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# LARGE SMALL MODULES OVER VON NEUMANN REGULAR RINGS

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Abstract. The paper summarizes and partially deepens known results about von Neumann regular rings over which there exist infinitely generated small modules i.e. such modules $M$ that the functor $\mathrm{Hom}(M, -)$ commutes with direct sums. As consequences of listed facts we prove a criterion of existence of infinitely generated small modules over continuous regular rings and sufficient conditions over general simple and right self injective rings.

Compactness conditions for module categories are studied from 60's in the context of representable equivalence of module categories; motivation for this research is connected with searching of generalizations of the Morita theorem. Among modules satisfying some version of compactness condition the notion of a small module is defined by very straightforward and natural way: a module $M$ is *small* provided the covariant functor $\mathrm{Hom}(M, -)$ commutes with direct sums of all modules. Hyman Bass in [3, p.54] made an observation that there are examples of infinitely generated small modules however they are not easy to find. Basic properties of small modules are proved in the pioneer work [11] and lattice theoretical approach is introduced in the papers [8, 9]. Recall that a module $M$ is small if and only if it is not a union of a countable infinite strictly increasing chain of its submodules. This notion is studied under various terms, all are inspired by one of the equivalent definition conditions: module of type $\Sigma$ [11], dually slender [6, 19, 12, 16, 18], $\Sigma$-compact, U-compact [2], and small module [14, 15, 5].

Obviously, the class of all small modules contains all finitely generated modules, however many examples of infinitely generated small modules are known. It leads to definition of *right steady* rings over which every right small module is finitely generated. Several interesting classes of rings are known to be right steady (right noetherian, right perfect, right semiartinian with countable socle length, polynomials in countably many variables over a field), and on the other hand, it is proved about large classes of rings that they are not right steady (such as infinite products of rings, endomorphism rings over an infinitely generated vector space, polynomials in uncountably many variables). However a module-theoretic criterion of steadiness is known [16], general ring-theoretic characterization remains to be an open question.

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Recall that a ring $R$ is called *Von Neumann regular* (*abelian regular*) if for every $x \in R$ there exists $y \in R$ such that $x = xyx$ ($x = x^2y$). Abelian regular rings can be characterized as precisely those regular rings whose all idempotents are central [7, Theorem 3.5]. Since the border between steady and non-steady rings divides the class of all regular rings (even commutative semiartinian regular, cf. [6]) into two large and relevant classes, the natural question arises, how infinitely generated small modules (and their representative classes) over von Neumann regular rings can look like. Besides known (and partially generalized) results from the papers [5, 6, 11, 12, 14, 15, 16, 18, 19] we present necessary conditions of steadiness for general simple rings (Proposition 4.3) and general right self-injective rings (Proposition 4.5) and two criterion for particular classes of rings. Corollary 4.6 contains an easy module-theoretic criterion of steadiness of right self injective rings and Theorem 4.7 characterizes right steady right continuous regular rings.

Throughout the paper a *ring* $R$ means an associative ring with unit, and a *module* means a right $R$-module. We will say regular instead von Neumann regular. We denote by $J(R)$ Jacobson radical and by $Soc(R)$ the right socle of any ring $R$. $E(M)$ means an injective envelope of $M$. $R$ is said to be semisimple if $R = Soc(R)$. A *primitive factor* of a ring is a factor modulo annihilator of a simple module. Obviously, every abelian regular ring has all primitive factors simple.

For further notation, we refer to [7] and [13].

## 1. Basics

Before we focus on classes of small modules over regular rings, we recall several general properties that we need (and use without explicit quotations) in the following sections. First, we list equivalent conditions of smallness:

**Lemma 1.1.** [14, Lemma 1.2][5, Lemma 1.1], [11, 1°] *Let $M$ be a module. Then the following conditions are equivalent:*

