ON MODULES AND RINGS WITH RESTRICTED MINIMUM CONDITION

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Abstract. A module M over a ring R satisfies the restricted minimum condition (RMC) if M/N is Artinian for every essential submodule N of M. A ring R satisfies right RMC if R_R satisfies RMC as a right module. It is proved that (1) a right semiartinian ring R satisfies right RMC if and only if $R/\mathrm{Soc}(R)$ is Artinian, (2) if a semilocal ring R satisfies right RMC and $\mathrm{Soc}(R) = 0$, then R is Noetherian if and only if the socle length of $E(R/J(R))$ is at most ω, and (3) a commutative ring R satisfies RMC if and only if $R/\mathrm{Soc}(R)$ is Noetherian and every singular module is semiartinian.

1. Introduction

Given a module over a ring R, we say that M satisfies *restricted minimum condition* (RMC) if for every essential submodule N of M, the factor module M/N is Artinian. It is easy to see that the class of all modules satisfying RMC is closed under submodules, factors and finite direct sums. A ring R satisfies *right RMC* (or we say it is an *RM-ring*) if R_R satisfies RMC as a right module. An integral domain R satisfying the restricted minimum condition is called an RM-domain, i.e. R/I is Artinian for all non-zero ideals I of R (see [4]). Recall that a Noetherian domain has Krull dimension 1 if and only if it is an RM-domain [5, Theorem 1]. The class of all RM-rings is closed under taking factors and finite products.

The purpose of the present paper is to continue in study of works [3],[4], [5], [7] and [10], in which basic structure theory of RM-rings and RM-domains was introduced, and the recent paper [1], which describes some properties of classes of torsion modules over RM-domains known and videly studied for corresponding classes of abelian groups. As the method of [1] appears to be fruitful, this paper is focused on study of structure of modules satisfying RMC, in particular singular ones. For a module M with the essential socle, we show that M satisfies RMC if and only if $M/\mathrm{Soc}(M)$ is Artinian. It is also proved, among other results, that if M is singular, then M is semiartinian for a module M over a right RM-ring R.

This tools allow us to obtain ring theoretical results for non-commutative as well as commutative rings. Namely, if a ring R satisfies right RMC and $\mathrm{Soc}(R) = 0$, we prove that R is a nonsingular ring of finite Goldie dimension. As a consequence, in Section 2, we obtain characterizations for various classes of rings which satisfies RMC via some well known and important rings (semiartinian, (Von Neumann) regular, semilocal, max, perfect) plus some (socle finiteness) conditions:

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In the case when R is a right semiartinian ring, we prove that R satisfies right RMC if and only if $R/\mathrm{Soc}(R)$ is Artinian.

In the case when R is a semilocal ring with right RMC and $\mathrm{Soc}(R) = 0$, we show that R is Noetherian if and only if $J(R)$ is finitely generated if and only if the socle length of $E(R/J(R))$ is at most ω.

If R is a right max ring satisfying right RMC, we prove that $R/\mathrm{Soc}(R)$ is right Noetherian.

Section 3 is focused on commutative rings R, it is shown that such a ring R satisfies RMC if and only if $R/\mathrm{Soc}(R)$ is Noetherian and every singular module is semiartinian.

Throughout this paper, rings are associative with unity and modules are unital right R-modules, where R denotes such a ring and M denotes such a module. We write $J(R), J(M), \mathrm{Soc}(R), \mathrm{Soc}(M)$ for the Jacobson radical of the ring R, for the radical of the module M, the socle of R and the socle of M, respectively. We also write $N \trianglelefteq M$ and $E(M)$ to indicate that N is an essential submodule of M and the injective hull of M, respectively.

2. The results for general rings

Before stating some consequences, recall the following useful folklore observation (see [8, Lemma 3.6]).

