Comment: We start with important consequences of 5.21:

Corollary 5.22 If $s \in L^*$, then $\{P \in \mathbb{P}_{L|K} \mid v_P(s) \neq 0\}$ is finite.

Proof: As $v_P(s^{-1}) = -v_P(s)$ $\forall P \in \mathbb{P}_{L|K}$ it is enough to prove finiteness of $\{P \mid v_P(s) > 0\}$; for $s \in \tilde{K}-\{0\}$: $v_P(s) = 0$ $\forall P \in \mathbb{P}_{L|K}$
let $s \in \tilde{K}-\{0\}$ (i.e. it is transcendental) and $P_1, \dots, P_r$ distinct: $v_{P_i}(s) > 0$ $\forall i$

$$\Rightarrow r \le \sum_{i=1}^{r} \underbrace{v_{P_i}(s)}_{\ge 1} \underbrace{\deg P_i}_{\ge 1} \overset{5.21}{\le} [L:K(s)] \overset{1.13}{<} \infty$$

Comment: $|\{P \in \mathbb{P}_{L|K} \mid v_P(s) \neq 0\}| \le 2 \cdot [L:K(s)]$ since $K(s) = K(s^{-1})$.

Corollary 5.23 If f is a WEP and L is given by $f(\alpha,\beta) = 0$, then $\exists!\, P_\infty \in \mathbb{P}_{L|K}: v_{P_\infty}(\alpha) < 0$. Then $\deg P_\infty = 1$ and $v_{P_\infty}(\alpha) = -2$, $v_{P_\infty}(\beta) = -3$.

Proof: by 5.13(3) $3v_P(\alpha) = 2v_P(\beta)$ $\forall P: \alpha^{-1} \in P \Rightarrow 2 \mid v_P(\alpha^{-1}) > 0$

$$\Rightarrow \sum_{P:\alpha^{-1}\in P} \underbrace{v_P(\alpha^{-1})}_{\ge 2} \underbrace{\deg P}_{\ge 1} \overset{5.21}{\le} [L:\underset{K(\alpha)}{K(\alpha^{-1})}] \overset{f \text{ is WEP}}{=} 3 \Rightarrow \exists!\, P_\infty \ \deg P_\infty = 1 \ \&$$

$v_P(\alpha^{-1}) = 2 \Rightarrow v_P(\alpha) = -2 \overset{5.13(3)}{\Rightarrow} v_P(\beta) = -3$

T&N The place from 5.23 is denoted by P_∞.

Comment: By combining 5.17 & 5.23 we have full description of places of degree one over WEP.

Observation

If f is WEP smooth at $\forall \gamma \in V_f(K)$ and L is given by $f(\alpha,\beta) = 0$, then $\{P \in \mathbb{P}_{L|K} \mid \deg P = 1\} = \{P_\gamma \mid \gamma \in V_f(K), \deg P = 1\} \cup \{P_\infty\}$ by 5.17 & 5.23.

Example 5.24 $f := y^2 + y - (x^3+1) \in \mathbb{F}_2[x] \Rightarrow f$ is WEP, $\alpha = x + (f)$, $\beta = y + (f)$

$\Rightarrow L = \mathbb{F}_2(\alpha,\beta)$ is an AFF over $\mathbb{F}_2$ given by $f(\alpha,\beta) = 0$

$V_f(\mathbb{F}_2) = \{(1,0),(1,1)\} \Rightarrow P \in \mathbb{P}_{L|K}, \deg P = 1 \rightarrow P \in \{P_{(1,0)}, P_{(1,1)}, P_\infty\}$

But $|\mathbb{P}_{L|\mathbb{F}_2}| = \infty$ by 5.20(1) $\Rightarrow$ other places are of degree > 1

e.g. $\forall m \in \mathbb{F}_2[x]: \deg m > 1$, m irreducible $\exists P \in \mathbb{P}_{L|\mathbb{F}_2}: m(\alpha) \in P \rightarrow \deg P > 1$

6. Divisors

Comment: We introduce new notions to obtain new tools (and other notions) for dealing with WEP

In the sequel L is an AFF over K and $\tilde{K}$ is the field of constants.

Definition: Let $\mathrm{Div}(L|K) = \{\sum_{P \in \mathbb{P}_{L|K}} a_P P \mid a_P \in \mathbb{Z}\}$ denotes the free abelian group with the free bases $\mathbb{P}_{L|K}$ ($\Rightarrow |\{a_P \mid a_P \neq 0\}| < \infty$) and operations $\sum a_P P + \sum b_P P = \sum (a_P + b_P) P$, $\underline{0} = \sum 0 P$. A formal sum $\sum a_P P \in \mathrm{Div}(L|K)$ is called a *divisor* (of the AFF).

Example 6.1 $\forall r\in L^*$: $\sum_{P\in \mathbb{P}_{L|K}} v_P(r)P$ is a divisor as $|\{P \mid v_P(r)\neq 0\}|<\infty$ by 5.2

$\boxed{\text{T\&N}}$ The *degree of a divisor* is defined by $\deg_K(\sum a_P P) = \sum a_P \deg_K P \in \mathbb{Z}$

Comment: Degree of a place and of a divisor depends on the underlying field K, if it is clear, we will write only $\deg A$.

