, $seg(A) = e_pse(a\mathbb{Z}) = 0$, and so $seg = 0$. □

We recall the well known fact that the central idempotents e_p of $\mathrm{End}(A)$ are uniquely determined by the p-component.

Proposition 3.4. *Let A and B be d-subisomorphic C2 abelian groups and $\varphi : B \to A$ be a monomorphism such that $\varphi(B)$ is a direct summand in A. If A_p is finite for every $p \in \mathbb{P}$ such that A_p is a non-divisible p-component, then φ is an isomorphism.*

Proof. First, note that A_p and B_p are homococyclic by [5, Theorem 8] and d-subisomorphic by Lemma 2.2(1) for each $p \in \mathbb{P}$.

Let E and F be maximal divisible subgroups of A and B, respectively. Then E and F are d-subisomorphic by Lemma 2.2(3), and hence $E \cong F$ by [3, Theorem]. Note that E and F are direct summands of A and B respectively and A/E and B/F are d-subisomorphic groups by Lemma 2.2(3) containing no nonzero divisible subgroup. Thus we may suppose without loss of the generality that A is reduced and A_p is finite for all $p \in \mathbb{P}$. Now it remains to show that $A \cong B$ for such A.

Let $D := \varphi(B)$ be a direct summand of A which is isomorphic to B. Then, $D_p \subseteq A_p$ is finite, $D_p = \varphi(B_p) \cong B_p$, and $A_p \cong B_p$ by Lemma 2.3, which shows that $A_p = D_p$ for each $p \in \mathbb{P}$. Since there exists a direct summand X of A satisfying $X \oplus D = A$ and $\bigoplus_{p\in\mathbb{P}} A_p \subseteq D$, we get that

$$A/\bigoplus_{p\in\mathbb{P}} A_p \cong X \oplus (D/\bigoplus_{p\in\mathbb{P}} A_p),$$

where the term on the right side is divisible by Lemma 3.3(2). Hence X is a divisible subgroup of A. As A is reduced, $X = 0$ and so $A = D = \varphi(B)$. □

As in the case of ADS groups, also any direct summand of C2 group is C2, which allows us to formulate the following consequence:

Corollary 3.5. *Every C2 abelian group which has every non-divisible p-component finite satisfies the Schröder-Bernstein property.*

Note that, in C2 abelian groups, we can replace the notion ″d-subisomorphic direct summand″ by ″subisomorphic direct summand″.

Proposition 3.6. *Let A and B be d-subisomorphic C2 abelian groups. If A is reduced and there are only finitely many primes p for which A_p is infinite, then A and B are isomorphic.*

Proof. Denote by $p_1, \dots, p_n$ all primes such that A_{p_i} is infinite. Since A_{p_i} and B_{p_i} are homococyclic by Lemma 3.3 and non-divisible by the hypothesis, there exist $k_1, \dots, k_n \in \mathbb{N}$ such that $p_i^{k_i} A_{p_i} = 0$ for each i. Furthermore A_{p_i} and B_{p_i} are d-subisomorphic by Lemma 2.2(1), and hence they are isomorphic by Lemma 2.3 which implies $p_i^{k_i} B_{p_i} = 0$.

Put $r := \prod_{i=1}^n p_i^{k_i}$. Then $A \cong rA \oplus \bigoplus_{i=1}^n A_{p_i}$ and $B \cong rB \oplus \bigoplus_{i=1}^n B_{p_i}$. By Lemma 2.2(2), rA and rB are d-subisomorphic groups with finite p-components for all $p \in \mathbb{P}$. Hence $rA \cong rB$ by Proposition 3.4. □

Corollary 3.7. *Every C2 abelian group which has only finitely many non-zero p-components satisfies the Schröder-Bernstein property.*

Proposition 3.8. *If A and B are d-subisomorphic C2 abelian groups such that there are only finitely many primes p for which each A_p is non-divisible infinite, then A and B are isomorphic.*

Proof. It is easy to say that $A = R_A \oplus D_A$ and $A = R_B \oplus D_B$ for a pair of reduced groups (R_A, R_B) and a pair of divisible groups (D_A, D_B) where the both pairs (R_A, R_B) and (D_A, D_B) are d-subisomorphic by Lemma 2.2(3),(4). Then $R_A \cong R_B$ by Proposition 3.6 and $D_A \cong D_B$ by [3, Theorem]. □

Corollary 3.9. *Every C2 abelian group containing only finitely many non-divisible infinite p-components satisfies the Schröder-Bernstein property.*

4. On more reduced abelian groups and the C2-condition

Recall that a ring R is said to be right C2 if the module R_R is C2. Let us formulate an elementary description of such a ring.

Lemma 4.1. *A ring R is right C2 if and only if the right ideal seR is generated by an idempotent for every $e, s \in R$ such that e is an idempotent and $r_R(se) = (1-e)R$.*

Proof. Note that a right ideal I is a direct summand in R_R if and only if $I = eR$ for an idempotent e.

