Theorem 7.3 If $K=\tilde{K}$, then [L is an AFF over K of genus g with $\tilde{K}$] [5.5]

(1) $\dim_L(\Omega_{L|K})=1$

Comment: We have 1-1-correspondences between (1) $L \leftrightarrow \Omega_{L|K}$ (2) $\mathcal{L}(B) \leftrightarrow \Omega_{L|K}((\omega)-B)$

(2) if $\omega \in \Omega_{L|K} \setminus \{0\}$, $A \in \mathrm{Div}(L|K)$, then $\psi_{\omega A}: \mathcal{L}((\omega)-A) \to \Omega_{L|K}(A)$ given by $\psi_{\omega A}(\lambda)=\lambda\cdot\omega$ is a K-isomorphism.

Proof: (1) Consider ψ_ω from 7.2 and let $\omega, \tilde{\omega} \in \Omega_{L|K}\setminus\{0\} \overset{7.2(3)}{\Longrightarrow}$

$\Rightarrow \exists B \in \mathrm{Div}(L|K): 0 \neq \psi_\omega(\mathcal{L}((\omega)-B)) \cap \psi_{\tilde{\omega}}(\mathcal{L}((\tilde{\omega})-B)) \subseteq \Omega_{L|K}(B) \Rightarrow$

$\Rightarrow \exists a, \tilde{a} \in L^*: \tilde{a}\tilde{\omega} = \psi_{\tilde{\omega}}(\tilde{a}) = \psi_\omega(a) = a\omega \Rightarrow \tilde{\omega} = \frac{a}{\tilde{a}}\cdot\omega$ where $\frac{a}{\tilde{a}} \in L^*$

$\Rightarrow \Omega_{L|K} = \mathrm{Span}_L(\omega) \Rightarrow \dim_L \Omega_{L|K} = 1$

Comment: $\forall$ Weil differential $\neq 0$, generates L-space $\Omega_{L|K}$

(2) Note that $\psi_{\omega A} = \psi_\omega|\mathcal{L}((\omega)-A)$ & it is injective K-linear into $\Omega_{L|K}(A)$ by 7.2(2)

Suppose $\tilde{\omega} \in \Omega_{L|K}(A)$ and we search $\lambda \in \mathcal{L}((\omega)-A)$ such that $\psi_{\omega A}(\lambda)=\tilde{\omega}$.

If $\tilde{\omega}=0$ then $\lambda:=0$; let $\tilde{\omega}\neq 0 \Rightarrow \exists \lambda \in L^*: \tilde{\omega}=\lambda\omega \in \Omega_{L|K}(A)$ by (1) $\overset{7.1}{\Longrightarrow}$

$\Rightarrow (\lambda\omega)-A \overset{7.2}{=} (\lambda)+(\omega)-A \geq 0 \Rightarrow \lambda \in \mathcal{L}((\omega)-A)$

Corollary 7.4 Let $K=\tilde{K}$. The canonical divisors form one coset modulo $\mathrm{Princ}(L|K)$ (i.e. if W is canonical, then $A\sim W \Leftrightarrow A$ is canonical).

Proof: Let $\omega \in \Omega_{L|K}\setminus\{0\}$, then $(\omega)\sim A \Leftrightarrow \exists \lambda \in L^*: A=(\lambda)+(\omega) \overset{7.2(1)}{=} (\lambda\omega) \Leftrightarrow$

$\overset{7.3(1)}{\Leftrightarrow} \exists \tilde{\omega} \in \Omega_{L|K}: A=(\tilde{\omega})$

Comment: As an another consequence of 7.3 we formulate Riemann-Roch Theorem:

Theorem 7.5 (Riemann-Roch) If $K=\tilde{K}$ and W is a canonical divisor, then

$$\ell(A)=\deg A+\ell(W-A)+1-g \quad \forall A \in \mathrm{Div}(L|K)$$

Proof: Let $W=(\omega)$ for $\omega \in \Omega_{L|K}\setminus\{0\} \overset{7.3}{\Longrightarrow} \mathcal{L}(W-A) \cong \Omega_{L|K}(A) \Rightarrow$

$\ell(W-A) = \dim(\mathcal{L}(W-A)) \overset{\text{obs.}(1)}{=} i(A) = g-1-\deg A+\ell(A)$

Corollary 7.6 Let $K=\tilde{K}$ and $A, W \in \mathrm{Div}(L|K)$, then:

(1) If W is canonical $\Rightarrow \ell(W)=g$, $\deg(W)=2g-2$, $i(W)=1$,

(2) (main consequence of R.-R. Thm): If $\deg A \geq 2g-1 \Rightarrow \ell(A)=\deg(A)+1-g$

Proof (1) follows from 7.5 for $A=0$ & $A=W$

(2) (1) $\Rightarrow \deg(W-A) \leq 2g-2-(2g-1) = -1$, $\overset{(R8)}{\Longrightarrow} \ell(W-A)=0 \overset{7.5}{\Longrightarrow} \ell(A)=\deg A+1-g$

Lemma 7.7 Let $K=\tilde{K}$, $A\in \mathrm{Div}(L|K)$, then

Comment: We are searching how to recognize canonical divisors among all divisors

(1) If $\deg A=2g-2$, $\ell(A)\geq g \Rightarrow A$ is canonical

(2) if $g=1 \Rightarrow A$ is canonical if and only if A is principal.

Proof: (1) $i(A) = \underbrace{\ell(A)}_{\geq g} - \underbrace{\deg A}_{=2g-2} + g-1 \geq 1 \overset{\text{obs.}(1)}{\Longrightarrow} \exists \omega \in \Omega_{L|K}(A)-\{0\}$

$\overset{7.1}{\Longrightarrow} A \leq (\omega)$

$\overset{7.6(1)}{\Longrightarrow} \deg A = \deg(\omega)) \Rightarrow A=(\omega)$

Proposition 7.8 Let $K=\tilde{K}$, $A, B \in \mathrm{Div}(L|K)$, $g=0$.

