In the rest of the section L is an AFF over k given by $f(\alpha,\beta)=0$, $\deg f \geq 2$, which is simultaneously given by $w_\sigma(u,v)=0$ where $w_\sigma = y\,g(x,y) + h(x) + y$, $h \in k[x]$, $g \in k[x,y]$, $\deg h \geq 2$, $\operatorname{mult} h \geq 2$, $\operatorname{mult} g \geq 1$, $\sigma \in \mathrm{Aff}_2(k)$, $w_\sigma = (\sigma^{-1})^*(f)$, and $(u,v) = \sigma(\alpha,\beta) = (\sigma^*(x)(\alpha,\beta), \sigma^*(y)(\alpha,\beta))$.

If f is smooth at $\gamma \in V_f(k)$ and $A = \begin{pmatrix} b_1 & b_2 \\ a_1 & a_2 \end{pmatrix} \in GL_2(k)$ for $a_1 = \frac{\partial f}{\partial x}(\gamma)$, $a_2 = \frac{\partial f}{\partial y}(\gamma)$, then σ could be given by $\sigma := \vartheta_A \tau_{-\gamma}$ of the Proposition 5.7. (! beware: I changed numbers to correct as !)

Comment: It is very important we can change polynomial giving L over k to the polynomial of "w_σ-type" which works ~~for~~ in every smooth point of V_f (! we do not need any change $w \to f$ as there are no requirements concerning f!). It allows us to compute $v_P(\ell(\alpha,\beta))$ for an arbitrary line ℓ ~~order~~ using machinery of 5.8 & 5.9 when P is the unique $\in \mathbb{P}_{L|k}$ from 5.8 & 5.9. ~~In~~ The aim of the rest of the section is to show that $P_\gamma \in \mathbb{P}_{L|k}$ are those P from 5.8/9.

$\boxed{\text{T\&N}}$ Let $p \in k[x]$, $\gamma \in k$. Recall that multiplicity (of the root) γ of (the polynomial) p is $r \geq 0$ for which $(x-\gamma)^r \mid p$ & $(x-\gamma)^{r+1} \nmid p$.

Comment: We have noticed that multiplicity of 0 of $p(x)$ is exactly $\operatorname{mult} p$, now we need a general observation. The aim is to shift "everything" to the point $\underline{0}=(0,0)$ and to compute with usual multiplicities.

Observation: Let $p, s \in k[x]$, $g \in k[x,y]$, $\gamma \in k$, and $\tau_\gamma \in \mathrm{Aff}_1(k)$.

(1) multiplicity of γ of p is $r \iff \operatorname{mult} \tau^*_{-\gamma}(p) = r$

(2) if $s(\gamma)=0$, $\hat{g}(x) := g(x-\gamma, s(x)) \Rightarrow$ multiplicity of γ of $\hat{g}(x) \geq \operatorname{mult} g$.

Proposition 5.9: Let $\gamma=(\gamma_1,\gamma_2) \in V_f(k)$, $\frac{\partial f}{\partial y}(\gamma) \neq 0$, $\lambda, \mu \in k$ such that $\gamma_2 = \lambda\gamma_1 + \mu$. Then $\exists!\, P \in \mathbb{P}_{L|k}$ for which $\{\alpha-\gamma_1, \beta-\gamma_2\} \subset P$ and

$v_P(\beta - \lambda\alpha - \mu) =$ multiplicity of γ of the polynomial $\hat{f}(x) = f(x, \lambda x + \mu)$.

Comment: The condition $\gamma_2 = \lambda\gamma_1 + \mu$ means that γ is an element of the line $V_{y-\lambda x-\mu}(k)$, so we ~~the~~ will describe how to measure v_P at points of lines (note that the rule is easy to test in algorithms!)

Proof: Denote $\Delta = \Delta_{\underline{\gamma}}(f)$ the tangent of f at $\underline{\gamma}$, $a_1 := \frac{\partial f}{\partial x}(\underline{\gamma})$, $a_2 := \frac{\partial f}{\partial y}(\underline{\gamma}) \neq 0$ (by hypothesis)

$\Rightarrow \Delta = a_1(x-\gamma_1) + a_2(y-\gamma_2) = a_2\left(y + \frac{a_1}{a_2}x - (\gamma_2 + \frac{a_1}{a_2}\gamma_1)\right)$

put $\hat{\Delta}(x) = \Delta(x, \lambda x + \mu) \in K[x] \Rightarrow \deg \Delta(x) \leq 1$ & $\hat{\Delta}(\gamma_1) = \Delta(\gamma_1, \underbrace{\lambda\gamma_1+\mu}_{=\gamma_2}) = \Delta(\gamma_1, \gamma_2) = 0$

$\Rightarrow$ either $\hat{\Delta} = c(x-\gamma_1)$ for $c \in K \setminus \{0\}$ or $\hat{\Delta} = 0$.

