Recall that every WEP is absolutely irreducible (4.9) and if $w \in k[x,y]$ is absolutely irreducible, then (4.10) $k = \tilde{k}$ in $k(V_w)$. Thus:

<u>Corollary 4.11</u> If C is a Weierstrass curve, then each $\gamma \in k(C) \setminus k$ is transcendental over k.

<u>Example 4.12</u>: $w = y^2 + yx + x^3 + 1 \in \mathbb{F}_2[x]$ is WEP

$L :=$ the fraction field of $\mathbb{F}_2[x,y]/(w) \cong \mathbb{F}_2[x,y]_{(w)}/(w)$

$\left(\overset{\shortparallel}{\mathbb{F}_2}(V_w)\right)$ the localization of $\mathbb{F}_2[x,y]$ in (w)

Then $\widetilde{\mathbb{F}_2} = \mathbb{F}_2$ are all (only 2!) algebraic elements of L over $\mathbb{F}_2$

Then e.g. X^2+X+1 or X^3+X+1 are irreducible polynomials over L.

Similarly: $w \in \mathbb{F}_{2^m}[x,y]$ (the same polynomial) gives $\widetilde{\mathbb{F}_{2^m}} = \mathbb{F}_{2^m}$ in $\mathbb{F}_{2^m}(V_w)$.

$\boxed{T\&N}$ Let $w \in K[x,y]$, L be an AFF over K, $\alpha,\beta \in L$.
We say that an AFF L is given by (the equation) $w(\alpha,\beta)=0$
if (1) $L = K(\alpha,\beta)$
(2) w is irreducible
(3) $w(\alpha,\beta)=0$ in L.

Observation Let $L := K(V_w)$ for irreducible $w \in K[x,y]$ and $\alpha := x+(w)$, $\beta := y+(w)$. Then L is an AFF given by $w(\alpha,\beta)=0$.

5. Places

In this section K is a field and
$w = y\,g(x,y)_* + h(x) + y \in K[x,y]$ where $h \in K[x]\setminus\{0\}$, $g \in K[x,y]$,
$m := \mathrm{mult}(h) \geq 2$ and $\mathrm{mult}(g) \geq 1$

$\boxed{T\&N}$ Let $a = \sum_{\substack{i\geq 0\\ j\geq 0}} a_{ij} x^i y^j \in K[x,y] \setminus \{0\}$

$\mu(a) := \mathrm{mult}(a(x, y^m)) = \min\{i + jm \mid a_{ij} \neq 0\}$ "m-weighted multiplicity"

$\Delta(a) := \{(i,j) \mid i + jm = \mu(a),\ i \geq 0, j \geq 0\}$

$S(a) := \sum_{(i,j)\in\Delta(a)} a_{ij} x^i y^j \in K[x,y] \setminus \{0\}$ "m-socle"

Observation Let $a, b \in K[x,y] \setminus \{0\}$

(1) $\mathrm{mult}(a \cdot b) = \mathrm{mult}(a) + \mathrm{mult}(b)$,

if $\mathrm{mult}(a) < \mathrm{mult}(b) \Rightarrow \mathrm{mult}(a+b) = \mathrm{mult}(a)$

(a technical exercise, hint: use the lexicographic order on indices (i,j) of a_{ij}; it is analogy of properties of deg)

(2) $\mu(a \cdot b) = \mathrm{mult}(a(x,y^m) \cdot b(x,y^m)) \overset{(1)}{=} \mathrm{mult}(a(x,y^m)) + \mathrm{mult}(b(x,y^m)) =$

$= \mu(a) + \mu(b)$

if $\mu(a) < \mu(b)$ i.e. $\mathrm{mult}(a(x,y^m)) < \mathrm{mult}(b(x,y^m)) \overset{(1)}{\Rightarrow}$

$\Rightarrow \mu(a+b) = \mu(a)$

(3) if $(i+jm)+(r+lm) = \mu(a)+\mu(h) \overset{(a)}{=} \mu(a\cdot h)$ & $i+jm > \mu(a)$