Lemma 2.1. *Let a module M satisfy RMC.*

1. *Then Goldie dimension of $M/\mathrm{Soc}(M)$ is finite.*
2. *If K and N be submodules of M such that $K \trianglelefteq N$, then N/K is Artinian.*

Proof. (1) Put $S_0 := \mathrm{Soc}(M)$ and fix a submodule S_1 of the module M such that $S_0 \subseteq S_1$ and $S_1/S_0 = \mathrm{Soc}(M/S_0)$. By Zorn's Lemma, we may choose a maximal set of elements $m_i \in M$ satisfying the condition $S_1 \cap \bigoplus_{i\in I} m_i R = 0$. It is easy to see that $S_1 \oplus \bigoplus_{i\in I} m_i R \trianglelefteq M$. Since $\bigoplus_{i\in I} m_i R \cap S_0 = 0$, every module $m_i R$ has zero socle. Hence $m_i R$ is not simple and any maximal submodule of $m_i R$ is essential in $m_i R$. For every $i \in I$, let N_i be a fix maximal submodule in $m_i R$. As $\bigoplus_{i\in I} N_i \trianglelefteq \bigoplus_{i\in I} m_i R$, the module $L = S_0 \oplus \bigoplus_{i\in I} N_i$ is essential in M. Applying RMC, we get that M/L is an Artinian module containing an isomorphic copy of $(S_1/S_0) \oplus \bigoplus_{i\in I} m_i R/N_i$ which implies that I is finite and S_1/S_0 is a finitely generated semisimple module. By [9, Proposition 6.5], we obtain that the uniform dimension of $M/\mathrm{Soc}(M)$ is finite.

(2) If we chose a submodules A for which $N \cap A = 0$ and $N \oplus A \trianglelefteq M$, then $K \oplus A \trianglelefteq M$. Hence $M/(K \oplus A)$ and $(N \oplus B)/(K \oplus B) \cong N/K$ are Artinian modules. □

Proposition 2.2. *Let M be a module with the essential socle. Then M satisfies RMC if and only if $M/\mathrm{Soc}(M)$ is Artinian.*

Proof. If M satisfies RMC, then $M/\mathrm{Soc}(M)$ is right Artinian by the definition. Since every essential submodule N of M contains $\mathrm{Soc}(M)$, M/N is Artinian whenever $M/\mathrm{Soc}(M)$ is Artinian, the converse is clear. □

A module M is said to be *semiartinian* if every non-zero factor of M has a non-zero socle and a ring R is called *right semiartinian* if every non-zero right R-module has a non-zero socle.

Given a semiartinian module M, the socle chain of M is the increasing chain of submodules $(S_\alpha \mid \alpha \geq 0)$ defined by the following way: put $S_0 = 0$ and, recursively, $S_{\alpha+1}/S_\alpha = \mathrm{Soc}(M/S_\alpha)$ for each ordinal α and $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ if α is a limit ordinal. The first ordinal σ such that $S_\sigma = M$ is *the socle length* of M.

Corollary 2.3. *Let R be a right semiartinian ring. Then R satisfies right RMC if and only if $R/\mathrm{Soc}(R)$ is Artinian.*

The following example shows that the class of modules satisfying RMC is not closed under extensions.

Example 2.4. Let R be a right semiartinian ring of socle length 3 and such that $R/\mathrm{Soc}(R)$ is non-Artinian, hence R is not an RM-ring by Corollary 2.3. Then $R_0 = R/\mathrm{Soc}(R)$ is a right RM-ring by Corollary 2.3 because $R_0/\mathrm{Soc}(R_0)$ is semisimple. Clearly $\mathrm{Soc}(R)$ satisfies RMC as well. Hence the short exact sequence

$$0 \to \mathrm{Soc}(R) \to R \to R/\mathrm{Soc}(R) \to 0$$

shows that the class of all modules satisfying RMC is not closed under extensions.

Remark 2.5. In particular, we can fix a field F and take R_1 as an F-subalgebra of a natural F-algebra F^ω generated by $F^{(\omega)}$. Then R_2 is defined as an F-subalgebra of a natural F-algebra R_1^ω generated by $R_1^{(\omega)}$. It is easy to see that R_2 is a right semiartinian ring of socle length 3 and $R_2/\mathrm{Soc}(R_2)$ is non-Artinian.

