All steps should be explained in detail (preferably by references to assertions, examples, or/end exercises).

1. CFF Homework

To be submitted till 30th March, 9 AM.

1.1. Let V be a vector space over a field K, A and B subspaces of V and recall that $A^o = \{f \in V^* \mid f(A) = 0\}$. Prove that $A^o \cap B^o \subseteq (A+B)^o$.

5 points

1.2. For the extension $\mathbb{F}_2 \subseteq \mathbb{F}_8$ of Galois fields (of orders 2 and 8) show that $\mathbb{F}_8(x)$ is an AFF over $\mathbb{F}_2$ and determine its field of constants $\tilde{\mathbb{F}}_2$. Describe set of all polynomials $p \in \mathbb{F}_8[x] \subseteq \mathbb{F}_8(x)$ such that $[\mathbb{F}_8(x) : \mathbb{F}_2(p)] < \infty$.

5 points

1.3. If L is an AFF over K, P is its place, $a \in L$ satisfies $a^2 \in P^3 \setminus P^6$ and ν_P is a NDV determined by the place P, compute (a) $\nu_P(a)$, (b) $\nu_P(a^{-2}+1+a^5)$, (c) $\nu_P(a^3(1+a^2)^2)$.

5 points

2. CFF Homework

To be submitted till 20th April, 9 AM.

2.1. Find a short WEP which is $\mathbb{F}_3$-equivalent to the WEP

$$w = y^2 + y(2x+1) - (x^3 + 2x^2 + 2x) \in \mathbb{F}_3[x,y].$$

5 points

2.2. Find all singularities of the WEP $w \in \mathbb{F}_3[x,y]$ from 2.1 and one maximal ideal of $\mathbb{F}_3[x,y]$ containing w.

5 points

2.3. Decide whether the WEP is $y^2 - (x^3 + 4x^2 - x - 4) \in K[x,y]$ is smooth if
(a) $K = \mathbb{Q}$, (b) $K = \mathbb{F}_5$.

5 points

3. CFF Homework

To be submitted till 17th May, 5:30 PM.

3.1. Let $w = y^2 - (x^3 + 2x^2 + 1) \in \mathbb{R}[x,y]$ be a WEP and L an AFF over $\mathbb{R}$ given by $w(\alpha,\beta) = 0$ for $\alpha = x + (w)$ and $\beta = y + (w) \in \mathbb{R}[x,y]/(w)$. If ν is a NDV on the AFF L over $\mathbb{R}$ such that $\nu(\alpha - 1) > 0$ and $\nu(\beta - 2) > 0$, determine all $(l_0, l_1, l_2) \in \mathbb{R}^3$ for which $\nu(l_0 + l_1\alpha + l_2\beta) = 1$.

5 points

3.2. Let $f = y(x+2y)+(x^3+x^5)+y \in \mathbb{F}_5[x,y]$, $u = x+(f)$, $v = y+(f)$ and $L = \mathbb{F}_5(u,v)$ be an AFF over $\mathbb{F}_5$ given by $f(u,v) = 0$. If P is a place containing u, v, then compute $\nu_P(u)$, $\nu_P(v)$, $\nu_P(u^4+v)$, $\nu_P(uv+1)$.

5 points

3.3. Prove Lemma 10.6(3): If L is an AFF over K given by $w(\alpha,\beta) = 0$ for a WEP w, $P \in \mathbb{P}_{L/K}$ and $K[\alpha,\beta] \nsubseteq \mathcal{O}_P$, then $3\nu_P(\alpha) = 2\nu_P(\beta) < 0$. *(Hint: Apply Lemma 10.6(2) and the hypothesis $f(\alpha) = \beta(g(\alpha,\beta)+\beta)$ where $w = y^2 + yg(x,y) - f(x)$ for $\deg f = 3$ and $\deg g \leq 1$ to show that assumptions $\nu_P(\alpha) < 0 \leq \nu_P(\beta)$, $\nu_P(\alpha) \geq 0 > \nu_P(\beta)$ and $\nu_P(\alpha) \leq \nu_P(\beta) < 0$ implies contradictions.)*

5 points

4. CFF Homework

To be submitted before the end of June, but not later then during your first attempt to pass the exam

4.1. Determine divisor $(\alpha - \beta - 3)$ as an element of free group $Div(L/\mathbb{Q})$ in the AFF L over $\mathbb{Q}$ given by $f(\alpha,\beta) = 0$, if you know that f a smooth WEP and there are no $\gamma \in V_f(\mathbb{Q})$ satisfying $l(\gamma) = f(\gamma) = 0$ for $l(x,y) = x - y - 3$.

Hint: Proceed as in 12.8 and use that $\nu_P(\alpha-\beta-3) < 0$ if ν_P is negative for a summand and compute negative $\nu_P(\alpha-\beta-3)$ using 11.7. Then show by 9.7 that there is no place of degree 1 containing $\alpha-\beta-3$ and compute positive part of $(\alpha-\beta-3)$.

5 points

4.2. Compute $|\{P \in \mathbb{P}_{L/\mathbb{F}_2} \mid \beta^{-3} \in P\}|$ for the AFF L given by $w(\alpha,\beta) = 0$ over $\mathbb{F}_2$ for $w = y^2 + y + x^3 + 1 \in \mathbb{F}_2[x,y]$.

5 points

4.3. Prove Lemma 13.1(4): If $A = \sum_{P \in \mathbb{P}_{L/K}} a_p P$, then $\dim_K(\mathcal{A}_{L/K}/(\mathcal{A}_{L/K}(A) + L)) = i(A)$.

(Hint: First, suppose that $i(A) = 0$ and prove that $\mathcal{A}_{L/K} = \mathcal{A}_{L/K}(A) + L$: for $f \in \mathcal{A}_{L/K}$ define $D = \sum_{P \in \mathbb{P}_{L/K}} d_p P$ by $d_p := \max(0, a_p, -\nu_P(f(P)))$, and then show that $f \in \mathcal{A}_{L/K}(D) + L = \mathcal{A}_{L/K}(A) + L$ using 13.1(2). Finally, for general A, find B such that $B \geq A$, $i(B) = 0$ using (E5), then by applying first part for B and 13.1(2) prove the assertion.)

5 points