3. CFF Homework, series 3, to be submitted till 4th May

All steps should be explained in detail (preferably by reference to the class assertions).

Let L_i is an AFF over $\mathbb{F}_2$ given by $f_i(\alpha, \beta) = 0$ for $i = 1, 2$ where

$$f_1 = y^2 + x^3 + x + 1 \in \mathbb{F}_2[x, y] \quad \text{and} \quad f_2 = y^2 + y + x^3 + 1 \in \mathbb{F}_2[x, y].$$

3.1. Show that f_1 is singular at $(1, 1) \in V_{f_1}(\mathbb{F}_2)$ and find an element $a \in L_1$ for which both elements $a, a^{-1} \notin \mathcal{O}_{(1,1)}$.

Hint: Use the definition of singularity (3.12 is not applicable!) and construct a as in the proof of 5.11 (using the shift $(1, 1) \to (0, 0)$).

5 points

3.2. Check that $V_{f_2}(\mathbb{F}_2) = \{(1, 0), (1, 1)\}$ and
(a) prove that $\{P_{(1,0)}, P_{(1,1)}\} \subseteq \mathbb{P}_{L_2/\mathbb{F}_2}$,
(b) and find a generator p of the principal ideal $P_{(1,0)}$ of the DVR $\mathcal{O}_{P_{(1,0)}}$ of L_2.

Hint: (a) Apply 5.16 and the definition of smoothness.
(b) Apply the fact that $\mathcal{O}_{P_{(1,0)}}$ is a DVR by 2.15 and that p is a generator of a place P iff $\nu_P(a) = 1$. Then such an element could be found for example by applying 5.8.

5 points

3.3. Determine the number of places:
(a) $|\{P \in \mathbb{P}_{L_1/\mathbb{F}_2} \mid \alpha^{-5} \in P\}|$ of L_1,
(b) $|\{P \in \mathbb{P}_{L_2/\mathbb{F}_2} \mid \alpha \in P\}|$ of L_2,
(c) $|\{P \in \mathbb{P}_{L_2/\mathbb{F}_2} \mid \alpha + 1 \in P\}|$ of L_2,
(d) $|\{P \in \mathbb{P}_{L_2/\mathbb{F}_2} \mid \deg P = 1\}|$ of L_2.

Hint: (a) Using observations $u \in P$ iff $\nu_P(u) \geq 1$ and $\nu_P(\alpha^{-5}) = -5\nu_P(\alpha)$ prove that $\alpha^{-5} \in P \Rightarrow \alpha \in P \Rightarrow \nu_P(\alpha) < 0$ and then apply 5.23.
(b) Compute $[L_2 : K(alpha)]$ by 4.6 and use the exercise 2, 5.17 and 5.23 to show that there is no place of degree 1 containing α and then apply 5.21 and an observation that every element of an AFF is contained in some place.
(c) Show that $\alpha + 1 \in P_{(1,0)} \cap P_{(1,1)}$ and apply exercise 2 and 5.21 along with 4.6.
(d) Use (c) and 5.21 to determine degree of places from (c) and then apply 5.17 and 5.23.

10 points