If R_R is $C2$ and $r_R(se) = (1-e)R$ for an element s and an idempotent e, then the right multiplication by s induces a monomorphism $eR_R \to R_R$. Since the image seR is a direct summand, it is generated by an idempotent. For the converse, we assume that $\varphi : eR \to R$ is an embedding. Then there exists $s \in E$ such that $sr = \varphi(e(r))$. Since $r_R(se) = (1-e)E$, we get an idempotent generating the image seR by the hypothesis. □

Proposition 4.2. *The following conditions are equivalent for a reduced abelian group A and $E = \mathrm{End}(A)$:*

(1) *A is C2,*
(2) *E is right C2,*
(3) *For each $p \in \mathbb{P}$ there exist a central idempotent $e_p \in E$, $n_p \in \mathbb{N}$, and a cardinal κ_p such that $e_p(A) = A_p \cong \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$, the map $\varepsilon : E \to \prod_{p\in\mathbb{P}} e_p E$ given by $\varepsilon(r) = (e_p r)_{p\in\mathbb{P}}$ is a ring embedding, and for every $e, s \in E$ such that e is an idempotent and $r_{e_pE}(e_p se) = e_p(1-e)E$ for all $p \in \mathbb{P}$, there exist idempotents $f_p \in e_pE$ satisfying $f_pe_pE = e_pseE$ for $p \in \mathbb{P}$ such that $(f_p)_{p\in\mathbb{P}} \in E$.*

Proof. (1)⇒(3) The properties of A_p, $p \in \mathbb{P}$ and ε follows from Lemma 3.3(1) and (3). Note that $e(A)$ is a direct summand of the C2 group A and the restriction of the endomorphism $s \in \mathrm{End}(A)$ to $e(A)$ forms a homomorphism $e(A) \to A$. If $s(e(a)) = 0$ for $e(a) \neq 0$, then there exists $g \in E$ such that $eg \neq 0$ and $seg = 0$ by Lemma 3.3(4). This implies that $0 \neq eg \in r_E(se)$ which contradicts to the hypothesis (i.e. $r_E(se) = (1-e)E$). Therefore $se(A)$ is a monomorphic image of $e(A) = B$, which is a direct summand of A as A is C2. Thus there exists an idempotent $f \in E$ such that $f(A) = se(A)$ which implies that $fE = seE$. Now it remains to put $f_p = e_p f$ for each $p \in \mathbb{P}$.
(3)⇒(2) This follows immediately from Lemma 4.1 where the desired idempotent is of the form $(f_p)_{p\in\mathbb{P}}$.
(2)⇒(1) Let B be a direct summand of A and $\varphi : B \to A$ be an embedding. Then there exist $s \in E$ and an idempotent $e \in E$ satisfying $B = e(A)$ and $s(a) = \varphi(e(a))$. Clearly, $r_E(se) = (1-e)E$, which implies the existence of an idempotent $f \in E$ such that $fE = seE$ by Lemma 4.1. Now, $f(A) = se(A) = \varphi(B)$ is a direct summand of A, which proves that A is C2. □

Note that the equivalence of the first two conditions does not hold for general abelian groups.

Example 4.3. *Let $p \in \mathbb{P}$ and $A = \mathbb{Z}_{p^\infty}$. Note that A is divisible and so is C2. Then $\mathrm{End}(A) = \hat{\mathbb{Z}}_p$ is the ring of p-adic integers which is not C2 by Lemma 4.1 since $\hat{\mathbb{Z}}_p$ is a non-trivial local domain.*

We formulate two consequences of Proposition 4.2.

Corollary 4.4. *Let A be an abelian group and D be the maximal divisible subgroup of A. The following conditions are equivalent:*

(1) *A is C2,*
(2) *$\mathrm{End}(A/D)$ is right C2.*

Proof. (2)⇒(1) Since direct summand of C2 groups are C2 and so A/D is a reduced group which is isomorphic to the direct summand of A, the claim follows from Proposition 4.2.
(1)⇒(2) Let us remark that $A \cong t(D) \oplus D_f \oplus A/D$ where $t(D)$ is torsion divisible, D_f is torsion-free divisible and A/D is $t(D)$-automorphic. Hence $\oplus D_f \oplus A/D$ is $t(D)$-automorphic. Now it remains to apply [5, Lemma 11]. □

Corollary 4.5. *Suppose A is a reduced abelian group, $E = \mathrm{End}(A)$ and there exists a central idempotent $e_p \in E$ such that $A_p = e_p(A)$ is homococyclic for every $p \in \mathbb{P}$. If $\varepsilon : E \to \prod_{p\in\mathbb{P}} e_pE$, given by $\varepsilon(r) = (e_pr)_{p\in\mathbb{P}}$, is an isomorphism, then A is C2.*

