Corollary 5.27. If $s \in L^*$, then $\{P \in \mathbb{P}_{L|K} \mid v_P(s) \neq 0\}$ is finite. $\boxed{\tilde{K} \text{ - a field of constants}}$ 1

Proof: As $v_P(s^{-1}) = -v_P(s)$ $\forall P \in \mathbb{P}_{L|K}$

it is enough to prove $\{P \mid v_P(s) > 0\}$ is finite.

if $s \in \tilde{K} \setminus \{0\} \Rightarrow v_P(s) = 0$ $\forall P \in \mathbb{P}_{L|K}$

let $s \in L - \tilde{K} \overset{1.13}{\Longrightarrow} [L : K(s)] < \infty$

let $P_1 \dots P_r : v_{P_i}(s) > 0$ $\forall i \Rightarrow \underline{r} \leq \sum_{i=1}^{r} \underbrace{v_{P_i}(s)}_{\geq 1} \underbrace{\deg P_i}_{\geq 1}$

$\underset{\boxed{5.21}}{\leq} [L : K(s)] < \infty$

$\Rightarrow r$ is bounded by $[L : K(s)]$

<u>Corollary 5.23</u> If f is a WEP and L is given by $f(\alpha, \beta) = 0$, then $\exists!\, P_\infty \in \mathbb{P}_{L|K}$ such that $v_{P_\infty}(\alpha) < 0$. Furthermore $\deg P_\infty = 1$, $v_{P_\infty}(\alpha) = -2$, $v_{P_\infty}(\beta) = -3$.

<u>proof</u>: by 5.13(3): $3v_P(\alpha) = 2v_P(\beta)\ \forall P : \alpha^{-1} \in P$

$$\Rightarrow 2 \mid v_P(\alpha^{-1}) > 0 \Rightarrow \sum_{\alpha^{-1} \in P} \underbrace{v_P(\alpha^{-1})}_{\geq 2} \underbrace{\deg P}_{\geq 1} \underset{\boxed{5.21}}{\leq} [L : \overbrace{K(\alpha^{-1})}^{= K(\alpha)}] \underset{f \text{ WEP}}{=} 3$$

$$\Rightarrow \exists!\, P \ \&\ \deg P = 1,\ 2 \mid v_P(\alpha^{-1}) \Rightarrow v_P(\alpha) = -2$$

$$\overset{5.13(3)}{\Longrightarrow} v_P(\beta) = -3$$

$\boxed{\text{T\&N}}$ The only place P containing α^{-1} for WEP is denoted by P_∞.

<u>Observation</u> If f is WEP smooth at $V_f(K)$ and L is given by $f(\alpha,\beta)=0$, then by 5.17, 5.23

$$\{P\in\mathbb{P}_{L|K} \mid \deg P=1\} = \{P_\gamma \mid \gamma\in V_f(K), \deg P_\gamma = 1\} \cup \{P_\infty\}$$

<u>Example 5.24</u> $f = y^2 + y - (x^3+1) \in \mathbb{F}_2[x]$

f is WEP $\alpha := x+(f)$, $\beta := y+(f) \in \mathbb{F}_2(\alpha,\beta)$

$L = \mathbb{F}_2(\alpha,\beta)$ is an AFF over $\mathbb{F}_2$ given by $f(\alpha,\beta)=0$

$V_f(\mathbb{F}_2) = \{(1,0),(1,1)\} \Rightarrow P\in\mathbb{P}_{L|K}$ $\deg P = 1 \Rightarrow$

$P\in\{P_{(1,0)}, P_{(1,1)}, P_\infty\}$

But $|\mathbb{P}_{L|\mathbb{F}_2}| = \infty$ by 5.20(1)

other places are of $\deg > 1$ e.g. $\forall m\in\mathbb{F}_2[x]$ irreducible $\exists P\in\mathbb{P}_{L|K}$ $\deg m>1$, $m(\alpha)\in P$, $\deg P>1$

6. Divisors

Let L be an AFF over K, $\tilde{K}$ be the field of constants

Definition: Let $Div(L|K) = \{\sum_{P \in \mathbb{P}_{L|K}} a_P P \mid a_P \in \mathbb{Z}\}$ denotes the free abelian group with the free basis $\mathbb{P}_{L|K}$ ($\Rightarrow |\{P \mid a_P \neq 0\}| < \infty$) and operations $\sum_{P \in \mathbb{P}_{L|K}} a_P P \pm \sum_{P \in \mathbb{P}_{L|K}} b_P P = \sum_{P \in \mathbb{P}_{L|K}} (a_P \pm b_P) P$, $\underline{0} := \sum 0 \cdot P$. A formal sum $\sum a_P P \in Div(L|K)$ is called a divisor (of the AFF). Degree of a divisor is defined by $\deg_K(\sum a_P P) := \sum a_P \deg_K P \in \mathbb{Z}$ (over K).

Example 6.1 $\forall r \in L^*$: $\sum_{P \in \mathbb{P}_{L|K}} v_P(r) P$ is a divisor since $\{P \mid v_P(r) \neq 0\}$ is finite by 5.22.

