In the sequel $L$ is an AFF over $K$ given by $f(\alpha,\beta)=0$ for $\alpha,\beta$ transcendental over $K$

Observation: Let $\omega : K[x,y] \to K[\alpha,\beta]$ be defined by $\omega(m)=m(\alpha,\beta)$ and $P\neq 0$ be a prime ideal of $K[\alpha,\beta]$

(1) $\omega$ is a surjective ring homomorphism and $\operatorname{Ker}\omega=(f)$.

(2) $(f) \subsetneqq \omega^{-1}(P)$ - is a prime ideal of $K[x,y]$

$\overset{4.3}{\Longrightarrow} \exists\, \mathfrak{p} \in V_f$ such that $P=\omega(I_{\mathfrak{p}})$

(3) $\hat{K} := K[\alpha,\beta]/P$ is an algebraic extension of $K$ since $P$ is maximal, $\hat{K}=K[\alpha+P,\beta+P] \Rightarrow [\hat{K}:K]<\infty$.

(4) $[\hat{K}:K]=1 \overset{1.28(2)}{\Longleftrightarrow} K[x,y]/_{I_{\mathfrak{p}}} \cong K \overset{4.3}{\Longleftrightarrow} \mathfrak{p}\in V_f(K)$.

**Lemma 5.15** Let $P \in \mathbb{P}_{L|K}$, $\tilde{P} := P \cap K[\alpha,\beta]$.

(1) if $K[\alpha,\beta] \subseteq \mathcal{O}_P \Rightarrow \tilde{P}$ is a maximal ideal of $K[\alpha,\beta]$,

$\dim_K K[\alpha,\beta]/\tilde{P} < \infty$, $v_P(\alpha) \geq 0$, $v_P(\beta) \geq 0$

(2) if $K[\alpha,\beta] \nsubseteq \mathcal{O}_P \Rightarrow \tilde{P} = 0$ and either $v_P(\alpha) < 0$ or $v_P(\beta) < 0$.

(3) if $K[\alpha,\beta] \nsubseteq \mathcal{O}_P$ & $f$ is a WEP, then $v_P(\alpha) < 0$ and $v_P(\beta) < 0$,

and $3v_P(\alpha) = 2v_P(\beta)$.

proof: (1) $\alpha,\beta \in \mathcal{O}_P \Rightarrow v_P(\alpha) \geq 0$, $v_P(\beta) \geq 0$

?? $\tilde{P} = 0 \Rightarrow K[\alpha\beta] \setminus \{0\} \subseteq \mathcal{O}_P \setminus P = \mathcal{O}_P^* \Rightarrow K(\alpha,\beta) \subseteq \mathcal{O}_P \subsetneq K(\alpha,\beta)$

a contradiction

Thus $\tilde{P} \neq 0 \Rightarrow \tilde{P}$ is maximal & $\dim_K (K[\alpha,\beta]/\tilde{P}) < \infty$ by Observ. (3)

(2) ?? $\tilde{P} \neq 0$, $a \in K[\alpha,\beta] \setminus \mathcal{O}_P \Rightarrow v_P(a) < 0$

by Observation (3) $a + \tilde{P}$ is algebraic over $K \Rightarrow \exists\, m \in K[x] \setminus K : \underbrace{m(a) \in \tilde{P} \subseteq P}_{v_P(m(a)) \geq 0}$

$\underline{0 \leq v_P(m(a))} \underset{2.17(3)}{=} \underbrace{\deg m}_{>0} \cdot \underbrace{v_P(a)}_{<0} \underline{< 0} \Rightarrow$ a contradiction $\Rightarrow \tilde{P} = 0$

$v_P(\alpha) \geq 0$, $v_P(\beta) \geq 0 \Rightarrow \alpha,\beta \in \mathcal{O}_P \Rightarrow K[\alpha,\beta] \subseteq \mathcal{O}_P$

(3) Let $f = y^2 + y\,g(x) - h(x)$ where $g, h \in K[x]$, $\deg g \le 1$, $\deg h = 3$