Proof. It is enough to check the hypothesis of Proposition 4.2(3). Let $e, s \in E$,where e is an idempotent, and $r_{e_pE}(e_pse) = e_p(1-e)E$ for each $p \in \mathbb{P}$. Since e_pse induces a monomorphism $B = e_pe(A) \to e_p(A)$, where B is a projective $\mathbb{Z}_{p^{n_p}}$-module, we obtain $e_pse(B)$ is a projective module over the Frobenius ring $\mathbb{Z}_{p^{n_p}}$. Thus $e_pse(B)$ is injective, hence there exists an idempotent $f_p \in e_pE$ satisfying $f_p(e_p(A)) = e_pse(B)$ for each $p \in \mathbb{P}$. Since $\varepsilon(E) = \prod_{p\in\mathbb{P}} e_pE$, we get $(f_p)_{p\in\mathbb{P}} \in E$, and hence A is C2 by Proposition 4.2. □

Recall that e_p denotes the uniquely defined central idempotent such that $e_p(A) = A_p$. Furthermore, we will identify $E = End(A)$ with its image $\varepsilon(E)$ in the ring $\prod_{p\in\mathbb{P}} e_pE$.

Theorem 4.6. *Let A be a reduced abelian group and $E = \mathrm{End}(A)$. If, for every $p \in \mathbb{P}$, there exists a central idempotent $e_p \in E$ such that $A_p = e_p(A)$ is homococyclic and $E = \prod_{p\in\mathbb{P}} e_pE$, then A is a C2 group satisfying the Schröder-Bernstein property.*

Proof. By Corollary 4.5, the reduced abelian group A is C2. Since A_p satisfies the Schröder-Bernstein property by Lemma 2.3, we obtain that $e_pE \cong \mathrm{End}(A_p)$ satisfies it by [7, Theorem 2.4(a)] for each $p \in \mathbb{P}$. Therefore $E = \prod_{p\in\mathbb{P}} e_pE$ and hence A satisfies the Schröder-Bernstein property by [7, Theorem 2.4(d),(a)]. □

Recall that the class of abelian groups satisfying the Schröder-Bernstein property was not closed under the factor.

Proposition 4.7. *Let M be an abelian group and D be its maximal divisible subgroup. The following conditions are equivalent:*

(1) *M satisfies the Schröder-Bernstein property.*
(2) *M/D satisfies the Schröder-Bernstein property.*

Proof. (2)⇒(1) Assume A and B are d-subisomorphic direct summands of M. We denote by R_A and R_B reduced subgroups and D_A and D_B (maximal) divisible subgroups satisfying $A = R_A \oplus D_A$ and $B = R_B \oplus D_B$. Clearly, D_A and D_B are direct summands of D and $R_A \cap D = R_B \cap D = 0$, which implies that R_A and R_B are isomorphic to direct summands of M/D. Note that D_A and D_B are d-subisomorphic by Lemma 2.2(3) and R_A and R_B are d-subisomorphic by Lemma 2.2(4). Hence $D_A \cong D_B$ by [3, Theorem] and $R_A \cong R_B$ by the hypothesis.
(1)⇒(2) This implication follows from [7, Theorem 2.4(b)] since $M \cong D \oplus (M/D)$. □

Theorem 4.8. *Let A and D be abelian groups and $E = \mathrm{End}(A)$. If D is divisible and A is reduced C2 such that $E = \prod_{p\in\mathbb{P}} e_pE$, then $A \oplus D$ satisfies the Schröder-Bernstein property.*

Proof. By Theorem 4.6, A satisfies the Schröder–Bernstein property and hence the assertion follows from Proposition 4.7. □

Example 4.9. *Let* $A = \prod_{p\in\mathbb{P}} \mathbb{Z}_{p^{n_p}}^{(\kappa_p)}$ *for a system of natural numbers* n_p *and cardinals* κ_p *for each* $p \in \mathbb{P}$.
(1) A *is an abelian reduced group since* $\bigcap_{p\in\mathbb{P}} p^{n_p} A = 0$.
(2) By applying the idea of [15, Lemma 2.2 and Proposition 2.4]*, we can easily see that*

$$E = \mathrm{End}(A) = \prod_{p\in\mathbb{P}} e_p E \cong \prod_{p\in\mathbb{P}} \mathrm{End}(\mathbb{Z}_{p^{n_p}}^{(\kappa_p)})$$

where $e_q = (\delta_{pq})_{p\in\mathbb{P}}$ *for the Kronecker's* δ *and* $e_q E \cong \mathrm{End}(A_q)$, $q \in \mathbb{P}$. *Thus* A *is a C2 group satisfying the Schröder-Bernstein property by Theorem 4.6.*
(3) By Theorem 4.8, $A \oplus (\mathbb{Q}/\mathbb{Z})^{(\kappa)} \oplus \mathbb{Q}^{(\lambda)}$ *also satisfies the Schröder-Bernstein property for every cardinals* κ *and* λ.

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