$\Rightarrow r+lm < \mu(h) \Rightarrow b_{rl} = 0$, hence:

$$S(a)\cdot S(h) = \sum_{\substack{(i,j)\in s(a)\\(r,l)\in s(h)}} a_{ij}\cdot b_{rl}\, x^{i+r} y^{j+l} =$$

$$= \sum_{\substack{i,j,r,l:\\ i+jm+r+lm=\mu(ah)}} a_{ij}\, b_{rl}\, x^{i+k} y^{j+l} = S(a\cdot h)$$

(4) $\mu(a) = \mu(S(a)) \geq \mathrm{mult}(a)$,

if $\mu(a) < \mu(h) \overset{(2)}{\Rightarrow} S(a+h) = S(a)$

$\boxed{\text{IRN}}$ Define a K-homomorphism

$\Lambda : K[x,y] \to K[x,y]$ & iterate

$\Lambda(u(x,y)) := u(x, -h(x) - y\,g(x,y))$ (as substitution)

Lemma 5.1. For every $i,j\geq 0$ $\mu(\Lambda(x^iy^j)) = i+jm$ and $\exists\, \lambda\in K^*$ such that $S(\Lambda(x^iy^j)) = \lambda\cdot x^{i+jm}$

Proof: Note that: $\underline{\mu(-h)}=\mu(h)=\mathrm{mult}(h)=\underline{m}$

$$\underline{\mu(-yg)}\overset{\text{Obs.(2)}}{=}\underline{\mu(yg)}=\mu(y)+\underbrace{\mu(g)}_{\geq \mathrm{mult}(g)\geq 1}>\mu(y)=\mathrm{mult}(y^m)=\underline{m}$$

$\geq \mathrm{mult}(g)\geq 1$ by the hypotheses on w

$$\Rightarrow \quad \mu(-h-yg)\overset{\text{Ob.(2)}}{=}\mu(-h)=m$$

and $S(-h-yg)\overset{\text{Ob.(4)}}{=}S(-h)=-1\overset{\neq 0}{h_m}x^m$ (where $h=\sum_i h_i x^i$)

$$\Rightarrow S(\Lambda(x^iy^j)) = S(x^i(-h-yg)^j)\overset{\text{Ob.(3)}}{=}\underbrace{S(x)^i}_{x^i}\cdot\underbrace{S(-h-yg)^j}_{(-h_mx^m)^j}$$

$$=\underbrace{\boxed{(-h_m)^j}}_{\in K^*}x^{i+jm} \qquad \Rightarrow \mu(\Lambda(x^iy^j))=i+jm$$

(As the assertions are rather technical, illustrate them on an example:

Example 5.2 Let $\tilde{w} = (y+x+1)^2 - (x^3+2x+1) \in \mathbb{R}[x,y]$

Since $GCD(x^3+2x+1, \underbrace{(x^3+2x+1)'}_{3x^2+2}) = 1$, $\tilde{w}$ is a smooth RBP

Put $w := \frac{1}{2}\tilde{w} = \frac{1}{2}(y^2+x^2+2yx+2y-x^3) =$

$$= y\underbrace{\left(x+\frac{y}{2}\right)}_{g(x,y)} + \underbrace{\frac{1}{2}(x^2-x^3)}_{h(x)} + y$$

$mult(g) = 1$, $m := mult(h) = 2 \Rightarrow w$ is of a required type

$\mu(g) = mult(\frac{y^2}{2}+x) = 1$, $S(g) = x$

$\mu(h) = mult(h) = 2$ $S(h) = \frac{1}{2}x^2$

$\mu(x^3y^2) = 3+2\cdot 2 = 7$, $\mu(x^2y^3) = 2+3\cdot 2 = 8$

$\Rightarrow \mu(x^3y^2+x^2y^3) = \mu(x^3y^2) = 7$

$S(\Lambda(x^3y^2+x^2y^3)) = S(\Lambda(x^3y^2)) = \frac{1}{4}x^7$ (s the proof of 5.1)