Observation (A) Let $r := [\tilde{K} : K]$, $P\in \mathbb{P}_{L|K}$, $a\in L^*$

(1) L is an AFF over $\tilde{K}$, $\tilde{K}\subseteq \mathcal{O}_P$ by 2.15 $\Rightarrow \mathbb{P}_{L|\tilde{K}} = \mathbb{P}_{L|K}$, $\operatorname{Div}(L|K) = \operatorname{Div}(L|\tilde{K})$

(2) $\deg_{\tilde{K}} P = \dim_{\tilde{K}} \mathcal{O}_P/P = r\cdot \dim_K \mathcal{O}_P/P = r\cdot \deg_K P$ ($\Rightarrow r<\infty$ using 2.15)

(3) $\deg_K = r\cdot \deg_{\tilde{K}}$ is a homomorphism of the abelian groups $\operatorname{Div}(L|K)$ and $\mathbb{Z}$.

(4) $\sum v_P(a)P = 0 \Leftrightarrow v_P(a) = 0\ \forall P\in\mathbb{P}_{L|K} \Leftrightarrow a\in \tilde{K}^*$

$\boxed{\text{T\&N}}$ The divisor $\sum v_P(r)P$ for $r\in L^*$ from 6.1 is called *principal*, we will denote it by (r) and $\operatorname{Princ}(L|K) := \{(r) \mid r\in L^*\}\subseteq \operatorname{Div}(L|K)$.

$\boxed{\text{T\&N}}$ Let $A = \sum a_P P$, $B = \sum b_P P \in \operatorname{Div}(L|K)$. Let us denote:

$\max(A,B) := \sum \max(a_P,b_P)P$, $\min(A,B) := \sum \min(a_P,b_P)P$ $(\in \operatorname{Div}(L|K))$

$A_+ := \max(A,\underline{0})$, $A_- := \min(A,\underline{0})$, A is *positive* if $A = A_+$

Define relations $\leq$ (resp. opposite $\geq$) and $\sim$ on $\operatorname{Div}(L|K)$:

$A\leq B \overset{def}{\Leftrightarrow} a_P\leq b_P\ \forall P\in\mathbb{P}_{L|K}$ $(\Leftrightarrow A = \min(A,B) \Leftrightarrow B = \max(A,B))$

$A\sim B \overset{def}{\Leftrightarrow} A-B\in \operatorname{Princ}(L|K)$; $\mathcal{L}(A) := \{r\in L^* \mid (r)+A\geq \underline{0}\}\cup\{0\}$

Observation (B) Let $A,B,C,D\in \mathbb{P}_{L|K}$, $r,s\in L^*$

Comment: We will need to know more about divisors A: $A\geq\underline{0}$

(1) $(r\cdot s) = \sum \overbrace{v_P(r\cdot s)}^{=v_P(r)+v_P(s)} P = (r)+(s) \Rightarrow$ the map $r\to(r)$ is a group homomorphism $L^*\to\operatorname{Div}(L|K)$

(2) ~~thus~~ Thus $(r^{-1}) = -r$, $(1) = (0)$ and $\operatorname{Princ}(L|K)\leq \operatorname{Div}(L|K)$.
moreover $(r) = (s) \Leftrightarrow \exists \lambda\in\tilde{K}^*: r = \lambda s$

(3) $\sim$ is an equivalence on $\operatorname{Div}(L|K)$ compatible with group operations

(4) $\leq$ is an ordering on $\operatorname{Div}(L|K)$ & $\forall A\leq B, C\leq D: A+C\leq B+D$

(5) if $r\in L-\tilde{K} \Rightarrow \exists P,Q\in\mathbb{P}_{L|K}: v_P(r)>0, v_Q(r)<0 \Rightarrow r\not\geq \underline{0}$ $(r\not\leq\underline{0})$

(6) $\mathcal{L}(A)$ is a $\tilde{K}$-space ($\Rightarrow K$-space), $\mathcal{L}(\underline{0}) = \{r\in L^* \mid (r)\geq 0\}\cup\{0\} \overset{(5)}{\cong} \tilde{K}$.

$\boxed{\text{T\&N}}$ $\operatorname{Cl}(L|K) := \operatorname{Div}(L|K)/\operatorname{Princ}(L|K)$ is called the *class group* of the AFF

$A\in\operatorname{Div}(L|K)$: $\mathcal{L}(A)$ is *Riemann-Roch space* of A, $\underline{\ell(A) = \dim_{L|K} A := \dim_K \mathcal{L}(A)}$

If $K = \tilde{K}$, then L is called a *full constant AFF over K*

Observation (C) Let $i\leq j$, $P\in\mathbb{P}_{LK}$, $t\in P$ $v_P(t) = 1 \Rightarrow (t) = P$

$\forall r\in\mathbb{Z}$: $P^r := t^r\mathcal{O}_P$ a module generated by t^r

(1) $\varphi_j: \mathcal{O}_P/P\to P^{j-1}/P^j$ defined by $\varphi_j(a+P) = at^{j-1}+P^j$ is an ~~iso~~ isomorphism of K-spaces

(2) $\deg_K P = \dim_K \mathcal{O}_P/P \overset{(1)}{=} \dim_K P^{j-1}/P^j$

(3) $\boxed{\dim_K P^i/P^j \overset{(2)}{=} \sum_{r=i+1}^{j} \dim_K P^{r-1}/P^r = (j-i)\deg P}$