Lemma 6.9 Let $K=\tilde{K}$ and $A\in Div(L|K)$ such that $\deg A=0$, then (1) $\ell(A)\in\{0,1\}$, (2) $\ell(A)=1 \Leftrightarrow A\in Princ(L|K)$

Comment: Degree zero divisors are either principal or of zero Riemann-Roch space

Proof: Let $\ell(A)\ge 1 \xRightarrow{(D7)} \exists s\in L^*: A+(s)\ge 0 \xRightarrow{6.5}$
$\Rightarrow \deg(\underbrace{A+(s)}_{\ge \underline{0}}) = 0 + 0 = 0 \Rightarrow A+(s)=\underline{0} \Rightarrow A=(s^{-1}) \xRightarrow{(D9)} \ell(A)=\dim s^{-1}K=1$

Theorem 6.10 (Riemann) If $K=\tilde{K}$, then $\exists \gamma\in\mathbb{Z}: \gamma\ge 0$ such that
$\deg A-\ell(A)<\gamma\ \forall A\in Div(L|K)$

Comment: $\deg A-\ell(A)$ has an upper bound which characterises AFF; $\deg((s))-\ell((s))=-1\ \forall s\in L^*$

Proof: Since $A\le A_+ \xRightarrow{(D4)} \deg A-\ell(A)\le \deg A_+-\ell(A_+)$
$\Rightarrow$ it is enough to prove the claim for $A\ge \underline{0}$. $\boxed{\text{Let } A\ge\underline{0}}$
Let $s\in L\setminus K$ and denote $\boxed{C:=(s)_-}$ $\Rightarrow rC=(s^r)_-\ne 0\ \forall r$, $\deg C>0$
6.5 & 6.4(3) $\Rightarrow \exists B\ge\underline{0}: \deg(rC)-\ell(rC)=r[L:K(s)]-\ell(rC)\le\deg B-[L:K(s)]$
Put $\gamma:=\deg B-[L:K(s)]+1 \Rightarrow \boxed{\deg(rC)-\ell(rC)<\gamma}$; $A\ge 0\Rightarrow rC-A\le rC \Rightarrow$
$\xRightarrow{(D4)} \deg(rC-A)-\ell(rC-A)\le\deg(rC)-\ell(rC)<\gamma\quad \forall r>0 \Rightarrow$
$\Rightarrow \ell(rC-A) > \underbrace{-\gamma-\deg A}_{\text{a constant}} + r\underbrace{\deg C}_{>0}\quad \exists r_0: \forall r\ge r_0\ \ell(rC-A)\ge 1 \xRightarrow{(D6)}$
$\Rightarrow \deg A-\ell(A)\le\deg(rC)-\ell(rC)<\gamma$

Definition: The minimal γ from 6.10 for the AFF L over $\tilde{K}$ is called genus of L over K, it will be denoted $\boxed{g}$ in the sequel.

Observation (D) Let $A,D\in Div(L|K)$, $K=\tilde{K}$, $\deg D-\ell(D)=g-1$, then:
(1) $g>\deg 0-\ell(0)=-1\Rightarrow g\ge 0$

Comment: D is a divisor with the maximal possible $\deg(D)-\ell(D)$

(2) $\deg(A-D)-\ell(A-D)\le g-1$ (by the definition) $\Rightarrow \ell(A-D)\ge\deg A-\deg D-g+1$
(3) if $\deg A\ge\deg D+g \xRightarrow{(2)} \ell(A-D)\ge 1$
(4) if $\ell(A-D)\ge 1$ or $D\le A \xRightarrow{(D4),(D6)}$
$\Rightarrow g-1=\deg D-\ell(D)\le \deg A-\ell(A)\le g-1 \Rightarrow \deg A-\ell(A)=g-1$

Comment: If A is "large enough", then $\deg(A)-\ell(A)$ is maximal possible

Lemma 6.11: $\exists\gamma\in\mathbb{N}$ such that $\forall A\in Div(L|K)$ with $\deg(A)\ge\gamma$ it holds $\deg(A)=\ell(A)+g-1$

Proof: it is enough to put $\gamma:=\deg D+g$ from Observ. D and args (3), (4).