(1) A is principal $\Leftrightarrow \deg A=0$,

(2) $A\sim B \Leftrightarrow \deg A=\deg B$

(3) A is canonical $\Leftrightarrow \deg A=-2$

Proof of 7.8: (1) ($\Rightarrow$) if A is principal $\overset{6.8}{\Rightarrow}$ $\deg A = 0$

($\Leftarrow$) if $\deg A = 0 \overset{7.6(2)}{\Longrightarrow} \ell(A) = 1 \overset{6.9}{\Longrightarrow} A$ is principal.

(2) $A \sim B \Leftrightarrow A - B \in \mathrm{Princ}\,\mathcal{L}(K) \overset{(1)}{\Longleftrightarrow} \deg(A-B) = 0 \Leftrightarrow \deg A = \deg B$.

(3) follows from 7.6(1) & 7.7(1)

Comment: 7.9(4) shows the tool of determining genus of AFF

TaN $\mathbb{P}^{(1)}_{L|K} := \{P \in \mathbb{P}_{L|K} \mid \deg P = 1\}$

Lemma 7.9 Let $P \in \mathbb{P}^{(1)}_{L|K} \neq \emptyset$, $\ell \in \mathbb{Z} : \ell \geq 0$, $\Delta \in L$. Then

(1) $K = \tilde{K}$,

(2) $\Delta \in \mathcal{L}(iP) \setminus \mathcal{L}((i-1)P) \Leftrightarrow (\Delta)_- = iP \ \forall i \geq 1$,

(3) if $\exists k \geq 0$ such that $\ell(iP) \geq i - \ell + 1 \ \forall i \geq k \Rightarrow g \leq \ell$,

(4) if $\forall i \geq \ell + 1 \ \exists \Delta_i \in L$ such that $(\Delta_i)_- = iP \Rightarrow g \leq \ell$.

Proof: (1) $1 = \deg_{L|K}(P) = [\tilde{K} : K] \deg_{L|\tilde{K}}(P) \Longrightarrow [\tilde{K} : K] = 1$

(2) $\Delta \in \mathcal{L}(iP) \overset{\text{by the definition}}{\Longleftrightarrow} (\Delta) + iP \geq 0 \Leftrightarrow \begin{cases} v_Q(\Delta) \geq 0 \ \forall Q \in \mathbb{P}_{L|K} \setminus \{P\} \\ \& \\ v_P(\Delta) \geq -i \end{cases}$

Thus:

$\Delta \in \mathcal{L}(iP) \setminus \mathcal{L}((i-1)P) \Leftrightarrow v_Q(\Delta) \geq 0 \ \forall Q \in \mathbb{P}_{L|K} \setminus \{P\} \ \& \ v_P(\Delta) = -i \Leftrightarrow (\Delta)_- = iP$.

(3) Note that $\deg P = 1 \Rightarrow \deg(iP) = i$ and let $i \geq \max(k, 2g-1)$

$\Longrightarrow i - g + 1 = \deg(iP) - g + 1 \overset{7.6(2)}{=} \ell(iP) \overset{\text{by hypothesis}}{\geq} i - \ell + 1 \Rightarrow g \leq \ell$

(4) Let $i \geq \ell + 1 : (\Delta_i)_- = iP \overset{(2)}{\Longrightarrow} \Delta_i \in \mathcal{L}(iP) \setminus \mathcal{L}((i-1)P) \Rightarrow \mathcal{L}((i-1)P) \subsetneq \mathcal{L}(iP)$

$\Longrightarrow \tilde{K} \overset{(1)}{=} K = \mathcal{L}(0) \subseteq \mathcal{L}(\ell P) \subsetneq \mathcal{L}((\ell+1)P) \subsetneq \dots \subsetneq \mathcal{L}((\ell+j-1)P) \subsetneq \mathcal{L}((\ell+j)P) \subsetneq \dots$

compute dimensions: $1 = \ell(0) \leq \ell(\ell P) < \ell((\ell+1)P) < \dots < \ell((\ell+j-1)P) < \ell((\ell+j)P) < \dots$

$\Longrightarrow \ell(iP) \geq i - \ell + 1 \ \forall i \geq \ell + 1 \overset{(3)}{\Longrightarrow} g \leq \ell$

Example 7.10 Let x be a variable $\Rightarrow K(x)$ is an AFF over K

By 2.14 $\mathbb{P}_{K(x)|K} = \{P_p \mid p \in K[x], p \text{ irreducible}\} \cup \{P_\infty\}$

where $P_p = (p) = \{q \in K(x) \mid v_p(q) > 0\}$ is the maximal ideal of VR $K[x]_{(p)}$ (localization)

$P_\infty = \{q \in K(x) \mid v_\infty(q) > 0\}$ for $v_\infty(\frac{a}{b}) = \deg b - \deg a$, $a, b \in K[x] \setminus \{0\}$

$\Rightarrow v_{P_p} = v_p \ \& \ v_p(x^i) \geq 0 \ \forall i \geq 0 \ \& \ p \in K[x]$ irreducible

$v_\infty(x^i) = -\deg x^i = -i$ $\Longrightarrow (x^i)_- = iP_\infty \ \forall i \geq 1$

$\Rightarrow \deg P_\infty = \deg(x)_- \overset{6.8}{=} [K(x) : K(x)] = 1 \overset{7.9(4)}{\Longrightarrow} 0 \leq g \leq 0 \Rightarrow K(x)$ is of genus 0.