$\boxed{L \text{ is an AFF over } K \text{ of genus } g,\ \tilde{K} \text{ is the field of constants}}$

Lemma 6.13 Let $\mathcal{S} \subsetneq \mathbb{P}_{L|K}$, $P_1, \dots, P_n \in \mathcal{S}$ be pairwise distinct, and $a_1, \dots, a_n \in L$. Then $\forall r \in \mathbb{Z}\ \exists s \in L$:

$$v_{P_i}(s - a_i) > r \ \forall i = 1, \dots, n, \quad v_P(s) \geq 0 \ \forall P \in \mathcal{S} \setminus \{P_1, \dots, P_n\}.$$

Proof: Let $Q \in \mathbb{P}_{L|K} \setminus \mathcal{S}$ and $\forall m \in \mathbb{Z}$ define $B_m = \sum b_P P$ ($\in \mathrm{Div}(L|K)$) by the rules $\boxed{{}_m b_Q := m,\ {}_m b_{P_i} := -r-1 \ \forall i = 1, \dots, n,\ b_P := 0 \ \forall P \in \mathbb{P}_{L|K} \setminus \{P_1, \dots, P_n, Q\}}$

$\deg B_m = \deg B_0 + m \underbrace{\deg Q}_{>0} \overset{6.11}{\Longrightarrow} \exists k : \forall m \geq k \ i(B_m) = 0,$

put $B := B_k$ and $f \in \mathcal{A}_{L|K}$ such that $f(P_i) = a_i \ \forall i$, $f(P) = 0 \ \forall P \neq P_i$

$\underset{(i(B)=0)}{\overset{6.12(5)}{\Longrightarrow}} \exists s \in L, \exists \tilde{f} \in \mathcal{A}_{L|K}(B) : f = \tilde{f} - s \Rightarrow s + f = \tilde{f} \in \mathcal{A}_{L|K}(B)$

$\underset{\text{by def. of } \mathcal{A}_{L|K}(B)}{\Longrightarrow} v_{P_i}(s - a_i) \geq r + 1 \ \forall i = 1, \dots, n,\ v_P(s) \geq 0 \ \forall P \in \mathcal{S} \setminus \{P_1, \dots, P_n\}$

Theorem 6.14 (Strong approximation theorem):

Let $\mathcal{S} \subsetneq \mathbb{P}_{L|K}$, $P_1,\dots,P_n \in \mathcal{S}$ be pairwise distinct, $a_1,\dots,a_n \in L$ and $r_1,\dots,r_n \in \mathbb{Z}$. Then $\exists\, s \in L$ such that

$$v_{P_i}(s-a_i) = r_i \ \forall i=1..n, \quad v_P(s) \geq 0 \ \forall P \in \mathcal{S} \setminus \{P_1,\dots,P_n\}$$

Proof: we repeat the arguments of the proof of 5.19 replacing 5.18(2) by 6.13:

Let $v_i := v_{P_i}$ and $r := \max_{i=1..n}(r_i)$, chose $b_i \in L : v_i(b_i) = r_i$

$\xRightarrow[(2\times\text{applied})]{6.13}$ $\exists\, t, s \in L$ such that $v_P(t) \geq 0$, $\boxed{v_P(s) \geq 0 \ \forall P \in \mathcal{S} - \{P_1 .. P_n\}}$

& $v_i(t-b_i) > r \geq r_i$, $v_i(s-(t+a_i)) > r \geq r_i$ $\forall i=1..n \Longrightarrow$

$\Longrightarrow$ compute v_i: $\underbrace{s-a_i}_{=r_i} = \underbrace{(s-(t+a_i))}_{>r_i} + \underbrace{(t-b_i)}_{>r_i} + \underbrace{b_i}_{=r_i} \Big\} \Rightarrow \boxed{v_i(s-a_i) = r_i \ \forall i = 1, .. n}$

7. Weil differentials

$\boxed{18.N}$ Recall that V^* is the space of linear forms and $W^0 = \{\varphi \in V^* \mid \varphi(W) = 0\}$ for a k-space V and subspace W.