1. *$M$ is small;*
2. *the functor* $\mathrm{Hom}(M, -)$ *commutes with countable direct sums;*
3. *the functor* $\mathrm{Hom}(M, -)$ *commutes with (countable) direct sums of injective modules;*
4. *if $M = \bigcup_{i<\omega} M_i$ for an increasing chain of submodules $M_i \subseteq M_{i+1} \subseteq M$, $i < \omega$, there exists $n$ such that $M = M_n$;*
5. *if $M = \sum_{i<\omega} M_i$ for a system of submodules $M_i \subseteq M$, $i < \omega$, there exists $n$ such that $M = \sum_{i<n} M_i$.*

As we need finer tools for working with small modules, we define for every uncountable cardinal $\lambda$ a stronger notion; we say that a module $M$ is *$\lambda$-reducing*, provided every less-then-$\lambda$-generated submodule $N \subseteq M$ is contained in a suitable finitely generated submodule of $M$. If $\kappa \geq \lambda$, it is easy to see that $M$ is $\lambda$-reducing and small whenever it is a $\kappa$-reducing module. On the other hand, for a field $F$ every non-principal maximal ideal of the ring $F^\omega$ is an example of a small module which is not $\omega_1$-reducing by [15, Example 2.6].

The following observation is essential from the point of view of our approach:

**Proposition 1.2.** [17, Proposition 1.3] *Let $\lambda$ be an uncountable cardinal. The both classes of all $\lambda$-reducing and of all small modules are closed under taking homomorphic images, extensions and finite sums.*

Among weaker notions then smallness we recall the notion of a self-small module only, which is defined as a module $M$ such that the covariant functor $\mathrm{Hom}(M, -)$ commutes with direct powers of $M$.

Note that finitely generated modules satisfies all the conditions of compactness defined above and recall that a ring $R$ is *right (strongly) steady* if every right (self-small) small module is finitely generated. We obtain as an easy consequence of [4, Proposition 3.7] the following criterion of strong steadiness for regular rings:

**Proposition 1.3.** *Let $R$ be a regular ring. Then $R$ is right (left) strongly steady iff it is semisimple.*

*Proof.* As the maximal ring of quotients of every strongly steady regular ring $R$ is semisimple by [4, Proposition 3.7], $R$ contains no infinite orthogonal set of idempotents, hence $R$ is semisimple as well. The converse is trivial. □

Before we illustrate that the question of steadiness of regular rings is very far from being clear, we state some closure conditions of the class of general steady rings, which plays an important role for our approach.

**Proposition 1.4.** [5, Lemma 1.9], [15, Theorem 2.5], [6, Lemma 1.7] *The class of all right steady rings is closed under factorization, finite products and Morita equivalence.*

The following example shows that no infinite product of rings is right steady.

**Example 1.5.** *Let $(R_i|\ i < \kappa)$ be an infinite system of non-zero rings. Then the product $R = \prod_{i<\kappa} R_i$ is not right steady* [15, Theorem 2.5]. *Let $\kappa = \lambda^+ > \omega_1$ and for every $i < \kappa$ define an idempotent $e_i \in R$ by the conditions $e_i(j) = 1$ if $j \leq i$ and $e_i(j) = 0$ elsewhere. Then the ideal $\bigcup_{i<\kappa} e_i R$ is $\lambda$-reducing but it is not $\kappa$-reducing.*

The idea of [15, Lemma 2.2, Theorem 2.5] can be easily generalized in the following lemma, which we will need in the last section of this paper.

**Lemma 1.6.** *Let $R$ be a ring, $E$ the set of all central idempotents of $R$ and $X$ an infinite set. If there exists an injective homomorphism of the lattice $(P(X), \cup, \cap)$ to the lattice $(E, \cdot, \vee)$ where $e \vee f = e + f - ef$, then $R$ is neither right nor left steady.*