$\boxed{\text{T\&N}}$ Let $\mathbb{P}:=\mathbb{P}_{L|K}$ and consider $L^{\mathbb{P}}$ (Cartesian product) as an L-algebra with the operations $f\pm g(P)=f(P)\pm g(P)\ \forall f,g\in L^{\mathbb{P}}$, $\underline{0}(P):=0$, $\underline{1}(P)=1\ \forall P\in\mathbb{P}$ where $l\to l\cdot\underline{1}$ identifies L and $\{l\cdot\underline{1} \mid l\in L\}\subseteq L^{\mathbb{P}}$. Let $A=\sum_{P\in\mathbb{P}} a_P P\in Div(L|K)$:
$f\in L^{\mathbb{P}}$ is called an adèle if $|\{P\in\mathbb{P} \mid f(P)\notin O_P\}|<\infty$ and $\boxed{\mathbb{A}_{L|K}}$ denotes the set of all adèles over L/K.
$\mathbb{A}_{L|K}(A):=\{f\in L^{\mathbb{P}} \mid v_P(f(P))+a_P\ge 0\ \forall P\in\mathbb{P}\}$, $i(A):=g-1-\deg A+\ell(A)$.

$i(A) \geq 0$ and it is called the <u>index of speciality</u>.

A is <u>special</u> if $i(A) > 0$ and A is <u>nonspecial</u> if $i(A) = 0$.

Recall that $P^{z} = \{r \in L \mid v_P(r) \geq z\}\ \forall z \in \mathbb{Z},\ \forall P \in \mathbb{P}_{L|k}$

<u>Comment</u> P^z is a cyclic $\mathcal{O}_P$-module generated by t^z where $P = (t)$

<u>Observation (E)</u>: Let $\mathbb{P} = \mathbb{P}_{L|k}$, $r \in L$, $f \in L^{\mathbb{P}}$, $A = \sum_{P \in \mathbb{P}} a_P P \in \mathrm{Div}(L|k)$, then:

(1) $f \in \mathcal{A}_{L|k} \Leftrightarrow v_P(f(P)) < 0$ for only finitely many $P \in \mathbb{P} \overset{5.22}{\Rightarrow} r = r \cdot 1 \in \mathcal{A}_{L|k}$

(2) $\mathcal{A}_{L|k}$ is a L-subalgebra of the L-algebra $L^{\mathbb{P}}$

(3) $f \in \mathcal{A}_{L|k}(A) \Leftrightarrow v_P(f(P)) \geq -a_P\ \forall P \in \mathbb{P}$

<u>Comment</u>: $L^{\mathbb{P}}$ is L-algebra $\Rightarrow$ it is k-algebra, $\mathcal{A}_{L|k}$ is L & k-subalgebra

(4) $\mathcal{A}_{L|k}(A) = \prod_{P \in \mathbb{P}} P^{a_P}$ is a k-subspace of $\mathcal{A}_{L|k}$

(5) $\mathcal{A}_{L|k} = \bigcup_{B \in \mathrm{Div}(L|k)} \mathcal{A}_{L|k}(B)$

<u>Comment</u>: 6.12 describes correspondences between the index of speciality and $\mathcal{A}_{L|k}(A)$.

<u>Lemma 6.12</u> Let $K = \bar{k}$; $A = \sum a_P P$, $B = \sum b_P P \in \mathrm{Div}(L|k)$, $\Delta \in L^*$:

(1) if $A \leq B \Rightarrow \mathcal{A}_{L|k}(A) \subseteq \mathcal{A}_{L|k}(B)$ and $\dim_k(\mathcal{A}_{L|k}(B)/\mathcal{A}_{L|k}(A)) = \deg(B-A)$,

(2) if $A \leq B \Rightarrow \dim_k((\mathcal{A}_{L|k}(B)+L)/(\mathcal{A}_{L|k}(A)+L)) = i(A) - i(B)$,

(3) $\mathcal{A}_{L|k}(A) \cap \mathcal{A}_{L|k}(B) = \mathcal{A}_{L|k}(\min(A,B))$