Let $A \in \mathrm{Div}(L|K)$

$$\Omega_{L|K}(A) := (\mathcal{A}_{L|K}(A) + L)^0_K = \{\varphi \in (\mathcal{A}_{L|K})^*_K \mid \varphi(\mathcal{A}_{L|K}(A) + L) = 0\}$$

$$\Omega_{L|K} := \bigcup_{B \in \mathrm{Div}(L|K)} \Omega_{L|K}(B) = \{\varphi \in (\mathcal{A}_{L|K})^* \mid \varphi(L) = 0 \,\&\, \exists B \in \mathrm{Div}(L|K) : \varphi(\mathcal{A}_{L|K}(B)) = 0\}$$

Elements of $\Omega_{L|K}$ (i.e. K-linear forms on $\mathcal{A}_{L|K}$) are called <u>Weil differentials</u>

<u>Observation</u> Let $A, B \in \mathrm{Div}(L|K)$, $\Delta \in L^*$:

(1) $\dim_K(\Omega_{L|K}(A)) \overset{1.4(2)}{=} \dim(\mathcal{A}_{L|K} / (\mathcal{A}_{L|K}(A) + L)) \overset{6.12(4)}{=} i(A)$,

(2) if $A \leq B \overset{6.12(1)}{\Longrightarrow} \mathcal{A}_{L|K}(A) \subseteq \mathcal{A}_{L|K}(B) \overset{1.4(4)}{\Longrightarrow} \Omega_{L|K}(B) \subseteq \Omega_{L|K}(A)$,

(3) $\Omega_{L|K}(A) \cap \Omega_{L|K}(B) \overset{1.4(3)}{=} (\mathcal{A}_{L|K}(A) + \mathcal{A}_{L|K}(B) + L)^{\circ} \overset{6.12(3)}{=} \Omega_{L|K}(\max(A,B))$,

$\Omega_{L|K}(A) + \Omega_{L|K}(B) \overset{1.4(3)}{=} ((\mathcal{A}_{L|K}(A)+L) \cap (\mathcal{A}_{L|K}(B)+L))^{\circ} \boxed{\overset{1.4(1)}{\subseteq}} ((\mathcal{A}_{L|K}(A) \cap \mathcal{A}_{L|K}(B)) + L)^{\circ} \overset{6.12(3)}{=} \Omega_{L|K}(\min A,B)$,

(4) $\lambda \Omega_{L|K}(A) \overset{1.5(3)}{=} (\lambda^{-1} \mathcal{A}_{L|K}(A))^{\circ} \overset{6.12(6)}{=} \Omega_{L|K}(A + (\lambda))$,

(5) Let $\forall \omega \in \Omega_{L|K}$ define $0 \cdot \omega = 0$, $(\lambda \cdot \omega)(\lambda) = \omega(\lambda \lambda)$ $\forall \lambda \in L^*$, $\lambda \in L$

Then $\Omega_{L|K}$ is an L-space by (3), (4) & 1.5.

<u>Lemma 7.1</u> Let $\omega \in \Omega_{L|K} \setminus \{0\}$ and $K = \tilde{K}$. Then $\exists!\, W \in \mathrm{Div}(L|K)$ such that $\omega(\mathcal{A}_{L|K}(W)) = 0$ and $\forall A \in \mathrm{Div}(L|K)$ satisfying $\omega \in \Omega_{L|K}(A)$ it holds that $A \leq W$.

Proof: Note $\forall A \in \mathrm{Div}(L|K)$: $\omega \in \Omega_{L|K}(A) \iff \omega(\mathcal{A}_{L|K}(A)) = 0$

by the definition of $\Omega_{L|K}$ $\exists A \in \mathrm{Div}(L|K)$: $\omega \in \Omega_{L|K}(A)$

by 6.11 $\exists g$ such that $i(A) = 0$ if $A \in \mathrm{Div}(L|K)$: $\deg A \geq g$

by Observation (1) $\dim_k(\Omega_{L|k}(A)) = i(A) > 0$ (as $\Omega_{L|k}(A) \neq 0$)

$\Rightarrow \deg(A) < \gamma$: Fix $W \in \mathrm{Div}(L|k)$ is of the maximal degree such that $\omega(\mathcal{A}_{L|k}(W)) = 0$

Suppose $B \in \mathrm{Div}(L|k)$ such that $\omega \in \Omega_{L|k}(B) \Rightarrow$

$$\Rightarrow \omega \in \Omega_{L|k}(W) \cap \Omega_{L|k}(B) \overset{\text{Ob.(3)}}{=} \Omega_{L|k}(\max(W, B))$$

$\deg \max(W, B) \leq \deg W \Rightarrow B \leq W$ $\Rightarrow$ we have proved $\exists$

from the condition $B \leq W$ $\forall B : \omega(\mathcal{A}_{L|k}(B)) = 0 \Rightarrow$ unicity

$\boxed{\text{T\&N}}$ The divisor W from 7.1 uniquely determined by the Weil differential $\omega \neq 0$ is called a <u>canonical divisor</u> (of ω) and it is denoted by (ω).