(*) We claim: $\hat{\Delta} = 0 \Leftrightarrow \lambda x + \mu + \frac{a_1}{a_2}x - (\gamma_1 + \frac{a_1}{a_2}) = 0 \Leftrightarrow \Delta = a_2(y - \lambda x - \mu)$

$a_2 \neq 0 \Rightarrow \Delta \neq 0$ i.e. f is smooth at $\underline{\gamma} \overset{5.8}{\Rightarrow} \exists!\, P \in \mathbb{P}_{L|K}: \underbrace{v_P(\alpha-\gamma_1) > 0, v_P(\beta-\gamma_2) > 0}$

$\{\alpha-\gamma_1, \beta-\gamma_2\} \subseteq P$

$\Downarrow$

$A = \begin{pmatrix} 1 & 0 \\ a_1 & a_2 \end{pmatrix} \in GL_2(K)$; put $\sigma := \vartheta_A \tau_{-\underline{\gamma}}$ Comment: $\sigma^*(y) = a_1(x-\gamma_1) + a_2(y-\gamma_2) = \Delta$

(from 5.7(3)) which is used in the proof of 5.8

Compute $f(x,y) = \sigma^*(w_0) = \sigma^*(h + \Delta g + y) = h(x-\gamma_1) + \Delta(x,y)(g(x-\gamma_1, \Delta(x,y)) + 1)$

Substitute $y = \lambda x + \mu$: $\hat{f}(x) = f(x, \lambda x + \mu) = h(x-\gamma_1) + \hat{\Delta}(x) \cdot (g(x-\gamma_1, \hat{\Delta}(x)) + 1)$

Case (a) if $\Delta = a_2(y - \lambda x - \mu) \Rightarrow \hat{f}(x) = h(x-\gamma_1)$ by (*) as $\hat{\Delta} = 0$

$\Rightarrow v_P(\beta - \lambda\alpha - \mu) \overset{5.8 \& 5.5}{=} v_P(w) = \text{mult}(h) \underset{\text{Obs. (1)}}{\overset{\uparrow}{=}}$ multiplicity of γ_1 of $\hat{f}$

Comment: we use the translation of $K(\alpha,\beta) = L$ to $K(y,w) = L$ from the beginning

case (b) if $\Delta \neq a_2(y - \lambda x - \mu) \Rightarrow \hat{\Delta} = c(x-\gamma_1)$ for some $c \in K^*$ by (*)

$\Rightarrow \hat{f}(x) = h(x-\gamma_1) + c(x-\gamma_1) \cdot \underbrace{(g(x-\gamma_1, c(x-\gamma_1)) + 1)}$ $(x-\gamma_1) \nmid$ it by Observ. (2)

$(x-\gamma_1)^2 \mid h(x-\gamma_1)$ $\quad$ $\boxed{\gamma_1 \text{ is not a root of it}}$

$\overset{5.8}{=} (x-\gamma_1)^2 \cdot \tilde{h} + c(x-\gamma_1)\tilde{g}$ where $\tilde{g}(\gamma_1) \neq 0 \Rightarrow$ multiplicity of γ_1 of $\hat{f} = \underline{1}$

$\overset{5.8}{\Rightarrow} v_P(\beta - \lambda\alpha - \mu) = 1 \overset{5.8}{=}$ multiplicity of γ_1 of $\hat{f}$ $\square$

Example 5.10

Let $f = y^2 + xy + x^5 + 32 \in \mathbb{R}[x,y]$

by 4.9. f is absolutely irreducible $\Rightarrow$ it is irreducible

Put $\alpha := x + (f)$, $\beta := y + (f) \in \mathbb{R}[x,y]$, then $\boxed{L := \mathbb{R}(\alpha,\beta) \text{ is a AFF over } \mathbb{R} \text{ given by } f(\alpha,\beta) = 0}$

$(-2,2) \in V_f(\mathbb{R})$ because $f(-2,2) = 4 - 4 - 32 + 32$

$\left.\begin{array}{l} \frac{\partial f}{\partial x} = y + 5x^4 \Rightarrow \frac{\partial f}{\partial x}(-2,2) = 2 + 80 = 82 \\ \frac{\partial f}{\partial y} = 2y + x \Rightarrow \frac{\partial f}{\partial y}(-2,2) = 4 + (-2) = 2 \end{array}\right\} \Rightarrow \Delta = \Delta_{(-2,2)}(f) = 82x + 2y + 160$

Put $\Delta := \frac{1}{2}\Delta(\alpha,\beta) = \beta + 41\alpha + 80$, by 5.9 $\exists!\, P \in \mathbb{P}_{L|K}: \{\alpha+2, \beta-2\} \subseteq P$

compute $v_P(\Delta)$ by 5.9: $\hat{f}(x) = f(x, -41x - 80)$

Comment: Computing of $\hat{f}$, substitution to f and comp. the multiplication is algorithmic!