$f(\alpha,\beta) = 0 \Rightarrow \beta(\beta + g(\alpha)) = h(\alpha)$ Put $\underbrace{a := h(\alpha)}$, $\underbrace{b := \beta(\beta + g(\alpha))}$ and $v := v_p$

$\Rightarrow v(a) = v(b) \overset{DV1}{=} v(\beta) + v(\beta + g(\alpha))$, we know that either $v(\alpha) < 0$ or $v(\beta) < 0$

(a) Assume ?? $v(\alpha) < 0 \le v_p(\beta)$ Then $\boxed{\text{by computing rules 2.13, 2.17 \& (DV1)-DV3}}$

$\underset{\deg h}{3 v(\alpha)} \overset{4}{\underset{2.17}{=}} v(a) = v(b) = \underbrace{v(\beta)}_{> v(\alpha)} + \underbrace{v(\beta + g(\alpha))}_{> v(\alpha)} > 2 v(\alpha) \Rightarrow v(\alpha) > 0$ – a contradiction

(b) Assume ?? $v(\alpha) \ge 0 > v(\beta)$ Then

$0 \overset{2.17}{\le} v(a) = v(b) = \underbrace{v(\beta)}_{< 0} + \underbrace{v(\beta + g(\alpha))}_{v > v(\beta)} \overset{2.13}{=} 2 v(\beta) < 0$ $\Rightarrow$ a contradiction

Hence (a) & (b) $\Rightarrow v(\alpha) < 0$ & $v(\beta) < 0$

(c) Assume ?? $v(\alpha) \le v(\beta) \overset{\text{as in (a)}}{\Longrightarrow} 3 v(\alpha) = v(a) = v(\beta) + v(\beta + g(\alpha)) \ge 2 v(\alpha)$

$\Rightarrow v(\alpha) \ge 0$ – a contradiction

(c) $\Rightarrow v(\alpha) > v(\beta) \Rightarrow 3 v(\alpha) = v(a) = v(b) = 2 v(\beta)$

**Proposition 5.16** Let $P \in \mathbb{P}_{L/K}$, $\deg P = 1$ and $f$ be smooth at every $\underset{(\gamma_1,\gamma_2)}{\gamma} \in V_f(K)$. Then the following is equivalent:

(1) $K[\alpha,\beta] \subseteq \mathcal{O}_P$

(2) $\exists!\, \gamma \in V_f(K)$ such that $v_P(\alpha-\gamma_1) > 0$, $v_P(\beta-\gamma_2) > 0$.

(3) $\exists!\, \gamma \in V_f(K)$ such that $P = P_\gamma$.

Proof: (1) $\Rightarrow$ (2) $\exists \gamma \in V_f : \tilde{P} = P \cap K[\alpha,\beta] = \omega(I_\gamma)$ by 5.15 & Observ. (2)

$0 \neq K[\alpha,\beta]/\tilde{P} \overset{\text{(3. Iso. Theorem)}}{\cong} (K[\alpha,\beta]+P)/P$ is a subspace of $K$-space $\mathcal{O}_P/P \Rightarrow$

$\Rightarrow 0 < \dim_K (K[\alpha,\beta]/\tilde{P}) \leq \dim \mathcal{O}_P/P = \deg P = 1 \Rightarrow$

$\Rightarrow \dim_K (K[\alpha,\beta]/\tilde{P}) = 1 \overset{\text{Obs. (4)}}{\Longrightarrow} \gamma \in V_f(K)$

Unicity: if $\alpha-\gamma_1, \alpha-\tilde{\gamma}_1, \beta-\gamma_2, \beta-\tilde{\gamma}_2 \in P \Rightarrow \gamma_1-\tilde{\gamma}_1, \gamma_2-\tilde{\gamma}_2 \in \underset{\{0\}}{\underset{\shortparallel}{P \cap K}}$