<u>Observation</u> A Put $k := [\tilde{K} : K]$, $P \in \mathbb{P}_{L|K}$, $a \in L^*$

(1) L is an AFF over $\tilde{K}$, $\tilde{K} \subseteq \mathcal{O}_P$ by 2.15 $\Longrightarrow$

$$\mathbb{P}_{L|\tilde{K}} = \mathbb{P}_{L|K} \text{ and } \mathrm{Div}(L|K) = \mathrm{Div}(L|\tilde{K}),$$

(2) $\deg_{\tilde{K}} P = \dim_{\tilde{K}} \mathcal{O}_P/P = k \cdot \dim_K \mathcal{O}_P/P = \cancel{k} \cdot \deg_K P$

(Hence $k < \infty$ by 2.15),

(3) $\deg_K = k \cdot \deg_{\tilde{K}}$ is a homomorphism of the abelian groups $\mathrm{Div}(L|K)$ and $\mathbb{Z}$,

(4) $\sum_{P \in \mathbb{P}_{L|K}} v_P(a) P = 0 \iff v_P(a) = 0 \ \forall P \in \mathbb{P}_{L|K} \iff a \in \tilde{K}^*$

$\boxed{\text{T\&N}}$ $\sum_{P \in \mathbb{P}_{L|K}} v_P(r) P$ for $r \in L^*$ is called a <u>principal divisor</u>, it is denoted <u>(r)</u>, $\mathrm{Princ}(L|K) := \{(r) \mid r \in L^*\}$

$\boxed{T\&N}$ Let $A = \sum a_P P$, $B = \sum b_P P \in Div(L/K)$.

$max(A, B) := \sum max(a_P, b_P) P$, $min(A, B) := \sum min(a_P, b_P) P$.

$A_+ := max(A, \underline{0})$, $A_- := min(A, \underline{0})$

A is <u>positive</u> if $A = A_+$

Define relations $\leq (\geq)$ and $\sim$ on $Div(L/K)$:

$A \leq B \overset{def}{\equiv} a_P \leq b_P \ \forall P \in \mathbb{P}_{L/K}$ $(\Leftrightarrow A = min(A, B) \Leftrightarrow B = max(A, B))$

$A \sim B \overset{def}{\equiv} A - B \in Princ(L/K)$

$\mathcal{L}(A) := \{r \in L^* \mid (r) + A \geq \underline{0}\} \cup \{0\}$

<u>Observation</u> (B) Let $A, B, C, D \in \mathbb{P}_{L/K}$, $r, s \in L^*$.

(1) $A_s (r \cdot s) = \sum \overline{v_P(r \cdot s)} P = (r) + (s)$, the mapping $r \to (r)$ is a group homomorphism $L^* \to Div(L/K)$

$v_P(r) + v_P(s)$

(2) $-(r) = (r^{-1})$, $(1) = \underline{0}$, $\mathrm{Princ}(L|K)$ is a subgroup of $\mathrm{Div}(L|K)$

and $(r) = (s) \iff \exists \lambda \in \tilde{K}^* : r = \lambda s$

(3) $\sim$ is a congruence on $\mathrm{Div}(L|K)$ (i.e. an equivalence with compatible operations)

(4) $\leq$ is an ordering on $\mathrm{Div}(L|K)$ such that

if $A \leq B$, $C \leq D \Rightarrow A + C \leq B + D$

(5) if $r \in L - \tilde{K}$, then $\exists P, Q \in \mathbb{P}_{L|K} : v_P(r) > 0, v_Q(r) < 0$

hence $r \ngeq \underline{0}$

(6) $\mathcal{L}(A)$ is a $\tilde{K}$-space (and so a K-space)

$\mathcal{L}(\underline{0}) = \{ r \in L^* \mid (r) \geq \underline{0} \} \cup \{0\} \overset{(5)}{=} \tilde{K}$

$\boxed{\text{T\&N}}$ $\mathrm{Cl}(L|K) := \mathrm{Div}(L|K) / \mathrm{Princ}(L|K)$ is the <u>class group of the AFF</u>

$A \in \mathrm{Div}(L|K)$: $\mathcal{L}(A)$ – <u>Riemann-Roch space of A</u>

$\ell(A) = \dim_{L|K} A := \dim_K \mathcal{L}(A)$

<u>Observation</u> ⓒ Let $i \leq j$, $P \in \mathbb{P}_{L|K}$, $t \in P$: $v_P(t) = 1$

$\left(P^r := (t^r) = t^r \mathcal{O}_P \right)$

(1) $P = (t)$ and $\psi_j : \mathcal{O}_P / P \to P^{j-1}/P^j$ defined by $\psi_j(a+P) = a t^{j-1} + P^j$ is an isomorphism of K-spaces

(2) $\deg_K P = \dim_K \mathcal{O}_P / P \overset{(1)}{=} \dim_K P^{j-1}/P^j$

(3) $\dim_K P^i / P^j \overset{(2)}{=} \sum_{r=i+1}^{j} \underbrace{\dim_K P^{r-1}/P^r}_{= \deg P} = (j-i) \deg P$

$\boxed{\text{T\&N}}$ If $K = \tilde{K}$, then L is said to be a <u>full constant AFF</u> over K