<u>Lemma 5.11</u> If f is singular at $\underline{x} \in V_f(k)$, then $\mathcal{O}_{\underline{x}}$ is not a VR.

<u>Proof</u>: By 3.10 f is singular at $\underline{x} = \tau_{\underline{x}}(0,0) \Longleftrightarrow$

$\underline{0} := (0,0)$ $\qquad \tau_{\underline{x}}^*(f)$ is singular at $\underline{0}$

& Observation (5) ${}_f\mathcal{O}_{\underline{x}} = {}_{\tau_{\underline{x}}^*(f)}\mathcal{O}_{\underline{0}} \Rightarrow$ we may assume

w.l.o.g. that f is singular at $\underline{0}$

Then & 3.8 $L(f) = \Delta_{\underline{0}}(f) \overset{=0}{} \Rightarrow \operatorname{mult} f \geq 2$

?? Assume that $\mathcal{O}_{\underline{0}}$ is VR $\Rightarrow$ either $\frac{\alpha}{\beta} \in \mathcal{O}_{\underline{0}}$ or $\frac{\beta}{\alpha} \in \mathcal{O}_{\underline{0}}$

let $\frac{\alpha}{\beta} \in \mathcal{O}_{\underline{0}} \Rightarrow \exists\, a, b \in k[x,y]$: $\operatorname{mult} a \geq 1$, $\operatorname{mult} b \geq 1$

$$\& \ \exists\, \nu, \lambda \in k;\ \lambda \neq 0 \ :\ \frac{\alpha}{\beta} = \frac{a(\alpha,\beta)+\nu}{b(\alpha,\beta)+\lambda} \Longrightarrow$$

as $(b+\lambda)(\underline{0}) \neq 0$

$\Longrightarrow \alpha(b(\alpha,\beta)+\lambda) - \beta(a(\alpha,\beta)+\nu) = 0$

$\Longrightarrow \underbrace{\underset{\text{mult} \geq 2}{f} \;/\; \underbrace{x b + \overset{\neq 0}{\lambda x} - (y a + \nu y)}_{\text{mult}(\text{---}\text{''}\text{---}) = 1}}$ ~~a~~ as $\lambda x \neq 0$ and mult $h \geq 1$

$\Longrightarrow$ a contradiction

the argument for $\frac{\beta}{\alpha}$ is symmetric.

<u>Lemma 5.12</u> Let L be an AFF given by $w(u,v)=0$ (where $w = h(x) + y\,g(x,y) + y$, mult $g \geq 1$, $m :=$ mult $h \geq 2$ etc.

Suppose $P \in \mathbb{P}_{L|K}$ such that $u, v \in P$ $v_P(u) = 1$.

If $s \in K[u,v] \setminus \{0\}$, then $\exists\, a, b \in K[x,y]$ with $a(\underline{0}) \neq 0 \neq b(\underline{0})$ and $\frac{s}{u^{v_P(s)}} = \frac{a(u,v)}{b(u,v)} \in \mathcal{O}_{\underline{0}}^* = \mathcal{O}_{\underline{0}} \setminus P_{\underline{0}}$.

<u>proof</u>: Put $k := v_P(s) = \mu(s)$ by 5.5

by 5.4 $\exists\, c \in K[x,y]\ \exists\, \lambda \in K^* : s = \lambda u^k + c(u,v)$ & $\mu(c) > k$.

denote coefficients of c : $c(x,y) = \sum c_{ij} x^i y^j$

if $c_{ij} \neq 0 \underset{\mu(c) > k}{\Longrightarrow} i + j m > k \Rightarrow i + j m - k > 0$

mult $a = m \Rightarrow \tilde{a} := -\frac{a(x)}{x^m} \in K[x]$

$w(u,v) = 0 \Rightarrow \underbrace{a(u)}_{-\tilde{a}(u) \cdot u^m} + v(1 + g(u,v)) = 0 \Rightarrow \frac{v}{u^m} = \frac{\tilde{a}(u)}{1 + g(u,v)}$

Hence if $c_{ij} \neq 0 \Rightarrow \frac{u^i v^j}{u^k} = \left(\frac{v}{u^m}\right)^j \cdot u^{\overbrace{i+jm-k}^{>0}}$

Compute $\left(\frac{c(x,y)}{x^k}\right)(u,v) = \sum c_{ij} \frac{u^i v^j}{u^k} = \sum c_{ij} \left(\frac{v}{u^m}\right)^j u^{i+jm-k}$

$\boxed{\text{put } l := \max\{j \mid c_{ij} \neq 0\}}$ $= \sum c_{ij} \frac{\tilde{a}(u)^j u^{i+jm-k}}{(1 + g(u,v))^j}$

Define $b := (1 + g(x,y))^l \Rightarrow$ mult $b = 0$

as mult $g \geq 1$

$\exists\, d \in K[x,y]$ with mult $d \geq 1$:

$\left(\frac{c(x,y)}{x^k}\right)(u,v) = \frac{d(u,v)}{b(u,v)}$; put $a := \lambda b + d$

$\Rightarrow$ mult$(a) = 0$

and $\underbrace{\frac{s}{u^k}}_{} = \lambda + \frac{c(u,v)}{u^k} = \frac{\lambda b(u,v) + d(u,v)}{b(u,v)} = \underbrace{\frac{a}{b}(u,v)}_{}$

mult$(a) = 0 =$ mult$(b) \Rightarrow a(\underline{0}) \neq 0 \neq b(\underline{0})$

<u>Proposition 5.13</u> Let f be smooth at $\gamma \in V_f(k)$ and $P \in \mathbb{P}_{L|K}$ satisfies $v_P(\alpha - \gamma_1) > 0$, $v_P(\beta - \gamma_2) > 0$.

(1) $\exists\, u \in P_\gamma$: $v_P(u) = 1$ and $\forall\, s \in K[\alpha,\beta] \setminus \{0\}$ $\frac{s}{u^{v_P(s)}} \in \mathcal{O}_\gamma^*$,

(2) $P = P_\gamma$.

Proof: (1) We use transformation from 5.7 & 5.8 again: $f \mapsto w_\sigma$ $(\delta(\sigma^{-1})^*(f) = w_\sigma)$

$\underline{0} = (0,0)$ and note $(u,v) = \bar{\sigma}(\alpha,\beta)$ $\nu_p(u) = 1$ f.o.S.

Furthermore $\mathrm{Span}_K(u,v) = \mathrm{Span}(\alpha-\gamma_1, \beta-\gamma_2) \Rightarrow$ we can extend the Observation (5) to

$$\boxed{\text{Observ. (5')}: {}_f\mathcal{O}_\gamma = {}_{w_\sigma}\mathcal{O}_{\underline{0}} \ \& \ {}_fP_\gamma = {}_{w_\sigma}P_{\underline{0}}}$$

Since $u = x(u) \Rightarrow u \in {}_{w_\sigma}P_{\underline{0}} = {}_fP_\gamma$

By 5.12 $\frac{r}{u^{\nu_p(r)}} \in {}_{w_\sigma}\mathcal{O}_{\underline{0}}^* = {}_f\mathcal{O}_\gamma^*$

(2) By 2.5 $\exists Q \in \mathbb{P}_{L|K} : P_\gamma \subseteq Q, \mathcal{O}_\gamma \subseteq \mathcal{O}_Q$

$\alpha-\gamma_1, \beta-\gamma_2 \in P_\gamma \underset{5.8}{\Longrightarrow} P = Q \Rightarrow P_\gamma \subseteq P$

we prove $P \subseteq P_\gamma$: let $r \in P_{-\{0\}} \Rightarrow \exists r_1, r_2 \in K[\alpha,\beta]-\{0\}$ s.t. $r = \frac{r_1}{r_2}$

Then by (1) for $i=1,2$ $\exists\, \sigma_i \in \mathcal{O}_{\mathcal{X}}^* : r_i = u^{v_p(r_i)} \cdot \sigma_i$

$\Rightarrow r = \frac{r_1}{r_2} = \underbrace{\frac{\sigma_1}{\sigma_2}}_{\in \mathcal{O}_{\mathcal{X}}^* \subseteq \mathcal{O}_P^*} \cdot u^{v_p(r_1)-v_p(r_2)} \Rightarrow \underset{\Updownarrow\, r \in P}{0 < v_p(r)} = \underbrace{v_p\left(\frac{\sigma_1}{\sigma_2}\right)}_{\overset{\shortparallel}{0}} + v_p(r_1) - v_p(r_2)$

$\Rightarrow r = \underbrace{\frac{\sigma_1}{\sigma_2}}_{\in \mathcal{O}_{\mathcal{X}}} \cdot \underbrace{u}_{\in P_{\mathcal{X}}}^{\overbrace{v_p(r)}^{>0}} \in P_{\mathcal{X}}$ $\square$

<u>Example 5.14</u> Repeating 5.10 $f = y^2 + xy + x^5 + 32 \in \mathbb{R}[x,y]$

$(-2,2) \in V_f(\mathbb{R})$, $\Delta_{(-2,2)}(f) = 82x + 2y + 160$

$P = P_{(-2,2)} \in \mathbb{P}_{\mathbb{R}(\alpha,\beta)}$ s.t. $v_p(\alpha+2) = 1$ $\quad v_p(\beta - 2) > 1$ (= 2 by 5.10)

$\Rightarrow P = P_{(-2,2)} = (\alpha+2) = \boxed{\left\{ (\alpha+2) \frac{p(\alpha,\beta)}{q(\alpha,\beta)} \,\middle|\, p,q \in \mathbb{R}[x,y],\ q(-2,2) \neq 0 \right\}}$

↑ it is principal ideal by 2.10 & 2.15