Lemma 2.6. *If an R-module M satisfies RMC, then Krull dimension of $M/\mathrm{Soc}(M)$ is at most one.*

Proof. Note that a module has zero Krull dimension if it is an Artinian module. Suppose that $N_0 \supseteq N_1 \supseteq \dots$ is a sequence of submodules of $M/\mathrm{Soc}(M)$. As $M/\mathrm{Soc}(M)$ has a finite Goldie dimension by Lemma 2.1(1), there exists n such that for each $i \geq n$ either $N_i = 0$ or $N_{i+1} \trianglelefteq N_i$. Since N_{i+1}/N_i is Artinian by Lemma 2.1(2), we obtain that $M/\mathrm{Soc}(M)$ has a Krull dimension at most 1. □

We recall another well-known observation which will allows us to prove the following theorem.

Lemma 2.7. *Let M be an Artinian R-module. If $J(N) \neq N$ for every nonzero submodule N of M, then M is Noetherian.*

Proof. Assume that M is not Noetherian. Then it contains a semiartinian submodule of infinite socle length. As M is Artinian, there is a minimal submodule N of infinite socle length. Thus N contains no maximal submodule, i.e. $J(N) = N$. □

Theorem 2.8. *Let R be a right RM-ring, $S(R)$ the greatest right semiartinian ideal of R and put $A := R/\mathrm{Soc}(R)$ and $S(A) := S(R)/\mathrm{Soc}(R)$.*

(1) *$\bigcap_{n<\omega} J(A)^n$ is a nilpotent,*
(2) *$S(A) \cap J(A)$ is a nilpotent,*
(3) *$S(A)/(S(A) \cap J(A))$ is Noetherian.*

Proof. First note that $S(A)$ is the greatest right semiartinian ideal of $S(R)/\mathrm{Soc}(R)$.

(1) Since Krull dimension of A is equal to 0 or 1 by Lemma 2.6, we may directly apply [6, Theorem 7.26], which proved that $\bigcap_n J(A)^n$ is a nilpotent.

(2) Put $K := S(A) \cap (\bigcap_n J(A)^n)$ and $I := S(A) \cap J(A)$. Note that K is a nilpotent by (1). Since $S(A)$ is Artinian by Lemma 2.1, we obtain that I is

Artinian. Moreover, $I^n \subseteq J(A)^n$, and so $\bigcap_n I^n \subseteq K$. Since I Artinian, there exists n for which $I^n \subseteq K$, which finishes the proof.

(3) Note that $S(A)$ and so $M = S(A)/(S(A) \cap J(A))$ is Artinian and $J(M) = 0$. Hence $J(N) = 0$ for each submodule N of M. By applying Lemma 2.7 we get the conclusion. □

Recall that the *singular submodule* $Z(M)$ of a module M is defined by

$$Z(M) = \{m \in M : mI = 0 \textit{ for some essential right ideal } I \textit{ of } R\}.$$

The module M is called *singular* if $M = Z(M)$, and *nonsingular* if $Z(M) = 0$. We describe structure of singular modules over RM-rings.

Lemma 2.9. *Let R be a right RM-ring. Then $Z(M)$ is semiartinian for each right R-module M.*

Proof. Let $m \in Z(M)$. Then $r(m) \trianglelefteq R$, hence $mR \cong R/r(m)$ is Artinian and so semiartinian. □

Theorem 2.10. *Let R be a right RM-ring and M be a right R-module.*

1. *If M is singular, then M is semiartinian.*
2. *$E(M)/M$ is semiartinian.*
3. *If M is semiartinian, then $E(M)$ is semiartinian. In particular, $E(S)$ is semiartinian for every simple module S.*

Proof. Assume that M is singular. By Lemma 2.9, $Z(M) = M$ is semiartinian, hence (1) holds. Since $E(M)/M$ is a singular module by [9, Examples 7.6(3)] and the class of semiartinian modules is closed under taking essential extensions, (2) and (3) hold. □

Theorem 2.11. *If a ring R satisfies right RMC and $\mathrm{Soc}(R) = 0$, then R is a nonsingular ring of finite Goldie dimension.*

Proof. As R contains no simple submodule, we obtain $Z(R) = 0$ by Lemma 2.9, and the rest follows from Lemma 2.1. □

Recall that a ring R is *(Von Neumann) regular* if for every $x \in R$ there exists $y \in R$ such that $x = xyx$. The structure of regular RM-rings appears to be very lucid:

Theorem 2.12. *The following conditions are equivalent for a regular ring R.*

1. *R is an RM-ring,*
2. *$R/\mathrm{Soc}(R)$ is Artinian,*
3. *R is semiartinian of the socle length 2.*

Proof. (1) ⇒ (2) By Lemma 2.1(1), $R/\mathrm{Soc}(R)$ is of finite Goldie dimension. Since $R/\mathrm{Soc}(R)$ is a regular ring which cannot contain an infinite set of orthogonal set idempotents, we obtain that $R/\mathrm{Soc}(R)$ is Artinian.