*Proof.* Let $f : P(X) \to E$ be an injective lattice homomorphism. Denote by $\mathcal{F}$ the set of all filters on $X$ and by $\mathcal{I}$ the set of all right ideals of $R$. Similarly as in [15, Lemma 2.2] define a mapping $\varphi : \mathcal{F} \to \mathcal{I}$ by the rule $\varphi(F) = \sum_{Y\in F} f(Y)R$. Suppose that $\varphi(F) = \bigcup_{i<\omega} I_n$ for an increasing chain of right ideals $I_n$ and put $F_n = \{Y \subset X | \ f(Y) \in I_n\}$ for each $n$. We show that $F_n$ is a filter. If $Y \in F_n$ and $Y \subset Z$, then $f(Z) = f(Y \cup Z) = f(Y) \cdot f(Z) \in I_n$. If $Y_1, Y_2 \in F_n$, we see $f(Y_1 \cap Y_2) = f(Y_1) + f(Y_2) - f(Y_1)f(Y_2) \in I_n$. Hence $Z, Y_1 \cap Y_2 \in F_n$. Obviously $F = \bigcup_{n<\omega} F_n$, thus $\varphi(F)$ is small if $F$ is not a union of a strictly increasing chain of right ideals. Moreover, it is easy to see that $\varphi(F)$ is a finitely generated (right) ideal iff $F$ is a principal filter. Now, using the same argument as in [15, Lemma 2.4] we find an infinitely generated small right (and left) ideal as $\varphi(U)$ for a non-principal ultrafilter $U$ on $X$. □

## 2. Regular rings with primitive factors artinian

We start this section with a general module-theoretic criterion of steadiness.

**Theorem 2.1.** [16, Theorem 1.4] *Let $R$ be a ring and put $\kappa = card(R)^+$. Denote by $\mathcal{S}$ the representative set of all simple modules. Then $R$ is not right steady iff $\prod_{S\in\mathcal{S}} S^\kappa \oplus \bigoplus_{S\in\mathcal{S}} E(S)$ contains an infinitely generated small submodule.*

Since many module-theoretic properties of regular rings with primitive factors artinian are similar to the case of commutative regular rings, we may generalize [16, Proposition 1.6].

**Proposition 2.2.** *If $R$ is a regular ring with primitive factors artinian, each small module is embeddable in $\prod_{S\in\mathcal{S}} S^{(\omega)}$, where $\mathcal{S}$ is a representative set of simple modules.*

*Proof.* Denote by $\mathcal{M}$ the set of all maximal ideals of $R$. First, note that every simple module over regular ring with primitive factors artinian is injective (see e.g. [1]), which implies that Jacobson radical of each module is zero over such a ring. Moreover, $\bigcap_{I\in\mathcal{M}} MI = 0$.

Let $M$ be a small module and $I \in \mathcal{M}$. Since $M/MI$ is a small semisimple module (over $R$ and $R/I$), there exists a natural number $n_I$ such that $M/MI \cong (R/I)^{n_I}$. Now we obtain a composition of natural embeddings $M \hookrightarrow \prod_{I\in\mathcal{M}} M/MI \hookrightarrow \prod_{S\in\mathcal{S}} S^{(\omega)}$, which finishes the proof. □

The following criterion is formulated for commutative regular rings in [16], however it is actually proved for abelian regular rings:

**Theorem 2.3.** [16, Theorem 2.7] *Let $R$ be an abelian regular ring. Then $R$ is right steady if and only if the $R$-module $R^* = Hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ contains no infinitely generated small submodule.*

It is clear that noetherian regular rings (which are exactly semisimple rings) are steady. In the case of regular rings with primitive factors artinian we have a similar result for a weaker chain condition.

**Theorem 2.4.** [19, Theorem 9] *Let $R$ be a regular ring such that all ideals are countably generated (as two-sided ideals). If all primitive factor-rings of $R$ are artinian, then $R$ is right steady.*

As the following example illustrates, the hypothesis of Theorem 2.4 that $R$ has artinian primitive factors is essential.

**Example 2.5.** *Let a ring $R$ satisfy the condition that the right module $R^2$ is embeddable to $R$. Such a condition is true for the endomorphism ring of an infinitely generated vector space, moreover, there are known examples of countable directly infinite regular rings* [7, Example 5.16]*) for which it holds true that $R^2 \hookrightarrow R$ by* [7, Proposition 5.8]*).*

*Then every injective module over $R$ is $\omega_1$-reducing. Hence a representative class of small $R$-modules is proper and $R$ is not right steady.*

## 3. Semiartinian rings

Before we start to focus on the class of semiartinian rings, note that semiartinian rings need not be regular; for example every right perfect ring is right semiartinian. Nevertheless, we will see later in this section that the question about existence of infinitely generated small modules can be reduced at least in commutative case to the same question about a suitable regular factor of a semiartinian ring. First recall needful notions and their properties.