(2) $\Rightarrow$ (3) by 5.13

(3) $\Rightarrow$ (1) $\left.\begin{array}{l} \alpha-\gamma_1, \beta-\gamma_2 \in P_\gamma = P \\ \gamma_1, \gamma_2 \in K \end{array}\right\} \Rightarrow \alpha,\beta \in K + P_\gamma = \mathcal{O}_\gamma \subseteq \mathcal{O}_P \Rightarrow (1)$

<u>Corollary 5.17</u> If $f$ is ~~smooth~~ a WEP at all points of $V_f(k)$ and $P \in \mathbb{P}_{L|k}$ is of degree 1, then either $\exists\, \underline{x} \in V_f(k)$ for which $P = P_{\underline{x}}$ or $v_P(\alpha) < 0$ and $v_P(\beta) < 0$ (i.e. $\alpha^{-1}, \beta^{-1} \in P$)

~~Proof~~: by 5.16 & 5.15(3).

<u>Observation</u> Let $\tilde{k}$ be the field of constants of $L$ (i.e. $\tilde{k} = \{a \in L \mid a \text{ is algebraic over } k\}$)

(1) if $s \in L \setminus \tilde{k} \Rightarrow \exists P \in \mathbb{P}_{L|k} : v_P(s) > 0$ by 2.5

(2) $\tilde{k} = \{a \in L \mid v_P(a) = 0 \ \forall P \in \mathbb{P}_{L|k}\}$ where "$\supseteq$" follows from (1) and "$\subseteq$" holds by 2.15(1)

(3) if $a, b \in L$, $P \in \mathbb{P}_{L|k}$, $v_P(a) \neq 0 \neq v_P(~~b~~ h)$, then by 2.13 $v_P(a + b^r) = \min(v_P(a), r \cdot v_P(h))$ for all $r$ except at most one

$\Rightarrow \exists r_0 \ \forall r \geq r_0 \ v_P(a + b^r) = \min(v_P(a), r \cdot v_P(h))$

**Lemma 5.18** Let $P_1, \dots, P_n \in \mathbb{P}_{L|K}$ be pairwise distinct, $n \geq 1$, $v_i := v_{P_i}$, $a_1, \dots, a_n \in L$ and $r \in \mathbb{Z}$. Then:

(1) $\exists\, s \in L^*$: $v_1(s) > 0$ and $v_i(s) < 0$ $\forall i = 2, \dots, n$

(2) $\exists\, s_r \in L$: $v_i(s_r - a_i) > r$ $\forall i = 1, \dots, n$.

*proof*: will be proved next week. ((2) is technical result needed for the following theorem, (2) follows from (1)).

**Theorem 5.19** (Weak Approximation Theorem)

Let $n \geq 1$ and $P_1, \dots, P_n \in \mathbb{P}_{L|K}$ be pairwise distinct. If $a_1, \dots, a_n \in L$ and $r_1, \dots, r_n \in \mathbb{Z}$, then $\exists\, s \in L$ such that $v_{P_i}(s - a_i) = r_i$ $\forall i = 1, \dots, n$.

*proof*: put $v_i := v_{P_i}$, $r := \max\{r_i \mid i = 1, \dots, n\}$

Fix $b_i \in L$ such that $v_i(b_i) = r_i$ $\forall i = 1, \dots, n$

$$\left( \in P_i^{r_i} - P_i^{r_i+1} \neq \emptyset \right)$$

By 5.18(2) $\exists\, s \in L$ such that

$$v_i(s - b_i) > \Delta \geq r_i \quad \forall i$$

Again by 5.18(2) $\exists\, \Delta \in L$ :

$$v_i(\Delta - (s + a_i)) > \Delta \geq r_i \quad \forall i$$

Then: $\Delta - a_i = \underbrace{(\Delta - (s + a_i))}_{> r_i} + \underbrace{(s - b_i)}_{> r_i} + \underbrace{b_i}_{= r_i}$

compute $v_i$ :

$\overset{2.13}{\Longrightarrow}$ $v_i(\Delta - a_i) = v_i(b_i) = r_i$ .