(2) ⇒ (3) It is obvious because an Artinian regular ring is semisimple.

(3) ⇒ (1) It follows from Corollary 2.3. □

Recall that a ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian.

Lemma 2.13. *If R is a semilocal ring, then $J(R) + \mathrm{Soc}(R) \trianglelefteq R$.*

Proof. Let $J(R) + \mathrm{Soc}(R)$ is not essential in R. Then there exists a nonzero right ideal $I \subseteq R$ such that $I \cap (J(R) + \mathrm{Soc}(R)) = 0$. Since $\mathrm{Soc}(I) = \mathrm{Soc}(R) \cap I = 0$ and $R/J(R)$ contains an ideal which is isomorphic to I, we obtain that $\mathrm{Soc}(R/J(R)) \neq R/J(R)$. Hence R is not semilocal, a contradiction. □

The following example shows that the converse of Lemma 2.13 is not true.

Example 2.14. Suppose that R is a local commutative domain with the maximal ideal J. It is easy to see that J^ω is the Jacobson radical of the ring R^ω and it is essential in R^ω, however R^ω is not semilocal.

Recall that $J(R/J(R)) = \{0 + J(R)\}$ for an arbitrary ring R.

Proposition 2.15. *Assume that a ring R satisfies right RMC.*

1. *If $\mathrm{Soc}(R) = 0$, then $J(R) \trianglelefteq R$ if and only if R is semilocal.*
2. *If R is a semilocal ring, then $J(R)/\mathrm{Soc}(J(R))$ is finitely generated as a two-sided ideal.*

Proof. (1) Since $J(R) \trianglelefteq R$ and R satisfies right RMC, we obtain that $R/J(R)$ is an Artinian ring. On the other hand, $J(R/J(R)) = \{0 + J(R)\}$ implies that $R/J(R)$ is semisimple, and hence R is semilocal. The converse follows from Lemma 2.13.

(2) We note that there exists a finitely generated right ideal $F \subseteq J(R)$ such that $F + (\mathrm{Soc}(R) \cap J(R)) \trianglelefteq J(R)$ since $J(R)/(\mathrm{Soc}(R) \cap J(R))$ has a finite Goldie dimension by Lemma 2.1. Thus $RF + \mathrm{Soc}(R)$ is a two-sided ideal which is essential in R as a right ideal by Lemma 2.13. By the hypothesis, $R/(RF + \mathrm{Soc}(R))$ is a right Artinan ring. As $J(R)+\mathrm{Soc}(R)/(RF+\mathrm{Soc}(R))$ is finitely generated as a right ideal and

$$\begin{aligned}(J(R) + \mathrm{Soc}(R))/(RF + \mathrm{Soc}(R)) &\cong J(R)/(J(R) \cap (RF + \mathrm{Soc}(R)))\\ &= J(R)/(RF + (J(R) \cap \mathrm{Soc}(R)))\\ &= J(R)/(RF + \mathrm{Soc}(J(R))),\end{aligned}$$

the ideal $J(R)/\mathrm{Soc}(J(R))$ is finitely generated. □

Note that every Artinian module is semiartinian.

Lemma 2.16. *Let M be an Artinian R-module. Then the following are equivalent:*

1. *Socle length of M is greater then ω,*
2. *M contains a cyclic submodule with infinitely generated Jacobson radical,*
3. *M contains a cyclic submodule which is not noetherian.*

Proof. (1)⇒(2) Let M be an Artinian module of nonlimit infinite socle length and fix $x \in M$ such that xR has the socle length $\omega + 1$. Denote the α-th member of the socle sequence of xR by S_α. Since xR is Artinian, we obtain that $J(xR)$ is the intersection of finitely many maximal submodules, which implies that $xR/J(xR)$ is semisimple. As xR/S_ω is semisimple as well, we have $J(xR) \subseteq S_\omega$. Hence the socle length of $J(xR)$ is at most ω. Assume that $J(xR)$ is finitely generated. Then the socle length of $J(xR)$ is non-limit, hence finite. This implies that xR has a finite socle length, a contradiction, i.e. $J(xR)$ is infinitely generated.