A module $M$ is *semiartinian* if each non-zero factor-module of $M$ contains a simple submodule. We say that a strictly increasing chain of submodules $N_i$ of $M$, $i \leq \sigma$, is the *socle chain* of $M$ and an ordinal $\sigma$ is the *socle length* of $M$ if

- $Soc(M/N_i) = N_{i+1}/N_i$;
- $N_\alpha = \bigcup_{i<\alpha} N_i$ whenever $\alpha$ is a limit ordinal;
- $N_\sigma \neq N_i$ for every $i < \sigma$ and $Soc(M/N_\sigma) = 0$.

It is well known that $M$ is semiartinian if and only if $M$ is equal to the last member of the socle chain of $M$. A ring $R$ is called *right semiartinian* provided $R_R$ is a semiartinian module. Recall that every module over right semiartinian ring is semiartinian and its socle length is upper bounded by the socle length of $R_R$ [1, 10].

A similar chain condition as in Theorem 2.4 implies steadiness as the following claim, formulated for semiartinian rings, shows

**Theorem 3.1.** [6, Theorem 2.2] *Let $R$ be a right semiartinian ring of countable socle length. Then $R$ is right steady.*

Relevance of the question which semiartinian rings are right steady is illustrated in the paper [6], where large classes of examples of both steady and non-steady (commutative regular) semiartinian rings are presented.

**Example 3.2.** *Let $\sigma$ be an uncountable non-limit ordinal and $\kappa$ be an infinite cardinal such that $\kappa^+ \leq card(\sigma)$.*

(1) [6, Theorem 2.5] *There exists a commutative steady semiartinian regular ring of the socle length $\sigma$.*

(2) [6, Theorem 2.7] *There exists a right semiartinian unit-regular prime ring with the socle length $\sigma$ which contains an infinitely generated $\kappa$-reducing right ideal.*

An easy ring-theoretic criterion of steadiness of abelian regular semiartinian rings, which shows that the choice of an infinitely generated small module as an ideal in [6, Theorem 2.7] was not accidental, is proved in the paper [12].

**Theorem 3.3.** [12, Theorem 3.4] *Let $R$ be an abelian regular semiartinian ring. Then the following conditions are equivalent:*

(1) *$R$ is neither right nor left steady;*
(2) *There is an abelian regular factor-ring, $\bar{R}$, of $R$ and a member, $I$, of the socle chain of $\bar{R}$ such that $I$ is an infinitely generated small right or left $\bar{R}$-module.*

The following facts about semiartinian rings [10] and about steady ideals [19] are essential for characterization of steadiness of general commutative semiartinian rings:

**Lemma 3.4.** [12, Lemma 3.5] *Let $R$ be a ring.*

(1) *$R$ is left semiartinian iff $J(R)$ is right T-nilpotent and $R/J(R)$ is left semiartinian.*
(2) *If $R$ is left semiartinian, then $R$ is right steady iff $R/J(R)$ is right steady.*
(3) *Let $R$ be a commutative semiartinian ring. Then $R/J(R)$ is abelian regular and semiartinian.*

The next criterion shows that the class of all commutative semiartinian rings is very close to the class of all commutative regular rings from the point of view of steadiness.

**Theorem 3.5.** [12, Criterion A] *Let $R$ be a commutative semiartinian ring. Then the following conditions are equivalent:*

(1) *$R$ is not steady;*
(2) *There is a two-sided ideal, $L$, of $R$ such that $J(R) \subseteq L$ and a member, $I$, of the socle chain of the ring $R/L$ such that $I$ is an infinitely generated small right $R/L$-module.*

Theorem 3.3 can be generalized for semiartinian regular rings with primitive factors artinian. Following this purpose, we recall some useful notions with their basic properties.

An ideal $J$ of a regular ring is said to be *homogeneous* if there exists $n$ such that for every maximal ideal $I$ either $J/JI \cong M_n(K)$ for some skew-field $K$ or $J/JI = 0$.