$= x^5 + 40 \cdot 41 x^2 - 80 \cdot 81 x + 80^2 + 32$

$\hat{f}' = 5x^4 + 80 \cdot 41 x + 81 \cdot 80$, $\hat{f}'' = 20x^3 + 80 \cdot 41$

$\boxed{v_P(\Delta) = 2} \overset{5.9}{\Longleftarrow} \hat{f}(-2) = 0 = \hat{f}'(-2)$ & $\hat{f}''(-2) \neq 0$

$\boxed{\text{T\&N}}$ Let $\gamma \in \mathbb{A}^2(K)$: $\gamma \in V_f(K)$ ($\Rightarrow$ $(f) \subseteq I_\gamma = (x-\gamma_1, y-\gamma_2)$)

Denote by $R_\gamma := K[x,y]_{(I_\gamma)}$ - the localization of $K[x,y]$ at I_γ

$= \{\frac{a}{b} \in K(x,y) \mid a, b \in K[x,y],\ b(\gamma) \neq 0\}$

(I_γ) denotes the maximal ideal of R_γ ; $(I_\gamma) = \{\frac{a}{b} \in R_\gamma \mid a \in I_\gamma\}$

$\omega_\gamma : R_\gamma \to L$ is defined by the rule $\omega_\gamma(\frac{a}{b}) = \frac{a(\alpha,\beta)}{b(\alpha,\beta)}$

we will usually write $\frac{a}{b}(\gamma)$ instead $\omega_\gamma(\frac{a}{b})$ (i.e. $\frac{a}{b}(\gamma) := \omega_\gamma(\frac{a}{b})$)

Denote ${}_f\mathcal{O}_\gamma := \{\varrho \in L \mid \exists r \in R_\gamma : \omega_\gamma(r) = \varrho\}$ If f is fixed we will write

${}_fP_\gamma := \{\varrho \in L \mid \exists r \in (I_\gamma)\ \omega_\gamma(r) = \varrho\}$ $\mathcal{O}_\gamma := {}_f\mathcal{O}_\gamma$ and $P_\gamma := {}_fP_\gamma$

Let $r \in L$: $\mathrm{Dom}_f(r) = \{\gamma \in V_f(K) \mid r \in \mathcal{O}_\gamma\}$

Comment: We need the definition of $\mathcal{O}_\gamma$ and P_γ preferably. Other ones serve as tools for proving the properties of $\mathcal{O}_\gamma$ & P_γ

Comment: Our aim is to prove that $\mathcal{O}_\gamma$ are VR hence $P_\gamma \in \mathbb{P}_{L|K}$ $\forall \gamma$ smooth.

Observation: In the notation introduced above

(1) ω_γ is a well defined ring homomorphism (a technical exercise)

(2) $\mathcal{O}_\gamma = \omega_\gamma(R_\gamma)$, $P_\gamma = \omega_\gamma((I_\gamma))$

(3) $\mathcal{O}_\gamma$ is a local ring with the maximal ideal P_γ,
$\mathcal{O}_\gamma = K + P_\gamma$ and $\dim_K \mathcal{O}_\gamma / P_\gamma = 1$ (as it is a homomorphic image of the local ring R_γ)

(4) If $r = \omega_\gamma\left(\frac{a}{\prod b_i}\right)$ where $a, b_i \in K[x,y]$ b_i are irreducible, $b_i(\gamma) \neq 0$

$\Rightarrow V_f(K) \setminus \mathrm{Dom}_f(r) = \{\alpha \in V_f(K) \mid r \notin \mathcal{O}_\alpha\} \subseteq \bigcup_i V_{\{f, b_i\}}$ - is a finite set by 4.4.

$b_i \notin (f)$ as $b_i(\gamma) = 0$

(5) ${}_f\mathcal{O}_\gamma = {}_{\tau^*_\gamma(f)}\mathcal{O}_{(0,0)}$ and ${}_fP_\gamma = {}_{\tau^*_\gamma(f)}P_{(0,0)}$

Comment: Useful technical exercise. It says we can shift ${}_f\mathcal{O}_\gamma$ to ${}_{\tau^*_\gamma(f)}\mathcal{O}_{(0,0)}$ where is easier to recognize multiplicities of roots