(2)⇒(3) Clear.

(3)⇒(1) As a cyclic non-Noetherian Artinian module is of infinte non-limit socle length it have to be greater than ω. □

The next result characterizes semilocal rings satisfying right RMC further:

Theorem 2.17. *Assume a semilocal ring R satisfies right RMC and* $\mathrm{Soc}(R) = 0$. *Then the following conditions are equivalent:*

(1) *R is right Noetherian,*
(2) *$J(R)$ is finitely generated as a right ideal,*
(3) *the socle length of $E(R/J(R))$ is at most ω.*

Proof. (1)⇒(2) Obvious.

(2)⇒(3) First, note that every cyclic submodule of $E(R/J(R))$ is Artinian by Theorem 2.10. Suppose that the socle length of $E(R/J(R))$ is greater than ω. Hence it contains an Artinian submodule of the socle length greater than ω. By Lemma 2.16, there exists a cyclic module xR with infinitely generated Jacobson radical. Fix right ideals I_1 and I_2 such that $xR \cong R/I_1$, $I_1 \subseteq I_2$ and $I_2/I_1 = J(R/I_1)$. It is easy to see that I_2 is infinitely generated and $J(R) \subseteq I_2$. Since $I_2/J(R)$ is a right ideal of the semisimple ring $R/J(R)$, we obtain that $I_2/J(R)$ is finitely generated, and hence $J(R)$ is an infinitely generated right ideal.

(3)⇒(1) Let I be a right ideal. We show that I is finitely generated. By Lemma 2.1, there exists finitely generated right ideals F and G such that $F \trianglelefteq I$, $I \cap G = 0$ and $F + G \trianglelefteq R$. First we note that $R/(F + G)$ is an Artinian module with a submodule isomorphic to I/F and it is also easy to see that $R/(F+G)$ is isomorphic to a submodule of $\bigoplus_{i\leq n} E(S_i)$ for some simple modules $S_1, \dots S_n$. Since each $E(S_i)$ is isomorphic to some submodule of $E(R/J(R))$, we obtain that the socle length of $\bigoplus_{i\leq n} E(S_i)$ and so of $R/(F+G)$ is at most ω. As $R/(F+G)$ is a cyclic module, it is an Artinian module of finite socle length, which implies that $R/(F + G)$ is also a Noetherian module. Therefore I/F and so I are finitely generated modules. □

Recall that a ring R is called *right max* if every non-zero right module has a maximal proper submodule.

Theorem 2.18. *Let R be a right max ring satisfying right RMC. Then $R/\mathrm{Soc}(R)$ is right Noetherian.*

Proof. Let I be a right ideal of $R/\mathrm{Soc}(R)$. It is enough to show that I is finitely generated. If we apply Lemma 2.1 to I, we see that there exists a finitely generated right ideal F such that $F \trianglelefteq I$ and I/F is Artinian. Since R is a right max ring, every nonzero submodule of I/F contains a maximal submodule, and so I/F is Noetherian. By Lemma 2.7, it is finitely generated. Thus I is finitely generated as well. □

As right perfect rings are right max, we get

Corollary 2.19. *If R is a right perfect right RM-ring, then $R/\mathrm{Soc}(R)$ is right Noetherian.*

The following example shows that a perfect ring which has RMC needs not be a (right) Noetherian ring.

Example 2.20. Let F be a commutative field and V be a vector space over F. Consider the trivial extension $R = F \times V$. Then R is a local ring, hence it is perfect. The proper ideals of R are the $0 \times W$, where W is an F-subspace of V. Hence the only essential ideals of R are R and the maximal ideal $0 \times V$. Then R satisfies right RMC. We note that if V is infinite dimensional, then R is not Noetherian.

Moreover, since every left perfect ring is right Artinian, we can formulate the following consequence of Corollary 2.3.