**Lemma 3.6.** [18, Lemma 2.5] *Let $R$ be a regular semiartinian ring with primitive factors artinian, $J$ a homogeneous ideal and $x \in J$. Then $RxR$ is generated by a central idempotent.*

We say that an ideal $J$ is *2-dually slender* if $J$ is not a union of a countably infinite strictly increasing chain of ideals.

**Proposition 3.7.** [18, Proposition 2.7] *Let $R$ be a regular semiartinian ring with primitive factors artinian and let $J$ be an infinitely generated 2-dually slender ideal. Then there exists a factor ring $S$ of $R$, denote the natural projection $R \to S$ by $p$, such that $p(J)$ is an infinitely generated 2-dually slender homogeneous ideal.*

**Theorem 3.8.** [18, Theorem 2.8] *Let $R$ be a regular semiartinian ring with primitive factors artinian such that at least one factor of $R$ contains an infinitely generated 2-dually slender ideal. Then there exists a factor of $R$ containing an infinitely generated small right (left) ideal.*

Denote by $(S_\alpha \mid \alpha \le \sigma + 1)$ the right socle chain of $R$ and let $\{P_{\alpha\beta} \mid \alpha \le \sigma, \beta < \lambda_\alpha\}$ be the representative set of the class of all simple modules such that $P_{\alpha\beta}$ is embeddable to $R/S_\alpha$. Let $M$ be a module and $J$ an ideal. $M$ is said to be *$J$-saturated* provided that $P_{\alpha\beta}$ is a subfactor of $M$ for all those $\alpha$ less then the socle length of $M$ and all $\beta < \lambda_\alpha$ that $P_{\alpha\beta}$ is a subfactor of $J$.

**Lemma 3.9.** *Let $R$ be a regular semiartinian ring with primitive factors artinian which is not right steady.*

1. [18, Lemma 3.2] *If $M$ is a small module, $J$ an ideal such that $M = MJ$ and $M$ is $J$-saturated, then $J$ is 2-dually slender.*
2. [18, Lemma 3.3] *There exist a factor $S$ of the ring $R$, an infinitely generated small $S$-module $M$ and a homogeneous ideal $J \subseteq S$ such that $MJ = M$.*
3. [18, Lemma 3.4] *There exist a factor ring $S$ of $R$, an infinitely generated ideal $J \subseteq S$ and a $J$-saturated infinitely generated small $S$-module $M$ such that $MJ = M$.*

Now, the following criterion is a straightforward generalization of Theorem 3.3 using previous claims about homogeneous and 2-dually slender ideals.

**Theorem 3.10.** [18, Theorem 3.5] *Let $R$ be a regular semiartinian ring with primitive factors artinian. Then the following conditions are equivalent:*

1. *$R$ is right steady;*
2. *$R$ is left steady;*
3. *There exists no infinitely generated (as a two-sided ideal) 2-dually slender ideal of any factor-ring of $R$.*
4. *There exists no infinitely generated small right (left) ideal of any factor-ring of $R$.*

## 4. Regular and non-singular steady rings

First, we state a description of steadiness of simple and injective regular rings, and then we prove several consequences and generalization of this criterion.

**Theorem 4.1.** [6, Theorem 2.8][5, Proposition 11.1] *Let $R$ be a simple regular ring. Then $R$ is right steady iff $R$ is semisimple.*

[5, Proposition 11.1] even implies that every injective module over a simple regular ring is small. Hence a representative set of small modules over any simple regular ring is proper (cf. also Example 2.5 and [15, Example 2.8]).

**Corollary 4.2.** *Let $R$ be a regular ring. If there exists a maximal ideal $I$ such that $R/I$ is not semisimple, then $R$ is neither right nor left steady.*

We will denote by $Q_{max}(R)$ the maximal right ring of quotients of an ring $R$. Recall that $Q_{max}(R) \cong End(E(R))$ as rings, $Q_{max}(R)_R \cong E(R)$ as modules, and $Q_{max}(R)$ is a self-injective regular ring whenever $R$ is right singular (i.e. if the right annihilator of ever non-zero element from $R$ is not essential in $R$), for further details see [13, Chapter XII]. Not that every simple ring is singular.