<u>Lemma 7.2</u> Let $\omega, \tilde{\omega} \in \Omega_{L|K} \setminus \{0\}$, $K = \tilde{K}$, $A \in \mathrm{Div}(L|K)$, $\psi_\omega : L \to \Omega_{L|K}$ is defined by $\psi_\omega(\Delta) = \Delta \cdot \omega \ \forall \Delta \in L$. Then

(1) if $\Delta \in L^* \Rightarrow (\Delta\omega) = (\Delta) + (\omega)$

(2) ψ_ω is L and K-linear embedding and $\psi_\omega(\mathcal{L}((\omega) - A)) \subseteq \Omega_{L|K}(A)$

(3) $\exists B \in \mathrm{Div}(L|K)$: $\psi_\omega(\mathcal{L}((\omega) - B)) \cap \psi_{\tilde{\omega}}(\mathcal{L}((\tilde{\omega}) - B)) \neq \{0\}$.

Proof: (1) note that $A \leq (\Delta\omega) \overset{7.1.}{\Longleftrightarrow} \Delta\omega \in \Omega_{L|K}(A) \overset{\text{Obs}(4)}{\Longleftrightarrow}$
$\Leftrightarrow \omega \in \Omega_{L|K}(A - (\Delta)) \overset{7.1}{\Longleftrightarrow} A - (\Delta) \leq (\omega) \Longleftrightarrow A \leq (\Delta) + (\omega)$

$$\left.\begin{array}{l} \text{if we put } A := (\Delta\omega) \text{ we get } (\Delta\omega) \leq (\Delta) + (\omega) \\ \text{—— " —— } = (\Delta) + (\omega) \text{ — " — } (\Delta) + (\omega) \leq (\Delta\omega) \end{array}\right\} \Rightarrow (\Delta) + (\omega) = (\Delta\omega)$$

(2) $\text{Obs}(5) \Rightarrow \psi_\omega$ is L-linear, it is nontrivial $\Rightarrow$ it is injective

Since $\Delta \in \mathcal{L}((\omega) - A) \Leftrightarrow A \leq (\Delta) + (\omega) \overset{(1)}{=} (\Delta\omega) \Leftrightarrow \Delta\omega \in \Omega_{L|K}(A)$,

we obtain: $\psi_\omega(\mathcal{L}((\omega) - A)) \subseteq \Omega_{L|K}(A)$.

(3) Let $C \in \mathrm{Div}(L|K)$: $C > 0$; (1) $\Rightarrow \Psi_\omega(\mathcal{L}((\omega)+C)) \subseteq \Omega_{L|K}(-C)$

$-C < 0 \overset{(D8)}{\Longrightarrow} \ell(-C) = 0 \overset{\deg(1)}{\Longrightarrow} \dim_K(\Omega_{L|K}(-C)) = i(-C) = g - 1 \underbrace{-\deg(-C)}_{=\deg C} (+\overset{0}{\ell(-C)})$

by 6.11 $\exists \tilde{\gamma}$: if $\deg C \geq \tilde{\gamma}$: $i((\omega)+C) = 0 = i((\tilde{\omega})+C) \Longrightarrow$

$\ell((\omega)+C) = \deg((\omega)) + \deg C - g + 1 \overset{(2)}{=} \dim_K \Psi_\omega(\mathcal{L}(\omega)+C)$

$\ell((\tilde{\omega})+C) = \deg((\tilde{\omega})) + \deg C - g + 1 \overset{(2)}{=} \dim_K \Psi_{\tilde{\omega}}(\mathcal{L}(\tilde{\omega})+C)$

$\Psi_\omega(\mathcal{L}((\omega)+C))$ and $\Psi_{\tilde{\omega}}(\mathcal{L}((\tilde{\omega})+C)$ are K-subspaces of $\Omega_{L|K}(-C)$

if $\underbrace{\ell((\omega)+C) + \ell((\tilde{\omega})+C)}_{=2\deg C - 2(g-1) + \deg((\omega)) + \deg((\tilde{\omega}))} \overset{(*)}{>} \underbrace{\dim_K(\Omega_{L|K}(-C))}_{=g-1+\deg(+C)} \overset{L.A.}{\Longrightarrow} \Psi_\omega(\mathcal{L}((\omega)+C)) \cap \Psi_{\tilde{\omega}}(\mathcal{L}((\tilde{\omega})+C)) \neq \{0\}$

$\boxed{\text{whenever } \deg C \geq \tilde{\gamma}}$

if we chose C such that $\deg C > 3(g-1) - \deg((\omega)) - \deg((\tilde{\omega}))$

& $\deg C \geq \tilde{\gamma}$ $\Longrightarrow (*)$ is true

Thus we can put $\boxed{B := -C}$