Corollary 2.21. *If R is a left perfect right RM-ring, then $R/\mathrm{Soc}(R)$ is right Artinian.*

3. The results for commutative rings

The aim of this section is to study commutative rings and commutative domains that satisfy right RMC.

First, we recall the terminology that we need. Let P be a maximal ideal of a domain R. For every R-module M, the symbol $M_{[P]}$ denotes the sum of all finite length submodules U of M such that all composition factors of U are isomorphic to R/P. A module M is *self-small*, if the functor $\mathrm{Hom}(M,-)$ commutes with all direct powers of M. Recall that M is not self-small if and only if there exists a chain $M_1 \subseteq M_2 \subseteq \ldots \subseteq M$ of submodules such that $\bigcup_n M_n = M$ and $\mathrm{Hom}(M/M_n, M) \neq 0$ for each n. Denote by $\mathrm{Max}(M)$ a set of all maximal submodules of M.

We formulate in one criterion several results of [1] which will be generalized in the following for commutative RM-rings.

Theorem 3.1. *The following conditions are equivalent for a commutative domain R:*

1. *R satisfies RMC,*
2. *$M = \oplus_{P\in \mathrm{Max}(R)} M_{[P]}$ for all torsion modules M,*
3. *R is Noetherian and every non-zero (cyclic) torsion R-module has an essential socle,*
4. *R is Noetherian and every self-small torsion module is finitely generated.*

Proof. (1) $\Leftrightarrow$ (2) [1, Theorem 6].
(1) $\Leftrightarrow$ (3) [1, Lemma 3(2)].
(1) $\Leftrightarrow$ (4) [1, Lemma 3(2) and Theorem 9]. □

First, make an easy observation.

Lemma 3.2. *Every cyclic Artinian module is Noetherian over each commutative ring.*

The following example shows that the assumption of commutativity in Lemma 3.2 is not superfluous.

Example 3.3. Let F be a field and $I = N \cup \{\omega\}$ be a countable set (I consists of all natural numbers plus a further index ω). The ring R is the ring of non-commutative polynomials with coefficients in F and in the non-commutative indeterminates x_i, $i \in I$. The cyclic module will be a vector space V over F of countable dimension, with basis v_i, $i \in I$, over the field F.

We must say how R acts on V. For every $n \in N$, set $x_n v_i = v_n$ if $i \geq n$ and $i \in N$, $x_n v_i = 0$ if $i < n$ and $i \in N$, $x_n v_\omega = v_n$. Moreover, set $x_\omega v_i = 0$ for every $i \in N$, and $x_\omega v_\omega = v_\omega$. Thus we obtain a left R-module ${}_RV$. Now ${}_RV$ is cyclic generated by v_ω (because $x_n v_\omega = v_n$).

The R-submodules of ${}_RV$ are (each one is contained in the following one):

$$Rv_0 \subset Rv_1 \subset Rv_2 \subset \cdots \subset \bigcup_{i\in\mathbb{N}} Rv_i \subset Rv_\omega = V.$$

Thus the lattice of R-submodules of ${}_RV$ is isomorphic to $N \cup \{\omega\}$, that is, is order-isomorphic to the cardinal $\omega+1$. Thus the cyclic R-module ${}_RR$ is Artinian but not Noetherian.

The following observation generalizes [1, Lemma 3(2)].

Theorem 3.4. *Let R be a commutative ring. Then R satisfies RMC if and only if $R/\mathrm{Soc}(R)$ is Noetherian and every singular module is semiartinian.*

Proof. (:⇒) Let R be an RM-ring. If A is the greatest semiartinian ideal in R, then R/A has zero socle and $\mathrm{Soc}(R) \trianglelefteq A$. By Lemma 2.1, $A/\mathrm{Soc}(R)$ is Artinian, and so Noetherian by Lemma 3.2. It remains to show that R/A is Noetherian. Without loss of generality, we may suppose that $\mathrm{Soc}(R) = 0$. Let I be an ideal of R. We must show that it is finitely generated. Repeating the argument of the implication (3) ⇒ (1) of the proof of Theorem 2.17, we can find finitely generated ideals F and G such that $F \trianglelefteq I$, $I \cap G = 0$ and $F + G \trianglelefteq R$. Hence $R/(F+G)$ is Artinian and it has a submodule which is isomorphic to I/F. Since $R/(F+G)$ is Noetherian by Lemma 3.2, we have I/F as well as I are finitely generated. The rest follows from Lemma 2.9.