**Proposition 4.3.** *If $R$ is a right steady simple ring, then $Q_{max}(R)$ is isomorphic to a full matrix ring over a skew-filed.*

*Proof.* First, we will prove that $R$ has a finite right rank. Assume that $\bigoplus_{n<\omega} r_n R \subseteq R$ for a set of non-zero elements $\{r_n\} \subset R$ and let $E$ be an arbitrary injective module. Following the proof of [5, Lemma 1.10] we will show that $E$ is a small module. Assume that $E$ is not small, i.e. there is a countable chain of submodules $E_i \subsetneq E_{i+1} \subsetneq \cdots \subsetneq E$ such that $E = \bigcup_{i<\omega} E_i$. As $R$ is simple, $E_n r_n R = E_n$ for each $n$, hence there exists $m_n \in E_n$ such that $m_n r_n \in E_n \setminus E_{n-1}$. Since $E$ is injective, the naturally defined homomorphism $\bigoplus_{n<\omega} r_n R \to \sum_{n<\omega} e_n r_n R \subseteq E$ can be extended to a homomorphism $\varphi : R \to E$. As $\varphi(R)$ is a cyclic module, there exists $k$ such that all $e_n r_n \in E_k$, which is a contradiction with the choice of elements $e_n$.

We have proved that every injective module over a simple ring of an infinite rank have to be small, so each right steady simple ring has finite right rank. It implies that regular over-ring $Q_{max}(R)$ is of a finite rank, hence it is semisimple. Finally note that $Q_{max}(R)$ is simple because $R \trianglelefteq Q_{max}(R)$ and $R \cap I$ is a two-sided ideal of $R$ whenever $I$ is a two-sided ideal of $Q_{max}(R)$, which concludes the proof. $\square$

The following criterion for steady self-injective regular rings has also appeared as very useful for description of existence of infinitely generated self-small modules over non-singular rings [4, Proposition 3.7].

**Theorem 4.4.** [6, Theorem 2.8] *Let $R$ be a right self-injective regular ring. Then $R$ is right steady iff $R$ is semisimple.*

We prove several consequences of Theorem 4.4.

**Proposition 4.5.** *If $R$ is a right steady right self-injective ring, then $R$ is semiperfect.*

*Proof.* Applying [13, Corollary XIV.1.3] we get that $R/J(R)$ is right steady, right self-injective and regular, hence it is semisimple by Theorem 4.4. As all idempotents can be lifted modulo $J(R)$ by [13, Corollary XIV.1.5], $R$ is semiperfect. □

Applying the previous proposition and [16, Corollary 1.5] we obtain a module-theoretic criterion of steadiness of general self-injective rings.

**Corollary 4.6.** *Let $R$ be a right self-injective ring. Then $R$ is right steady iff $E(R/J(R))$ contains no infinitely generated small module.*

We say that a lattice $L(\wedge, \vee)$ is upper (lower) continuous provided $L$ is complete and $a \wedge \bigvee B = \bigvee\{a \wedge b|\ b \in B\}$ ($a \vee \bigwedge B = \bigwedge\{a \vee b|\ b \in B\}$) for each $a \in L$ and every linearly ordered subset $B \subset L$. A regular ring is called *right (left) continuous* if the lattice of all its right principal ideals is upper (lower) continuous.

As a consequence of the last theorem we can also characterize steadiness of right continuous regular rings:

**Theorem 4.7.** *Let $R$ be a right continuous regular ring. Then $R$ is right steady iff $R$ is semisimple.*

*Proof.* Let $R$ be right steady. By [7, Theorem 13.17] it is a direct product of a continuous abelian regular ring and a right self-injective ring. Since right steady right self-injective ring is semisimple by Theorem 4.4, it remains to prove that every continuous abelian regular ring is right steady.