$\mathcal{A}_{L|k}(A) + \mathcal{A}_{L|k}(B) = \mathcal{A}_{L|k}(\max(A,B))$

<u>Comment</u>: There is a Galois correspondence between $(\mathrm{Div}(L|k), \leq)$ and $(\{\mathcal{A}_{L|k}(A)\}, \subseteq)$

(4) $\dim_k(\mathcal{A}_{L|k}/(\mathcal{A}_{L|k}(A)+L)) = i(A)$

(5) $\mathcal{A}_{L|k}(A) + L = \mathcal{A}_{L|k} \Leftrightarrow i(A) = 0$

(6) $\Delta \cdot \mathcal{A}_{L|k}(A) = \mathcal{A}_{L|k}(A - (\Delta))$

<u>Proof</u>: (1) $A \leq B \Rightarrow \mathcal{A}_{L|k}(A) \overset{\text{Obs. } E(3)}{=} \{f \in L^{\mathbb{P}_{L|k}} \mid f(P) \geq -a_P (\geq -b_P)\} \subseteq \mathcal{A}_{L|k}(B)$

$\deg(B-A) = \dim(\prod_P P^{-b_P}/\prod_P P^{-a_P}) = \dim_k(\mathcal{A}_{L|k}(B)/\mathcal{A}_{L|k}(A))$

by Observ. C(3) using the arguments of the proof of 6.2 — Observation E(4)

(2) $A \leq B \Rightarrow \dim_k(\mathcal{A}_{L|k}(B)+L)/(\mathcal{A}_{L|k}(A)+L)) \overset{1.3(2)}{=} \dim(\mathcal{A}_{L|k}(B)/\mathcal{A}_{L|k}(A)) - \dim(\mathcal{L}(B)/\mathcal{L}(A))$

$(\mathcal{A}_{L|k}(B) \cap L) \quad \mathcal{A}_{L|k}(A) \cap L \quad \Vert \Leftarrow E(4) \Rightarrow \Vert$

$\overset{(1)}{=} \deg B - \deg A - (\ell(B) - \ell(A)) = i(A) - i(B)$

(3) Apply Obs. E(4): $\prod P^{-a_P} + \prod P^{-b_P} = \prod P^{\min(-a_P, -b_P)}$, $\min(-a_P,-b_P) = -\max(a_P, b_P)$

$\prod P^{-a_P} \cap \prod P^{-b_P} = \prod P^{\max(-a_P, -b_P)}$, $\max(-a_P,-b_P) = -\min(a_P, b_P)$

(4) (a) Let $i(A) = 0$ $\boxed{?\ \mathcal{A}_{L|k} = \mathcal{A}_{L|k}(A) + L}$ $\boxed{\supseteq} \Leftarrow$ Obs. E(1) & (5)

as $d_P \geq a_P$, $d_P + v_P(f(P)) \geq 0$

$\boxed{\subseteq}$ $f \in \mathcal{A}_{L|k}$, let $d_P := \max(a_P, 0, -v_P(f(P))\ \forall P$, $D := \sum d_P P \Rightarrow A \leq D$, $f \in \mathcal{A}_{L|k}(D)$

$\overset{\text{Obs. D(4)}}{\Rightarrow}$ $i(D) = 0 \Rightarrow f \in \mathcal{A}_{L|k}(D) + L \overset{(2)}{=} \mathcal{A}_{L|k}(A) + L$

(b) Let A be general $\Rightarrow$

$i(A) = 0$,

$\overset{6.11}{\Rightarrow} \exists B \geq A : i(B) = 0 \Rightarrow \mathcal{A}_{L|k} = \mathcal{A}_{L|k}(B) + L \Rightarrow$

$\Rightarrow \dim(\mathcal{A}_{L|k}/\mathcal{A}_{L|k}(A)+L) = \dim(\mathcal{A}_{L|k}(B)+L/\mathcal{A}_{L|k}(A)+L) \overset{(2)}{=} i(A)$

$\boxed{(5) \Leftarrow (4)}$

$f \in \Delta \mathcal{A}_{L|k}(A) \Leftrightarrow \Delta^{-1} f \in \mathcal{A}_{L|k}(A) \Leftrightarrow v_P(f(P)) + a_P - v_P(\Delta)\ \forall P \Leftrightarrow f \in \mathcal{A}_{L|k}(A-(\Delta))$