(⇐:) Let $R/\mathrm{Soc}(R)$ be Noetherian and every singular module be semiartinian. Fix an ideal $I \trianglelefteq R$. Then R/I is singular and so semiartinan by Lemma 2.9. Moreover, R/I is Noetherian and semiartinan, hence it is Artinian which finishes the proof. □

In light of Theorem 3.4, we ask the following.

Question 3.5. *Is $R/\mathrm{Soc}(R)$ Noetherian for each non-commutative RM-ring R?*

Lemma 3.6. *R be a commutative RM-ring and M a singular module. Then $M = \oplus_{P\in\ Max(R)} M_{[P]}$.*

Proof. Assume that $M \neq \oplus_{P\in \mathrm{Max}(R)} M_{[P]}$ and fix $m \in M \setminus \oplus_{P\in\mathrm{Max}(R)} M_{[P]}$. Since M is singular, mR is Artinian and $mR \cong R/r(m) \cong \prod_{r(m)\subseteq I} A_I$ where A_I are local commutative Artinian rings with maximal ideal I. Since $A_I \subseteq M_{[I]}$ and there are only finitely many $I \in \mathrm{Max}(R)$, we get a contradiction. □

Theorem 3.7. *The following conditions are equivalent for a commutative ring R:*

1. *R satisfies RMC,*
2. *$M = \oplus_{P\in Max(R)} M_{[P]}$ for all singular modules M,*
3. *$R/\mathrm{Soc}(R)$ is Noetherian and every self-small singular module is finitely generated.*

Proof. (1) ⇒ (2) It follows by Lemma 3.6.

(2) ⇒ (1) The proof is the same as the proof of [1, Theorem 6]. Let I be an essential ideal of R. Then R/I is a cyclic singular module, hence $R/I \cong \oplus_{P\in Max(R)} A_{[P]}$ where all $A_{[P]}$ are cyclic and only finitely many $A_{[P]}$ are non-zero. Since every cyclic module $A_{[P]}$ is a submodule of a sum of finite-length modules, it is Artinian. Thus R/I is Artinian and R satisfies RMC.

(1) ⇒ (3) By Theorem 3.4 and Lemma 2.16, we have $R/\mathrm{Soc}(R)$ is Noetherian and every singular module is semiartinian of socle length less or equal than ω. Let M be a self-small singular module. Then $M = \oplus_{P\in Max(R)} M_{[P]}$ by Lemma 3.6, hence $M_{[P]} \neq 0$ for only finitely many $[P]$. Since $\mathrm{Hom}(M_{[P]}, M_{[Q]}) = 0$ for all $P \neq Q$, we may suppose that $M = M_{[P]}$ for a single maximal ideal P by [11,

Proposition 1.6]. Denote by M_i the i-th member of the socle sequence of M. It is easy to see that $M_i = \{m \in M \mid mP^i = 0\}$. Assume that socle length of M is infinite, i.e. $M_i \neq M_{i+1}$ and $M = \bigcup_{i<\omega} M_i$. Then for each $i < \omega$, there exist $m_i \in M_{i+1} \setminus M_i$ and $p_i \in P^i$ such that $0 \neq m_i p_i \in \mathrm{Soc}(M)$. Then multiplication by p_i forms a nonzero endomorphism on M for which $M_i \subseteq \ker p_i$, a contradiction with the fact that M is self-small. We have proved there exists n such that $M_n = M$ and so M has a natural structure of a self-small module over commutative Artinian ring R/P^n. Hence M is finitely generated by [2, Proposition 2.9].

(3)⇒(1) The argument is similar as in the proof of [1, Theorem 9]. If I is an essential ideal of R, then $\mathrm{Soc}(R) \subseteq I$, hence R/I is Noetherian. Moreover, every self-small module over R/I is singular as an R-module, and so it is finitely generated. Now, the conclusion follows immediately from [2, Proposition 3.17]. □

Remark 3.8. Note that Theorem 3.1 is a direct consequence of Theorems 3.4 and 3.7 since singular modules over commutative domains are exactly torsion modules.

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