Suppose that $R$ is a continuous abelian regular ring. We will need two observations about $Q_{max}(R)$: first, that it is abelian regular by [7, Theorem 3.8] and that $R$ contains all idempotents of $Q_{max}(R)$ by [7, Theorem 13.13]. Now, fix an orthogonal set $\{e_i|\ i \in I\}$ of idempotents in $R$ such that $\bigoplus e_i R \trianglelefteq R_R$. Note that $Q_{max}(R) \cong \prod_i e_i R$ by [7, Proposition 9.10]. Assume $\{e_i|\ i \in I\}$ is infinite. As we have a lattice injective homomorphism of $P(I)(\cup, \cap)$ into the lattice of all (central) idempotents of $R$, we may apply Lemma 1.6, which says that $R$ is not right steady. Hence $R$ contains no infinite set of orthogonal idempotents, i.e. it is semisimple. □

Final two remarks are focused on regular subrings of right steady rings:

**Proposition 4.8.** [12, Corollary 4.2] *Let $S$ be a right steady ring and $R$ be its regular subring. Then $R$ is right steady.*

**Proposition 4.9.** [12, Proposition 4.3] *Let $R$ be a regular semiartinian ring. Assume that the center, $C$, of $R$ is not (right) steady. Then there exists an ideal in a factor-ring of $R$ which is infinitely generated and small as a right and left module.*

## References


[1] G. Baccella *Semiartinian V-rings and semiartinian von Neumann regular rings*, J. Algebra **173** (1995), 587–612.
[2] R. El Bashir, T. Kepka, P. Němec, *Modules commuting (via Hom) with some colimits*, Czechoslovak Math. J. **53**(128) (2003), 891–905.
[3] H. Bass *Algebraic K-theory*, Benjamin, New York, 1968.
[4] Breaz, S., Žemlička, J. When every self-small module is finitely generated. *J. Algebra* **315** (2007), 885–893.
[5] R. Colpi and J. Trlifaj *Classes of generalized $*$-modules*, *Comm. Algebra* **22** (1994), 3985–3995.
[6] P.C. Eklof, K.R. Goodearl and J. Trlifaj: *Dually slender modules and steady rings*, Forum Math. **9** (1997), 61–74.
[7] K. R. Goodearl: *Von Neumann Regular Rings*, London 1979, Pitman, Second Ed. Melbourne, FL 1991, Krieger.
[8] T. Head, *Embedded ordinals in modular algebraic lattices*, Algebra Universalis **1** (1971/72), 200–203.
[9] T. Head, *Preservation of coproducts by* $\mathrm{Hom}_R(M, -)$, Rocky Mountain J. Math. **2** (1972), 235–237.
[10] C. Năstăsescu and N. Popescu, *Anneaux semi-artiniens*, Bull. Soc. Math. France **96** (1968), 357–368.
[11] R. Rentschler, *Sur les modules $M$ tels que* $\mathrm{Hom}(M, -)$ *commute avec les sommes directes*, C.R. Acad. Sci. Paris, **268**, 1969, 930–933.
[12] P.Růžička , J. Trlifaj and J. Žemlička: *Criteria of steadiness*, Abelian Groups, Module Theory, and Topology, New York 1998, Marcel Dekker, 359–372.
[13] B. Stenström *Rings of quotients* Die Grundlehren der Mathematischen Wissenschaften, Band 217, Springer-Verlag, New York-Heidelberg, 1975.
[14] J. Trlifaj: *Almost $*$-modules need not be finitely generated*, Comm. Algebra, **21** (1993), 2453–2462.
[15] J. Trlifaj: *Steady rings may contain large sets of orthogonal idempotents*. Abelian groups and modules (Padova, 1994), Math. Appl., 343, Kluwer Acad. Publ., Dordrecht, 1995, 467–473.
[16] J. Žemlička: *Steadiness is tested by a single module*, Contemporary Mathematics, **273** (2001), 301–308.
[17] J. Žemlička: *Classes of dually slender modules*, Proceedings of the Algebra Symposium, Cluj, 2005, Editura Efes, Cluj-Napoca, 2006, 129–137.
[18] J. Žemlička: *Steadiness of regular semiartinian rings with primitive factors artinian*, J.Algebra, **304** (2006), 500–509.
[19] J. Žemlička and J. Trlifaj: *Steady ideals and rings*, *Rend. Sem. Mat. Univ. Padova*, **98** (1997), 161–172.



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